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Linear time-invariant solutions for LQ optimal control problems with terminal-state affine constraints

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Abstract— **We study the problem of steering the state of a system from a given initial condition towards a prescribed affine set while minimizing a quadratic cost functional. While the optimal solution is defined in terms of a time-varying open-loop control law, herein the problem is solved by limiting the search for the optimal input in the space of linear time-invariant feedback control laws. This choice preserves the LTI nature of the original plant in closed loop. Within this framework, the solution of the underlying optimal control problem hinges upon the solution of a nonlinear constrained optimization problem. Constructive algorithms and comparison with the time-varying optimal control law are discussed.**

Index Terms— **Optimal control, Optimization, Linear systems**

I. Introduction

Over the last century two main, somewhat alternative, methodologies for solving optimal control problems have been envisioned [1], [2], [3], [4], [5]. These are nowadays considered as the two cornerstones of the theory concerning the problem of controlling a dynamical systems in an optimal fashion: *Dynamic Programming* (DP) (see, *e.g.*, [6], [7]) and *Pontryagin's Minimum Principle* (PMP) (see, *e.g.*, [8]). Methods inspired by DP are particularly appealing for a number of reasons. In particular DP provides necessary and sufficient conditions of optimality, characterizing both the optimal (feedback) solution and the optimal cost for any initial condition in the state space. The optimal solution is obtained from the knowledge of the solution to the so-called *Hamilton Jacobi Bellman* partial differential equation (HJB-PDE). Obtaining a closed-form solution to the HJB-PDE is, in general, a daunting task. In the case of linear dynamics, however, such a solution revolves around the solvability of an algebraic equation, the celebrated *Algebraic Riccati Equation* (ARE), whenever the time horizon is infinite. In this specially structured setting the optimal solution is then yielded by a time-invariant state feedback. On the other hand, the use of a finite horizon of optimization affects the nature of the problem and its solution, so that the HJB-PDE reduces to a *Differential Riccati Equation* (DRE), leading to a time-varying state feedback.

The computational complexity intrinsically related to the computation of a solution to the HJB-PDE represents the major drawback of methods relying on DP, together with the fact that, within this framework, problems characterized by the presence of constraints involving the state variables and possibly in the presence of free terminal time are difficult to handle (see [9], [10]). On the other hand, such a class of problems becomes tractable within the framework provided by PMP. It is worth pointing out, however, that with respect to DP, methods based on PMP only provide necessary conditions of optimality and, moreover, the optimal solution is characterized, in the finite horizon setting, by the solution of a *two-point-boundary-value problem* (TPBVP). As a consequence this strategy yields an open-loop control law. Closed-form parameterization of the optimal open-loop solutions to such a problem, in the setting of fixed terminal time, have been provided in [11], together with results concerning the existence of a solution to such a class of problems. An extension of these results towards the case involving free terminal time has been provided in [12], although stated in the context of optimal control of hybrid systems.

Considering the setting of LQ optimal control problems with terminal-state affine constraints and free terminal time, the main contribution of this work is to provide a constructive characterization of an optimal solution within the set of linear time-invariant feedback control laws. The result is achieved by resorting to an equivalent formulation of the control task in terms of a nonlinear optimization problem. The derivation of such nonlinear programming problem from the original problem is thoroughly discussed and analyzed. The problem is characterized by the presence of an additional constraint, involving a *Lyapunov*-like matrix equation. Once solved, the solution of such an equivalent problem provides the optimal time-invariant gain matrix. This has the simple yet powerful consequence of preserving the linearity as well as the time-invariant nature of the underlying plant. Conversely, the candidate optimal solution, which is naturally given by the solution of a two-point boundary value problem, inevitably leads to an open-loop solution. When implemented on the original plant, such a solution induces a time-varying contribution to the closed-loop system, as discussed in [11] and [12].

The rest of the paper is organized as follows. The class of problems considered, together with some preliminaries,

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is introduced in Section II. The main result, mainly involving the constructive characterization of the nonlinear optimization problem equivalent to the original optimal control problem, providing the optimal time-invariant state feedback control law, is presented in Section III. Moreover, a gradient-based strategy, providing a locally optimal solution to the equivalent nonlinear optimization problem, is discussed in Section IV. Section V provides simulation results in which the proposed approach is compared against the one arising in the open loop scenario. Finally, a perspective on future work and some concluding remarks are given in Section VI.

II. Problem statement and preliminaries

The main purpose of this paper is to study the class of LQ optimal control problems described by the cost functional

$$\min_{u(\cdot),\, t_f} J(u(\cdot), t_f) = \min_{u(\cdot),\, t_f} \frac{1}{2}\int_{t_0}^{t_f} \|x(\tau)\|_Q^2 + \|u(\tau)\|_R^2 \, d\tau, \quad (1)$$

subject to the linear dynamics

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(t_0) = x_0, \quad (2)$$

and to the terminal-state affine constraints

$$\chi(x(t_f)) := Sx(t_f) + d = 0. \quad (3)$$

The variable $x : \mathbb{R} \to \mathbb{R}^n$ describes the state of the system, while $u : \mathbb{R} \to \mathbb{R}^m$ denotes the control input which acts on the system. Moreover $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $t_f > t_0 \geq 0$ is the final time. The matrices $Q \in \mathbb{R}^{n\times n}$, $Q = Q^\top$ and $R \in \mathbb{R}^{m\times m}$, $R = R^\top$, are assumed to be positive semi-definite and positive definite, respectively. Finally $S \in \mathbb{R}^{l\times n}$, with $l < n$, is assumed to be full-row rank and $d \in \mathbb{R}^l$. The following structural assumption is required in what follows.

Assumption 1. The pair (A, B) is reachable. ○

By introducing the *costate variable* $\lambda : \mathbb{R} \to \mathbb{R}^n$ and the (minimized) *Hamiltonian function* associated to the problem (1), (2), (3), namely the function

$$H(x, \lambda) = \frac{1}{2}x^\top Qx - \frac{1}{2}\lambda^\top BR^{-1}B^\top\lambda - \lambda^\top Ax, \quad (4)$$

it is well known (see, *e.g.* [2], [3]) that, for a prescribed initial condition $x_0 \in \mathbb{R}^n$, necessary conditions of optimality for the problem defined by (1), (2), (3) are provided by the two-point boundary value problem described by the equations

$$\begin{aligned} &\dot{x} = Ax - BR^{-1}B^\top\lambda, && x(t_0) = x_0, \\ &\dot{\lambda} = -Qx - A^\top\lambda, && \lambda(t_f) =: \lambda_f = -S^\top\mu, \\ &Sx_f + d = 0, && \\ &H(x_f, \lambda_f) = 0, && \end{aligned} \quad (5)$$

where $x_f = x(t_f)$ and $\mu \in \mathbb{R}^l$. Then, a candidate optimal control law is provided in terms of the open-loop policy

$$u^\star(t) = -R^{-1}B^\top\lambda(t). \quad (6)$$

Remark 1. An optimal control problem similar to (1), (2), (3) has been considered in [11], although in the presence of a prescribed value for the terminal time $t_f \in \mathbb{R}_{>0}$. It is shown therein (see [11, Prop. 1]) that the problem admits an optimal solution for specific t_f. Therefore, existence of the optimal control law for the cost *functional* (1), subject to (2) and (3), is intimately related to the properties of the *function* $t_f \mapsto \hat{J}(t_f) := \min_u J(u(\cdot), t_f)$. ▲

Two relevant observations are in order about the computation and the structure of the candidate optimal solution provided by (5) and (6). As far as the former aspect is concerned, (5) entails that the solution revolves around the solution of an ordinary differential equation with *split* boundary conditions, which may not be straightforward to find. Furthermore, regarding the latter aspect, the intrinsically open-loop nature of the control law derived in (6) may not be particularly desirable in practice, as it is not robust to uncertainties or perturbations. The above reasoning motivates the results discussed herein, which consist in addressing and solving the optimal control problem defined by (1), (2), (3) while further constraining the set of admissible control laws to the class of linear, time-invariant, feedback policies. The following statement introduces formally the problem investigated in the paper.

Problem 1. Let $x_0 \in \mathbb{R}^n$ be given and suppose that Assumption 1 holds. Suppose that u belongs to the set of feedback policies $\mathcal{F} := \{u(\cdot) = Kx(\cdot), K \in \mathbb{R}^{m\times n}\}$, namely

$$u(t) = Kx(t), \quad (7)$$

for all $t \in [t_0, t_f]$, with $K \in \mathbb{R}^{m\times n}$. Find, if they exist, a final time $t_f^\star > t_0$ and a constant gain matrix $K^\star = K^\star(x_0)$ such that $u^\star = K^\star x$ minimizes (1) with respect to all $u \in \mathcal{F}$, with t_f replaced by $t_f^\star$, along the trajectories of (2), (7), and such that $\chi(x(t_f^\star)) = 0$. ○

The restriction to the class of time-invariant state feedback described by (7) allows characterizing the optimal terminal time $t_f^\star$ and the optimal time-invariant gain matrix $K^\star(x_0)$ by relying on the solution to an equivalent constrained nonlinear optimization problem. This is precisely the purpose of the next section, in which such an equivalent problem is formally introduced and discussed. In the following, without loss of generality, it is assumed that $t_0 = 0$.

III. Time-invariant feedback solution

The objective of this section is to provide a constructive characterization of the solution to Problem 1. As discussed in the previous section, the latter constitutes, in turn, a time-invariant feedback approximate solution to the optimal control problem defined by (1), (2), (3).

Proposition 1. *Fix $x_0 \in \mathbb{R}^n$ and consider the nonlinear* *optimization problem*

$$\min_{K,t_f,L} \frac{1}{2}\|x_0\|_L^2 \quad (8a)$$

$$\text{s.t.} \quad Se^{A_K t_f}x_0 + d = 0 \quad (8b)$$

$$A_K^\top L + LA_K = e^{A_K^\top t_f}Q_K e^{A_K t_f} - Q_K \quad (8c)$$

$$L = L^\top \geq 0,\, t_f \geq t_0 \quad (8d)$$

with $A_K := A+BK$, $Q_K := Q+K^\top RK$. *Suppose that there exists an optimal solution* $(K^\star, t_f^\star, L^\star)$ *to* (8). *Then the pair* $(t_f^\star, K^\star)$ *constitutes a solution for Problem 1. Moreover, the optimal cost provided by implementing* $u^\star = K^\star x$ *is given by* $J^\star = \frac{1}{2}\|x_0\|_{L^\star}^2$. ◇

Remark 2. As implicitly suggested by the statement of Proposition 1, the additional constraint imposed to the control law, namely the requirement $u \in \mathcal{F}$, is such that Problem 1 cannot admit a solution for $x_0 = 0$ whenever $d \neq 0$ in (3). In fact, with $x_0 = 0$ and $u \in \mathcal{F}$, the closed-loop dynamics (2), (7) become $\dot{x}(t) = A_K x(t), x_0 = 0$. This in turn implies that $x(t) = 0$ for all $t \geq 0$ and hence $Sx(t_f) = 0$ violates the terminal constraint unless $d = 0$. ▲

Remark 3. Although the nonlinear programming problem (8) provides a finite-dimensional equivalent formulation of Problem 1, its solution appears to be a challenging task in practice. This is mainly due to the presence of exponential constraints involving the product of two unknown variables, namely t_f and K. To overcome, or at least circumvent, such a computational bottle-neck it may be possible to envision iterative strategies that ensure (local) convergence to a solution of (8). This approach is however enabled only by the ability of efficiently computing the *time-to-impact* function[1] $\varphi_K(x_0)$ for a fixed value of the gain matrix K (see Section IV-A) and, subsequently, of suitably updating the current estimate of the optimal feedback matrix (see Section IV-B). These aspects are addressed in the following section. ▲

IV. Locally-convergent gradient descent strategy

The main purpose of this section is to provide a constructive, iterative, strategy that ensures local convergence to a solution of the nonlinear programming problem (8). More precisely, as detailed below, convergence is guaranteed provided the initial condition of (2) is sufficiently close to the affine set described by the constraint $\chi(x) = 0$ and the matrix K sought for is sufficiently small. The following standing assumption is required to hold throughout the rest of this section.

Assumption 2. The terminal constraint is defined as in (3) with $l = 1$, namely $\chi(x) = 0$ defines a $(n-1)$-dimensional affine subset of $\mathbb{R}^n$. ○

[1] Given an autonomous linear system $\dot{x} = Ax$ and a prescribed affine set $\mathcal{S}$, the time-to-impact function $\varphi(x_0) : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is defined as the smallest time instant τ_I such that $e^{A\tau_I}x_0 \in \mathcal{S}$, whereas $\varphi(x_0) = \infty$ whenever the trajectory ensuing from x_0 does not intersect $\mathcal{S}$.

A. Computation of the time-to-impact function

Suppose that there exists a sufficiently small constant $\rho \in \mathbb{R}_{>0}$ such that $\|Sx_0 + d\| < \rho$, namely the initial condition x_0 is sufficiently close to the affine set described by (3). The above hypothesis, together with the selection of a sufficiently small matrix K, implies that the matrix exponential $e^{(A+BK)t}$ may be approximated, for small values of t, via the Taylor expansion of order $\nu \in \mathbb{N}$ according to

$$e^{(A+BK)t} \approx \sum_{i=0}^{\nu} \frac{1}{i!}(A+BK)^i t^i \,. \quad (9)$$

Therefore, provided Assumption 2 holds, the time-to-impact function $\varphi_K(x_0)$ of (2), (7), for fixed K, with respect to (8b) can be approximated on the basis of the following observation.

The substitution of the matrix exponential $e^{(A+BK)t}$ as in (9) into the constraint (8b) yields the *scalar* equation

$$\begin{aligned} 0 = d + (Sx_0)t + \frac{1}{2}(SA_Kx_0)t^2 + ... + \frac{1}{\nu!}(SA_K^\nu x_0)t^\nu \\ =: \pi_K(t) \end{aligned} \quad (10)$$

with respect to the scalar variable $t \in \mathbb{R}$ (recall that $A_K = A + BK$). In fact, $S(A+BK)^i x_0 \in \mathbb{R}$ for all $i \in \mathbb{N}$. Define

$$\Xi := \{\tau_I \in \mathbb{R}_{>0} : \pi_K(\tau_I) = 0\}, \quad (11)$$

namely the set of positive real roots of the polynomial (10). Then, the time-to-impact function is approximated by

$$\varphi_K(x_0) = \begin{cases} \min\{\Xi\} & \text{if } |\Xi| > 0 \\ \infty & \text{otherwise} \end{cases} \quad (12)$$

where $|\Xi|$ denotes the cardinality of the set Ξ.

Remark 4. By truncating the Taylor series expansion (9) at terms of order $\nu = 2$, the closed-form expression of the solutions to (10) is given by

$$\tau_I^{\pm} = \frac{-Sx_0 \pm \sqrt{(Sx_0)^2 - 4(SA_Kx_0)d}}{(SA_Kx_0)}. \quad (13)$$

Therefore, the approximation of the time-to-impact function (12) yields a finite value, provided that both solutions τ_I^- and τ_I^+ are real and that at least one of them belongs to the set Ξ. Moreover closed-form solutions which can be given in terms of elementary operations are available up to order $\nu = 4$, whereas for order $\nu \geq 5$, and only for certain polynomial equations, one must resort to the so-called *Galois Theory* (see [13, Chap. 22]). In general, for large ν practical methods such as the *Newton method* (see [14, Chap. 5]) are more appealing when solving the problem of finding the roots of a given polynomial. ▲

Example 1. The constructions around equations (10)-(12) are illustrated via the following numerical simulation involving the system (2) with

$$A = \begin{bmatrix} 0 & -2 & -1 & -1 \\ 1 & -1 & 0 & 2 \\ -2 & -2 & 0 & -2 \\ -1 & -1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}. \quad (14)$$

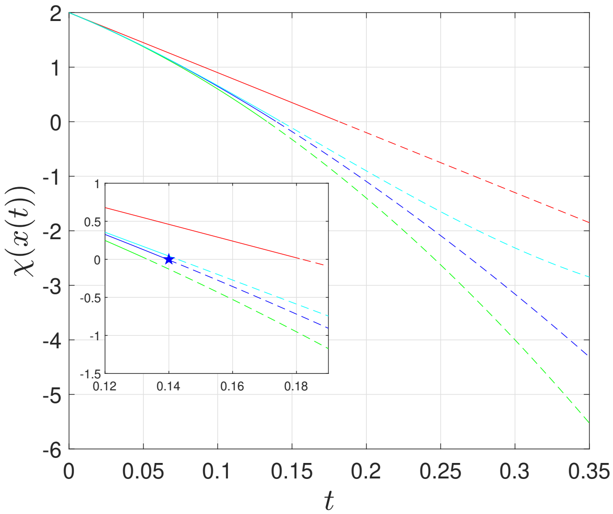

Fig. 1: Time histories of the map $\chi(x(t))$ (blue line) and of its successive approximations, obtained by truncating the Taylor series expansion (9) at order $\nu = 1$ (red line), $\nu = 2$ (green line), $\nu = 3$ (cyan line). The point indicated by a star denotes the positive real root of the actual $\chi(x(t))$, whereas the dashed portions of the trajectories highlight the zero-crossing of $\chi(x(t))$.

Suppose that $S = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}$, $d = 3$ and $x_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^\top$. Suppose that K is selected according to $K = \begin{bmatrix} 0.1250 & -18 & 1.125 & 6.875 \end{bmatrix}$. Figure 1 depicts the time histories of the function $\chi : \mathbb{R}^n \to \mathbb{R}$, $t \mapsto \chi(x(t))$ (blue line) and of its successive approximations, obtained by evaluating the Taylor series expansion (9) respectively at order $\nu = 1$ (red line), $\nu = 2$ (green line), $\nu = 3$ (cyan line). In the specific case study, the function $t \mapsto \chi(x(t))$ intersects the zero value at $t = 0.14$ (blue star), as illustrated in Figure 1. The latter figure further highlights how the third-order approximation is already sufficient to capture the behavior of the function χ near its root with respect to the time variable. It must be stressed, however, that this approximation better captures such a behavior for sufficiently small values of time t. Therefore, in the presence of large values of the matrix K, an expression based on the Taylor series expansion (9) may fail to provide an accurate approximation of the time to impact, even for large values of ν. △

B. Gradient descent update

The constructions in Section IV-A allow to address the problem of satisfying (8b) for a fixed value of the matrix K. On the other, towards an iterative approach to compute a solution to (8), one must also be able to compute a suitable update of the value of K such that the cost induced by (8c) is decreased. To this end, consider the time histories of the closed-loop, autonomous, dynamics (2), (7), namely

$$x(t) = e^{A_K t} x_0, \tag{15}$$

Algorithm 1

Require: Matrices A, B, Q, R, S, vector d, initial condition x_0, initial guess on the gain matrix K_p, real scalar $\varepsilon > 0$, order ν of the Taylor series expansion of e^{At}, step-size $\gamma \in \mathbb{R}_{>0}$, $\gamma \in (0, 1)$

Begin Initialization

1. $A_{K_p} \leftarrow A + BK_p$, $Q_{K_p} \leftarrow Q + K_p^\top R K_p$
2. get t_{f_p} as the minimal root of $\pi_{K_p}(t) = 0$ in Ξ
3. get L_p as the positive definite solution of

$$A_{K_p}^\top L_p + L_p A_{K_p} = e^{A_{K_p}^\top t_{f_p}} Q_{K_p} e^{A_{K_p} t_{f_p}} - Q_{K_p}$$

4. $J_p \leftarrow \frac{1}{2} x_0^\top L_p x_0$

End Initialization

Do

For $i = 1 : m, j = 1 : n$

5. $K_{ij}^v \leftarrow K_p + \varepsilon E_{ij}$,
$A_{K_{ij}^v} \leftarrow A + BK_{ij}^v$, $Q_{K_{ij}^v} \leftarrow Q + (K_{ij}^v)^\top R K_{ij}^v$
6. get $t_{f_{ij}^v}$ as the minimal root of $\pi_{K_{ij}^v}(t) = 0$ in Ξ
7. get L_{ij}^v as the positive definite solution of

$$A_{K_{ij}^v}^\top L_{ij}^v + L_{ij}^v A_{K_{ij}^v} = e^{A_{K_{ij}^v}^\top t_{f_{ij}^v}} Q_{K_{ij}^v} e^{A_{K_{ij}^v} t_{f_{ij}^v}} - Q_{K_{ij}^v}$$

8. $J_{ij}^v \leftarrow \frac{1}{2} x_0^\top L_{ij}^v x_0$
9. approximate (i, j)-th partial derivative as in (19)

End

10. approximate $\nabla_K J$ according to (20)
11. $K_{p+1} \leftarrow K_p - \gamma \nabla_K J$,
$A_{K_{p+1}} \leftarrow A + BK_{p+1}$, $Q_{K_{p+1}} \leftarrow Q + K_{p+1}^\top R K_{p+1}$
12. get $t_{f_{p+1}}$ as the minimal root of $\pi_{K_{p+1}}(t) = 0$ in Ξ
13. get L_{p+1} as the positive definite solution of

$$\begin{aligned} &A_{K_{p+1}}^\top L_{p+1} + L_{p+1} A_{K_{p+1}} \\ &= e^{A_{K_{p+1}}^\top t_{f_{p+1}}} Q_{K_{p+1}} e^{A_{K_{p+1}} t_{f_{p+1}}} - Q_{K_{p+1}} \end{aligned}$$

14. $J_{p+1} \leftarrow \frac{1}{2} x_0^\top L_{p+1} x_0$, $J_p \leftarrow J_{p+1}$, $K_p \leftarrow K_n$

While $\|\nabla_K J\|_F \geq 10^{-4}$

and

$$J(Kx(\cdot), t_f) =: J_{K,t_f} = \frac{1}{2} x_0^\top \left(\int_0^{t_f} e^{A_K^\top \tau} Q_K e^{A_K \tau} d\tau \right) x_0, \tag{16}$$

obtained by replacing (15) into the control law (7), and, in turn, the latter into the cost functional (1). Since the direct evaluation from (16) of $\nabla_K J \in \mathbb{R}^{m \times n}$, representing the main ingredient enabling a gradient descent strategy, is in general a daunting task, due to the non-trivial dependence of J from K, an approach to circumvent such a difficult task is presented herein, based on the approximation of the derivatives of J with respect to the (i, j)-th element of K. To begin with, let (K_p, t_{f_p}, L_p) be an admissible solution for (8) at a certain iteration p and let $\varepsilon \in \mathbb{R}_{>0}$ be sufficiently small. Consider the variation of K_p along the direction given by

$E_{i,j}$ the (i,j)-th element of the canonical basis[2] of $\mathbb{R}^{m\times n}$, namely

$$K_{ij}^v := K_p + \varepsilon E_{ij}. \tag{17}$$

To compute the sensitivity of the cost induced in the direction associate to K_{ij}^v, it is required to first ensure feasibility of such a variation. This objective is achieved by determining a solution $t_f \in \Xi$ via the approximation of the equation characterizing the constraint on the terminal state, provided in the Section IV-A. Let $t_{f_{i,j}^v}$ denote such a solution. It is then possible to solve the Lyapunov-like equation (8c), with t_f and K replaced by $t_{f_{i,j}^v}$ and K_{ij}^v, respectively. On the basis of the knowledge of the latter solution, denoted as $L_{i,j}^v$, one may then determine the sensitivity of the cost induced by $K_{i,j}^v$, hence in the direction E_{ij} according to

$$J_{i,j}^v = \frac{1}{2} x_0^\top L_{i,j}^v x_0, \tag{18}$$

from which one can approximate the partial derivative of J with respect to $K_{i,j}^v$ as

$$\frac{\partial J}{\partial K_{ij}^v} \approx \frac{1}{\varepsilon}(J_{ij}^v - J_p). \tag{19}$$

By repeating the previous reasoning for all $i = 1, \dots, m$, $j = 1, \dots, n$, it is then possible to obtain the approximation of $\nabla_K J$ as

$$\nabla_K J(K_p) \approx \begin{bmatrix} \frac{\partial J}{\partial K_{11}^v} & \cdots & \frac{\partial J}{\partial K_{1n}^v} \\ \vdots & \ddots & \vdots \\ \frac{\partial J}{\partial K_{m1}^v} & \cdots & \frac{\partial J}{\partial K_{mn}^v} \end{bmatrix}, \tag{20}$$

from which it is finally possible to provide, for the current iteration p, the *gain update law*

$$K_{p+1} = K_p - \gamma \nabla_K J. \tag{21}$$

Therefore, by the gradient nature of the update law (21), there exists a step size $\gamma \in (0, 1)$ such that K_{p+1} constitutes an updated value of the gain matrix with the property that the cost J is decreased, provided K_{p+1} is feasible for the overall optimization problem (8). This is, in turn, ensured by the existence of $t_{f_{p+1}} \in \Xi$, hence satisfying (8), together with the existence of a positive definite solution to (8c).

C. Locally convergent algorithm

Building on the constructions discussed in the two previous sections, the purpose herein is to summarize the proposed iterative method by means of Algorithm 1. Given the initial guess on the gain matrix K_p, the initialization phase (steps 1-4) provides the corresponding initial guesses t_{f_p}, obtained according to the constructions introduced in Section IV-A, and L_p, thus ensuring the feasibility of the initial guess K_p. The updated value of the gain matrix K_p is then iteratively evaluated following the steps introduced in Section IV-B (steps 5-11), until the stopping criterion (last line), requiring that the *Frobenius norm*[3] $\|\nabla_K J\|_F$ is less than a given tolerance, is satisfied. These iterations in turn, provide a sequence of $\{K_p\}$ along which the corresponding values of the sequence of updated costs $\{J_p\}$ (each term obtained by following steps 12-14) are decreasing, thus converging to a local extremum of (8), provided it exists. Finally note that, by construction, such an extremum cannot correspond to a local maximum.

V. A SIMPLE CASE STUDY

The effectiveness of the proposed strategy is corroborated by means of numerical simulations discussed in this section. Towards this end, consider a LTI system defined as in (2) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{22}$$

The desired cost functional is defined as in (1) with the weighting matrices described by $Q = I$ and $R = 1$, where I denotes the identity matrix. Furthermore, it is assumed that the constraint on the terminal state is captured by the scalar-valued function $\chi : \mathbb{R}^2 \to \mathbb{R}$, namely with $l = 1$, as in (3) with $S = [1\ 1]$ and $d = -2$. Therefore, the constraint defines the affine set consisting of the straight line in the (x_1, x_2) plane passing through the points $(0, 2)$ and $(2, 0)$. Consistent with the comments discussed in Remark 2, the set $(0,\ 2] \times (0,\ 2]$ is considered for admissible initial conditions. Figure 2 depicts the minimal value of the cost functional (1) provided by the optimal open loop control law (5), (6) (solid blue surface), together with the minimal value of the cost obtained by implementing the time-invariant feedback control law obtained by solving the nonlinear optimization problem (8) (light-gray transparent surface). The dashed red line depicts the set $\chi(x) = 0$. Figure 3 instead shows the time histories of the trajectories obtained by implementing the optimal open-loop (dashed blue line) and the time-invariant feedback (solid yellow line) control laws, initialized at $x_0 = (0.5,\ 0)$ (star) and $x_0 = (1,\ 1.8)$ (diamond). It is interesting to observe that, for the former initial condition, the trajectory induced by the time-invariant feedback essentially recovers the trajectory yielded by the implementation of the optimal open-loop control law. In this case, the solution to (8) yields $K^\star = [1.1979, -0.4407]$, $t_{f_{FB}}^\star = 1.14113s$ and $J_{FB} = 0.8285$, with the optimal open-loop solution yielding $t_{f_{OL}}^\star = 1.4100s$ and $J_{OL} = 0.8267$. For the latter initial condition, on the other hand, a different behavior can be observed for the two resulting trajectories, with the one arising from the implementation of the feedback control law moving away from the one obtained by implementing the optimal open-loop solution,

[2]The matrix $E_{i,j}$ given by

$$(E_{i,j})_{r,c} = \begin{cases} 1 & \text{if } r = i,\ c = j \\ 0 & \text{otherwise} \end{cases}$$

represents the (i,j)-th element of the *canonical basis* of $\mathbb{R}^{m\times n}$.

[3]Given a matrix $M \in \mathbb{R}^{m\times n}$, its *Frobenius norm* is defined as $\|M\|_F := \sqrt{\mathrm{tr}(M^\top M)}$, in which, for a given square matrix $\Gamma \in \mathbb{R}^{n\times n}$ the trace operator, $\mathrm{tr}(\cdot)$ is defined as the map $\mathrm{tr} : \mathbb{R}^{n\times n} \to \mathbb{R}$, $\mathrm{tr}(\Gamma) \mapsto \Gamma_{1,1} + \cdots + \Gamma_{n,n}$.

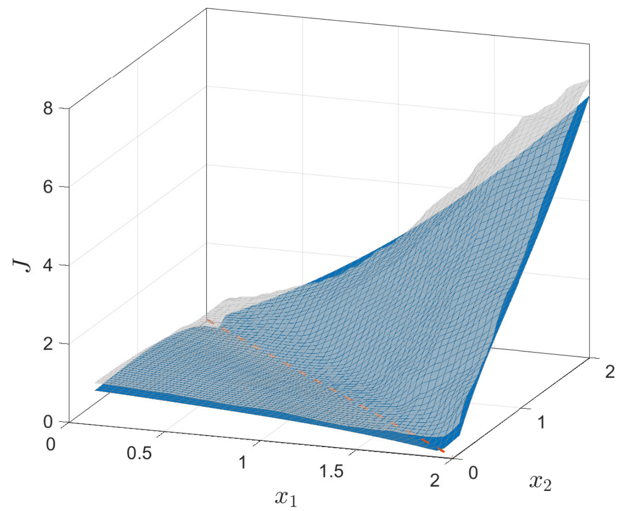

Fig. 2: Minimal cost for all initial conditions $x_0 \in (0, 2] \times (0, 2]$ for the problem defined by (1), (2), (3) obtained via the optimal open-loop solution (6) (blue solid surface) and by the linear time-invariant feedback (7) (gray transparent surface).

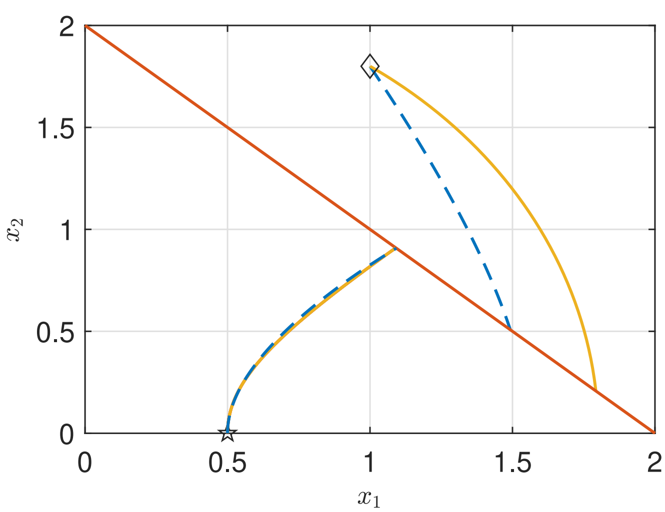

Fig. 3: Time histories of the trajectories obtained by implementing the optimal open-loop solution (6) (blue dashed line) and by implementing the linear time-invariant feedback (7) (solid yellow line), starting from the initial conditions $x_0 = (0.5,\ 0)$ (star) and $x_0 = (1,\ 1.8)$ (diamond).

thus resulting in different values for both the terminal times and the incurred cost. In particular the solution to (8) in this case provides $K^\star = [-1.4445,\ -6.0183]$, $t^\star_{f_{FB}} = 0.7372s$ and $J_{FB} = 3.1132$, whereas the solution to the two point boundary value problem (5) is such that $t^\star_{f_{OL}} = 0.4500s$ and $J_{OL} = 2.5856$.

VI. Conclusions and further work

This paper has studied the problem of transferring the state of a dynamical system from a given initial configuration to a desired affine set, while minimizing a certain cost functional along the induced trajectory. In terms of the framework of optimal control problem, the problem above can be formulated in terms of a finite-horizon optimal control problem with terminal state constraints and a free terminal time. While the solution to such a control problem can be obtained by relying on constructions inspired by PMP, the obtained solution may suffer from the drawback of being described in terms of an open loop control law. Therefore, herein we formulate a similar problem in which the set of feasible control laws is restricted to the class of LTI feedbacks from the current state. Within this framework, the solution of the underlying optimal control problem hinges upon the solution of a nonlinear constrained optimization problem. An iterative method to tackle the latter nonlinear programming problem has been proposed and corroborated by numerical simulations in a simple case study.

Further work aims at envisioning strategies that allow reducing the computational complexity related to the solution of the proposed nonlinear programming problem, which would enable its use in a more general formulation of the problem involving linear hybrid systems.

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