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A Model Reference Adaptive Controller Based on Operator-Valued Kernel Functions

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***Abstract*—This paper extends recent results on model reference adaptive control using reproducing kernel Hilbert space (RKHS) learning techniques for some general cases of multi-input systems. We leverage recent results on error bounds for nonlinear observers in a vector-valued RKHS to design adaptive model reference adaptive controllers (MRAC) that are induced by operator-valued kernels. This paper formulates a model reference adaptive control strategy based on a dead zone robust modification, and derives for this case conditions for the ultimate boundedness of the tracking error. The RKHS setting allows the control designer to influence the ultimate bound by selection and placement of operator-valued kernels. As in the scalar-valued setting, closed-form expressions are obtained for the ultimate upper bound. But in this case the upper bound depends on a generalization of the power function for an operator-valued kernel space. Finally, we provide a detailed illustration our results in practice for the case of attitude control of a streamlined tailed-controlled underwater vehicle.**

***Index Terms*—Adaptive Control, Autonomous Underwater Vehicles, Operator-Valued Reproducing Kernel Hilbert Spaces**

I. Introduction

A. Background

This paper proposes a model-reference adaptive controller that leverages properties a reproducing kernel Hilbert space (RKHS) to model functional uncertainty in the plant. Our work builds directly on a sequence of works [3], [12]–[14], [19], among others, that address native space RKHS embedding for adaptive estimation and control of uncertain ODE systems. We specifically build on [5] that addresses model reference adaptive control embedded in native RKHS spaces for scalar-valued uncertainty and control signals. The techniques in [5] rely on conventional scalar reproducing kernels. We generalize to the case of a vector-valued uncertainty and vector-valued control signals using a native space RKHS induced by operator-valued kernel functions. The approach in this paper also extends the work on nonlinear observers in [4] and [11] to express the model-reference adaptive control problem using operator-valued kernel functions. We establish sufficient conditions for boundedness of the state of the controlled plant and that of the reference model, and articulate a specific upper bound. Furthermore, we illustrate our results on a realistic case-study of attitude control of an streamlined tailed-controlled autonomous underwater vehicle. Such systems are often characterized by large uncertainties in hydrodynamics parameters, for which non-parametric data-driven models such as we derive herein are well-suited, and also characterized by coupling between control channels for which a multi-input control approach is required.

The use in control theory of reproducing kernel Hilbert space (RKHS) embedding methods, also known as native space embedding, has a rich history in the related fields of machine learning, model estimation, and statistical inference (e.g. [22], [21], [2], [6], [23], [9]). In these fields approaches typically address nonlinear regression using discrete data sets consisting of input and output observations. In contrast, RKHS methods for designing continuous-time adaptive controllers and observers leverage the reproducing and self-adjoint properties of the associated kernel to develop continuous update laws for which we can determine convergence rates for functional estimates and apply Lyapunov analysis techniques to the study of system stability [3], [14].

While the theoretical ground work for scalar-valued RKHS methods has existed since the mid-twentieth century [1], the study of vector-valued RKHS methods is less mature, generally dating to the work of Michelli and Pontil in the context of machine learning theory [17]. More recently, significant effort has been devoted to the study of the properties of RKHS embedding methods in the continuous time control setting. References [3] and [14] establish conditions required for functional estimate convergence and an RKHS analog of the persistent excitation condition for adaptive estimation using scalar-valued RKHS embedding. The work in [4] extends the persistent excitation and convergence results to vector-valued functions and presents uniform ultimate boundedness guarantees for observer error using RKHS embedding methods.

B. Summary of our contributions

We propose a model-reference adaptive controller for the case of vector-valued uncertainty and control signals using a native space induced from an operator-valued kernel function. Our work builds directly from recent results on model-reference adaptive control for scalar-valued functional uncertainty and control signals using native RKHS spaces induced from more typical scalar-valued kernel functions.

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Note that taking the Cartesian product of scalar-valued RKHS H to form a vector-valued space $H \times H \times \cdots \times H$ is one way to formulate estimation and control problems for vector-valued functional uncertainty, and this seems a natural approach to extend initial efforts using scalar-valued spaces to more general control problems. But many vector-valued native spaces cannot be generated in this way. From first principles the approach in this paper enables MRAC formulations for any operator-valued kernel, which includes the Cartesian product as a very special case. In this way the paper defines a very general strategy that is applicable to a much wider class of native spaces. This work addresses the case when a dead-zone modification is employed, for which we generate an ultimate bound between the state of the plant and the state of the reference model. We show in Proposition 1 that the ultimate bound is dependent on selection of the operator-valued kernel function as well as the number and location of kernel centers, which provide the control designer opportunities to reduce the bound. It is an explicit, known function that depends on the location and number of centers of the approximating subspace used to represent the functional uncertainty. It is shown following Corollary 1 that this explicit ultimate bound holds for all functional uncertainty f in the (generally infinite dimensional) uncertainty class $\mathcal{B}_R := \{f \in \mathbb{H} \mid \|f\|_{\mathbb{H}} \leq R\}$. In this sense it is a wider robustness guarantee than those stated for uncertainty associated with ranges of real parameters from a fixed uncertainty model of finite dimension. We illustrate our results by implementing an attitude controller for a tail-controlled underwater vehicle.

II. Problem Description

A. Background

We consider a nonlinear system

$$\dot{x}(t) = Ax(t) + B(u(t) + f(x)) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, and $u(t) \in \mathbb{R}^m$ is a control signal. We assume that $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are known constant matrices, and that $f : \mathbb{R}^n \to \mathbb{R}^m$ is an unknown function. We further assume that $f \in \mathbb{H}$, which is a native reproducing kernel Hilbert space induced by the operator-valued kernel $\mathbb{K}$ that maps $\mathbb{X}\times\mathbb{X}$ to the set of linear bounded functions $\mathcal{L}(\mathbb{Y})$. Throughout, we specialize to the case that $\mathbb{X} := \mathbb{R}^n$ and that $\mathbb{Y} := \mathbb{R}^m$.

Thorough treatments of RKHS and operator-valued kernels can be found in [20] and [11]. We provide herein only the background necessary for analysis of our specific model reference adaptive controller.

For any $x \in \mathbb{R}^n$, the evaluation operator is defined by $\mathbb{E}_x f = f(x) \in \mathbb{R}^m$. The vector-valued analog of the reproducing property for the vector-valued RKHS is the identity

$$\langle \mathbb{K}_x u, f\rangle_{\mathbb{H}} = \langle u, \mathbb{E}_x f\rangle_{\mathbb{R}^m}, \tag{2}$$

which is satisfied for each $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $f \in \mathbb{H}$. Here the operator $\mathbb{K}_x \in \mathcal{L}(\mathbb{R}^m, \mathbb{H})$ is defined by the identity

$$(\mathbb{K}_x u)(z) := \mathbb{K}(z, x)u.$$

From the reproducing property (2) and properties of the adjoint, we note that the adjoint of the evaluation operator satisfies

$$\mathbb{E}_x^* u = \mathbb{K}_x u$$

for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Therefore the adjoint of the operator $\mathbb{K}_x$ is in fact the evaluation operator. Finally, given the operator-valued kernel $\mathbb{K}$, the vector-valued native space is defined as the closed linear span

$$\mathbb{H} = \overline{\text{span}\{\mathbb{K}_x u | x \in \mathbb{R}^n, u \in \mathbb{R}^m\}}.$$

B. Finite Approximations

To construct a finite-dimensional approximation of $f \in \mathbb{H}$, we assume that a finite set of distinct points $x_i \in \mathbb{R}^n$ have been selected $\Omega_N = \{x_i \in \mathbb{R}^n | 1 \leq i \leq N\}$. The points in Ω_N are referred to as centers, and we define the space of finite-dimensional approximants

$$\mathbb{H}_N := \text{span}\{\mathbb{K}_{x_i} e_j | x_i \in \Omega_N, 1 \leq j \leq m\}$$

where $\{e_j\}_{j=1}^m$ is any basis for $\mathbb{R}^m$. We assume that the kernel function $\mathbb{K}$ that defines $\mathbb{H}$ is *strictly* positive definite. This implies that the dimension of $\mathbb{H}_N$ is always mN. An approximant $\hat{f}_N \in \mathbb{H}_N$ of $f \in \mathbb{H}$ is expressed

$$\hat{f}_N(\cdot) = \sum_{j=1}^{m}\sum_{i=1}^{N} \hat{\alpha}_k \mathbb{K}_{x_i}(\cdot)e_j, \;\; k = i + (j-1)N \tag{3}$$

for some set of scalar coefficients $\{\hat{\alpha}_k\}_{k=1}^{mN}$. In this work we consider time-varying approximants, and we write

$$\hat{f}_N(t, \cdot) = \sum_{j=1}^{m}\sum_{i=1}^{N} \hat{\alpha}_k(t) \mathbb{K}_{x_i}(\cdot)e_j$$

for time-varying coefficients $\{\hat{\alpha}_k(t)\}_{k=1}^{mN}$. We assume that the set of centers Ω_N are selected only once. However, the choice of centers affects approximation accuracy, and selecting centers in real-time is an on-going research challenge that arises in many streaming (real-time) applications of RKHS. Non-optimal center choice will adversely affect the achievable ultimate error bound. The error bound is proportional to the fill distance of the centers in Ω_N and is balanced by the need to maintain numerical stability of the kernel Gram matrix as characterized by its condition number which is inversely proportional to the fill distance of the centers. The problem of center selection is considered in greater detail in Chapter 3 of [11] and in [19].

For simplicity of notations, we omit dependency of functions on the spatial variable if there is no confusion (e.g. $f := f(\cdot)$ and $\hat{f}(t) := \hat{f}(t, \cdot)$). The orthogonal projection operator $\Pi_N : \mathbb{H} \to \mathbb{H}_N$ is characterized by [24]

$$\langle f - \Pi_N f, g\rangle_{\mathbb{H}} = 0, \;\forall\; g \in \mathbb{H}_N. \tag{4}$$

Define $f_N := \Pi_N f \in \mathbb{H}_N$. The estimation error between the function, f, and its finite dimensional approximant $\hat{f}_N$ is defined as

$$\tilde{f}(t, \cdot) := \hat{f}_N(t, \cdot) - f(\cdot),$$

and we note that $\tilde{f}(t,\cdot) \in \mathbb{H}$ for each fixed t. In our analysis, we find useful a specific decomposition of the estimation error into mutually orthogonal parts

$$\begin{aligned} \tilde{f}(t) &= \hat{f}_N(t) - (f - \Pi_N f + \Pi_N f) \\ &= \underbrace{(\hat{f}_N(t) - \Pi_N f)}_{\tilde{f}_N(t)} - \underbrace{(f - \Pi_N f)}_{\tilde{f}_R}. \end{aligned} \tag{5}$$

The first term

$$\tilde{f}_N(t) := \hat{f}_N(t) - \Pi_N f \in \mathbb{H}_N$$

is the error between the approximant $\hat{f}_N(t,\cdot)$ and the projection of the function f into the finite-dimensional subspace $\mathbb{H}_N$, and

$$\tilde{f}_R := f - \Pi_N f \perp \mathbb{H}_N$$

is the residual error due to projecting the function onto the subspace.

C. MIMO MRAC Using RKHS

We consider a nonlinear system in (1), expressed as

$$\dot{x}(t) = Ax(t) + B(u(t) + \mathbb{E}_{x(t)} f) \tag{6}$$

where $f \in \mathbb{H}$. Our approach follows a standard model-reference adaptive control(MRAC) formulation (see for example [16]). A reference model

$$\dot{x}_r(t) = A_r x_r(t) + B_r r(t) \tag{7}$$

is proposed that characterizes the desired closed-loop behavior of the system. Our challenge is to generate a control signal $u(t)$ for (6) such that $\|x(t) - x_r(t)\| \le \bar{\varepsilon}$ for an input signal $r(t)$ and $t \ge T$ for T sufficiently large, and where we have some influence over the value of $\bar{\varepsilon}$. For the reference model, $A_r \in \mathbb{R}^{n\times n}$ is Hurwitz, and $B_r \in \mathbb{R}^{n\times l}$. The control law has the form

$$u(t) = \hat{K}_x^{\mathsf{T}}(t)x(t) + \hat{K}_r^{\mathsf{T}}(t)r(t) - \hat{f}_N(t, x(t)) \tag{8}$$

where $\hat{K}_x^{\mathsf{T}}(t) \in \mathbb{R}^{m\times n}$, $\hat{K}_r^{\mathsf{T}}(t) \in \mathbb{R}^{m\times l}$, and $\hat{f}_N(t,\cdot) \in \mathbb{H}_N$ are updated online according to the adaptation laws. Specifically, the function $\hat{f}_N \in \mathbb{H}_N$ is an online estimate of the unknown dynamics f.

We explicitly consider the dead zone modification method of the standard approach to MRAC (see Section 11.2.1 in [16], Section 4.6.4 in [7], among many others). We denote the tracking error by

$$e(t) := x(t) - x_r(t) \tag{9}$$

and propose the parameter update laws

$$\dot{\hat{K}}_x(t) = \begin{cases} -M_x x(t) e^{\mathsf{T}}(t) PB & \text{if } \|e(t)\| \ge \bar{\epsilon} \\ 0 & \text{otherwise} \end{cases}, \tag{10}$$

$$\dot{\hat{K}}_r(t) = \begin{cases} -M_r r(t) e^{\mathsf{T}}(t) PB & \text{if } \|e(t)\| \ge \bar{\epsilon} \\ 0 & \text{otherwise} \end{cases}, \tag{11}$$

$$\dot{\hat{f}}_N(t,\cdot) = \begin{cases} \gamma_f \Pi_N \mathbb{E}^*_{x(t)} B^{\mathsf{T}} Pe(t) & \|e(t)\| \ge \bar{\epsilon} \\ 0 & \text{otherwise} \end{cases}. \tag{12}$$

In these equations, the positive definite matrices $M_x \in \mathbb{R}^{n\times n}$ and $M_r \in \mathbb{R}^{l\times l}$, as well as the positive constant $\gamma_f \in \mathbb{R}$ affect the adaptation rates, and $P \in \mathbb{R}^{n\times n}$ is the solution to the algebraic Lyapunov equation

$$A_r^{\mathsf{T}} P + PA_r = -Q \tag{13}$$

where $Q \in \mathbb{R}^{n\times n}$ is a positive definite matrix that is selected as part of the control design process.

D. Tracking error dynamics and matching conditions

The tracking error dynamics with the control law (8) satisfy

$$\begin{aligned} \dot{e}(t) = Ax(t) + B\left(\hat{K}_x^{\mathsf{T}} x(t) + \hat{K}_r^{\mathsf{T}} r(t) - \mathbb{E}_{x(t)}\hat{f}_N(t,\cdot)\right) \\ + B\mathbb{E}_{x(t)} f(\cdot) - A_r x_r(t) - B_r r(t). \end{aligned}$$

By imposing the classical *matching conditions*

$$\begin{aligned} A + BK_x^{\mathsf{T}} &= A_r, \\ BK_r^{\mathsf{T}} &= B_r, \end{aligned} \tag{14}$$

where K_x and K_r are the so-called ideal adaptation gains, the tracking error dynamics can be written

$$\dot{e}(t) = A_r e(t) + B\tilde{K}_x^{\mathsf{T}}(t)x(t) + B\tilde{K}_r^{\mathsf{T}}(t)r(t) - B\mathbb{E}_{x(t)}\tilde{f}_N(t,\cdot), \tag{15}$$

where the error variables are

$$\begin{aligned} \tilde{K}_x(t) &:= \hat{K}_x(t) - K_x, \\ \tilde{K}_r(t) &:= \hat{K}_r(t) - K_r, \\ \tilde{f}(t,\cdot) &:= \hat{f}_N(t,\cdot) - f(\cdot). \end{aligned}$$

The role of the matching conditions in control design are illustrated in the numerical example presented in Section IV.

III. Analysis of Closed-Loop Performance

The tracking error dynamics (15) combined with the update laws of (10)-(12) define a distributed parameter system whose state $\{e, \tilde{K}_x, \tilde{K}_r, \tilde{f}\}$ evolves in

$$\mathcal{A} := \mathbb{R}^n \times \mathbb{R}^{n\times m} \times \mathbb{R}^{l\times m} \times \mathbb{H}.$$

In this brief paper we always assume that the error equations that define this DPS are forward complete. That is, for any initial condition we assume that the maximal interval of existing is $[0,\infty)$. Existence and uniqueness of solutions have been discussed for a similar DPS in native space embedding in [3], [12]–[14], [19].

Asymptotic behavior of the closed-loop system is characterized by ultimate boundedness of the tracking error $e(t)$.

Proposition 1. *Suppose that there is a compact set $\Omega \supset \cup_{\tau\ge 0} x(\tau)$ that contains the closed loop trajectory and the deadzone $\bar{\varepsilon}$ in the update laws* (10)-(12) *satisfies*

$$\bar{\varepsilon} \ge \frac{2\|PB\|C}{\lambda_{\min}(Q)} \tag{16}$$

where $C = \sup_{x\in\Omega} \|\mathbb{E}_x(f - \Pi_N f)\|$. Then there exists T such that the tracking error $e(t)$ of (15) *combined with the update laws* (10)-(12) *satisfies*

$$\|e(t)\| \le \bar{\varepsilon} \tag{17}$$

for all $t \geq T$.

Proof. To establish ultimate boundedness of $e(t)$, we propose the Lyapunov function

$$\begin{aligned} v(e, \tilde{K}_x, \tilde{K}_r, \tilde{f}) = e^\mathsf{T} P e + \gamma_f^{-1} \langle \tilde{f}, \tilde{f} \rangle_{\mathbb{H}} \\ + \operatorname{trace}[\tilde{K}_x^\mathsf{T} M_x^{-1} \tilde{K}_x + \tilde{K}_r^\mathsf{T} M_r^{-1} \tilde{K}_r] \end{aligned} \tag{18}$$

where $\tilde{f} = \hat{f}_N - f$. We first consider the case that $\|e(t)\| \geq \bar{\varepsilon}$. Differentiating (18) along the trajectory of error equations and substituting

$$\begin{aligned} \dot{\hat{K}}_x &= -M_x x(t) e^\mathsf{T} P B, \\ \dot{\hat{K}}_r &= -M_r r(t) e^\mathsf{T} P B, \\ \dot{\hat{f}}_N(t, \cdot) &= \gamma \Pi_N \mathbb{E}^* B^\mathsf{T} P e, \end{aligned}$$

yields

$$\begin{aligned} \dot{v} = &- e^\mathsf{T} Q e + 2 e^\mathsf{T} P B \tilde{K}_x^\mathsf{T} x + 2 e^\mathsf{T} P B \tilde{K}_r^\mathsf{T} r - 2 e^\mathsf{T} P B \mathbb{E}_x \tilde{f} \\ &+ 2 \gamma_f^{-1} \langle \dot{\hat{f}}_N, \tilde{f} \rangle \\ &- 2 e^\mathsf{T} P B \tilde{K}_x^\mathsf{T} x - 2 e^\mathsf{T} P B \tilde{K}_r^\mathsf{T} r. \end{aligned}$$

This simplifies to

$$\dot{v} = -e^\mathsf{T} Q e + 2 \gamma_f^{-1} \left\langle \dot{\hat{f}}_N, \tilde{f} \right\rangle - 2 e^\mathsf{T} P B \mathbb{E}_x \tilde{f}. \tag{19}$$

Recalling the decomposition of $\tilde{f} = \tilde{f}_N - \tilde{f}_R$ in (5), the relationship in (4), and the function estimation dynamics in (12) noting that $\dot{\hat{f}}_N(t, \cdot) \in \mathbb{H}_N$, the inner product in (19) can be expressed

$$\begin{aligned} \langle \dot{\hat{f}}_N, \tilde{f} \rangle &= \left\langle \tilde{f}_N - \tilde{f}_R, \dot{\hat{f}}_N(t, \cdot) \right\rangle_{\mathbb{H}}, \\ &= \left\langle \tilde{f}_N, \gamma_f \mathbb{E}_x^* B^\mathsf{T} P e \right\rangle_{\mathbb{H}} - \left\langle \tilde{f}_R, \dot{\hat{f}}_N(t, \cdot) \right\rangle_{\mathbb{H}}, \\ &= \left\langle \mathbb{E}_x \tilde{f}_N, \gamma_f B^\mathsf{T} P e \right\rangle_{\mathbb{R}^m} - 0, \\ &= \gamma_f e^\mathsf{T} P B \mathbb{E}_x \tilde{f}_N. \end{aligned}$$

Again from (5), $\tilde{f}_R = f - \Pi_N f$, and we can write

$$\begin{aligned} \dot{v} &= -e^\mathsf{T} Q e + 2 e^\mathsf{T} P B \mathbb{E}_x \tilde{f}_R \\ &\leq \|e\| \left(-\lambda_{\min}(Q) \|e\| + 2 \|P\| \|B\| \|\mathbb{E}_x \left(I - \Pi_N \right) f\| \right). \end{aligned} \tag{20}$$

Thus $\dot{v} \leq 0$ whenever

$$\|e\| \geq \frac{2\|PB\|}{\lambda_{\min}(Q)} \sup_{x \in \Omega} \|\mathbb{E}_x (f - \Pi_N f)\| \tag{21}$$

where the right-hand side is a lower bound for the deadzone $\bar{\varepsilon}$. Thus (21) is satisfied whenever $\|e\| \geq \bar{\varepsilon}$. Following a standard argument in [7] or [16], for example, this is sufficient to ensure that $\|e\| \geq \bar{\varepsilon}$ for finite time. □

Proposition 1 is intuitively satisfying: if the deadzone $\bar{\epsilon}$ is scaled properly so that it provides a pointwise bound for the worst case approximation error $\|\mathbb{E}_x(f - \Pi_N f)\|$ for the functional uncertainty f, then the ultimate bound holds. One of the powerful properties of projection or interpolation operators in a native space is that such upper bounds are often readily available. In many of the recent papers [3], [12]–[14], [19] such bounds are obtained using the power function for a scalar-valued RKHS. In the paper [4], related ultimate bounds are derived for a class of nonlinear observers in vector-valued native spaces induced by an operator-valued kernel. These bounds make use of a generalization of the power function for operator-valued kernels described recently in [24]. These define the operator-valued power function given by

$$(P_N^\alpha(x))^2 := \langle (\mathbb{K}(x, x) - \mathbb{K}_N(x, x)) \alpha, \alpha \rangle_{\mathbb{R}^m}$$

for each $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}^m$. In this equation $\mathbb{K}_N(x, x)$ is the reproducing kernel of $\mathbb{H}_N$, as is given in Corollary 2.7 of [24]. This definition is of use in the present context since from Corollary 2.11 of [24] we have

$$|\langle \mathbb{E}_x (I - \Pi_N) f, \alpha \rangle_{\mathbb{R}^m}| \leq P_N^\alpha(x) \|f\|_{\mathbb{H}}$$

for all $x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m$, and $f \in \mathbb{H}$. This gives an immediate corollary that ties the placement of centers to a bound that is known in closed form.

Corollary 1. *Let the hypotheses of Proposition 1 hold. Then the tracking error satisfies the ultimate bound*

$$\|e(t)\| \leq O\left(\sup_{x \in \Omega} \sqrt{\|\mathbb{K}(x, x) - \mathbb{K}_N(x, x)\|} \|f\|_{\mathbb{H}} \right).$$

Proof. The proof of this corollary follows immediately from the comments above and the proof of Proposition 1. See [4] for the details. □

It should be emphasized that Corollary 1 can be understood as a statement of robust adaptive control. We define the functional uncertainty class

$$\mathcal{B}_R := \{ f \in \mathbb{H} \mid \|f\|_{\mathbb{H}} \leq R \},$$

and replace C in Equation 16 by

$$C := \sup_{x \in \Omega} \sqrt{\|\mathbb{K}(x, x) - \mathbb{K}_N(x, x)\|} R.$$

Then the ultimate bound of the Corollary holds for all functional uncertainty $f \in \mathcal{B}_R$ over the uncertainty class $\mathcal{B}_R$. Note that in contrast to approaches in real parametric adaptive control, the guarantee is over a (generally infinite dimensional) class of functional uncertainty, and not just over real parametric uncertainty in coordinate representations for a fixed dimensional model. In this sense the robustness guarantee is over a broader class of uncertain models than usually encountered in real parametric adaptive control.

Remark 1. We note that while the expressions of $\mathbb{K}$ and $\mathbb{K}_N$ are known in closed forms, computating centers Ω_N that attain the supremum on the right hand side involves combinatorial optimization. For scalar-valued case, greedy algorithms have been shown to reach near-optimal rate of convergence [10], [15].

IV. Numerical Illustration

For the purposes of numerical experimentation, we consider the challenge of designing an attitude control system for an autonomous underwater vehicle (AUV) that is streamlined in shape and controlled by articulated flaps in the rear of the AUV.

A. Underwater vehicle dynamics

We express the AUV dynamics with respect to the Earth-fixed reference frame. We use a typical 6-DOF model found in [8], [18], which take the form

$$M_\eta \ddot{\eta} + C_\eta \dot{\eta} + D_\eta \dot{\eta} + g(\eta) = \tau \tag{22}$$

where η is the Earth-fixed position of the vehicle, $\eta = \begin{bmatrix} x & y & z & \phi & \theta & \psi \end{bmatrix}$, where $\{x, y, z\}$ is position of the AUV in Earth-fixed coordinates, and $\{\phi, \theta, \psi\}$ is the attitude of the AUV expressed as Euler angles. $M_\eta(\eta)$ is the mass matrix, $C_\eta(\eta, \dot{\eta})$ is the Coriolis matrix, $D_\eta(\eta, \dot{\eta})$ is the damping matrix, $g(\eta)$ is the gravitational vector, and $\tau = \begin{bmatrix} X & Y & Z & K & M & N \end{bmatrix}^\mathsf{T}$ is the vector of external linear forces (X, Y, Z) and rotational moments (K, M, N). We use the specific model parameters that have been developed for the Virginia Tech 690 AUV, which are available in [18].

We are concerned with attitude control of an AUV with a single thruster and four control surfaces located in a t-configuration at the rear of the vehicle and dynamics given by (22). For simplicity we consider the problem of attitude control of a vehicle under constant forward speed thus we do not consider the thruster as a control input. The attitude states are given by $x = \begin{bmatrix} \phi & \theta & \psi & \dot{\phi} & \dot{\theta} & \dot{\psi} \end{bmatrix}^\mathsf{T} \in \mathbb{R}^6$ and the control inputs are the effective elevator angle δ_e created by the port and starboard control surfaces, the effective rudder angle δ_r created by the top and bottom control surface, and the roll angle δ_{roll} created by offsetting each fin by the same angle (see [18] for greater detail on control surface mapping). Thus the control inputs are given by $\delta = \begin{bmatrix} \delta_{roll} & \delta_e & \delta_r \end{bmatrix}^\mathsf{T} \in \mathbb{R}^3$.

Because we are only concerned with attitude control, we consider only the dynamics of (22) associated with the rotational moments. Decomposing the dynamics of (22) into linear forces and rotational moments

$$\begin{aligned} \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} M_{\eta_{11}} & M_{\eta_{12}} \\ M_{\eta_{21}} & M_{\eta_{22}} \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} C_{\eta_{11}} & C_{\eta_{12}} \\ C_{\eta_{21}} & C_{\eta_{22}} \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ x_2 \end{bmatrix} \\ + \begin{bmatrix} D_{\eta_{11}} & D_{\eta_{12}} \\ D_{\eta_{21}} & D_{\eta_{22}} \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \end{aligned} \tag{23}$$

where $\eta_1 \in \mathbb{R}^3 = \begin{bmatrix} x & y & z \end{bmatrix}^\mathsf{T}$ is the earth-fixed position and each block matrix is 3×3.

Because we are only concerned with attitude control, we consider only the dynamics of (22) associated with the rotational moments, that is the second row of (23)

$$\begin{aligned} \tau_2 = M_{\eta_{21}} \ddot{\eta}_1 + M_{\eta_{22}} \dot{x}_2 + C_{\eta_{21}} \dot{\eta}_1 + C_{\eta_{22}} x_2 \\ + D_{\eta_{21}} \dot{\eta}_1 + D_{\eta_{22}} x_2 + g_2(\eta). \end{aligned} \tag{24}$$

We make the additional simplifying assumptions that the center of gravity and center of buoyancy of the vehicle are collocated and that the vehicle's mass matrix is diagonal. The first assumption results in $g_2(\eta) = 0 \ \forall \ \eta$. The second assumption results in $M_{\eta_{21}} = 0 \ \forall \ \eta$. $M_{\eta_{22}}$ is a function of the roll and pitch due to the presence of a rotation matrix. Therefore, we additionally apply the small angle assumption which results in the rotation matrix being approximately equivalent to the identity matrix and reduces $M_{\eta_{22}}^{-1} = M_{22}^{-1}$ which is constant. Using these assumptions we can simplify the vehicle attitude dynamics of (24). Let $x_1 = \begin{bmatrix} \phi & \theta & \psi \end{bmatrix}^\mathsf{T}$ and let $x_2 = \begin{bmatrix} \dot{\phi} & \dot{\theta} & \dot{\psi} \end{bmatrix}^\mathsf{T}$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= M_{22}^{-1}((\tau_2 - (C_{\eta_{21}} + D_{\eta_{21}})\dot{\eta}_1 \\ &\quad - M_{22}^{-1}(C_{\eta_{22}} + D_{\eta_{22}})x_2 \end{aligned} \tag{25}$$

which is in the form required to apply the RKHS MRAC controller. In the subsequent illustration the functional uncertainty is given by

$$f(x, \dot{\eta}_1) = -(C_{\eta_{21}} + D_{\eta_{21}})\dot{\eta}_1 - (C_{\eta_{22}} + D_{\eta_{22}})x_2. \tag{26}$$

For the purposes of expressing the functional uncertainty we convert the earth reference frame linear velocities, $\dot{\eta}_1$ in (25) to body reference frame linear velocities, surge, sway and heave, denoted collectively ν_1. Under the assumption of constant forward speed the body reference frame velocities are also near constant which requires us to use fewer basis centers to cover the range of expected vehicle velocities in the numerical illustration below. Thus we can express the AUV dynamics in the required form

$$\dot{x} = Ax + B\big(\delta + f(x, \nu_1)\big) \tag{27}$$

where $A \in \mathbb{R}^{6 \times 6}$ and $B \in \mathbb{R}^{6 \times 3}$ are constant matrices and B is based on the vehicle moments of inertia and control surface effectiveness coefficients. The control surface effectiveness coefficients are estimated via hydrodynamic modeling discussed in [18]. For the purposes of this experiment, we neglect measurement error in the vehicle moments of inertia which results in a known B matrix. Additionally, the vehicle body reference frame linear velocities are included in the input space of the functional uncertainty, $f : \mathbb{R}^9 \to \mathbb{R}^3$.

B. Practical Control Considerations

For attitude control of an underwater vehicle, our control objective is reference signal tracking in the presence of the uncertainty discussed in the previous section. Practical attitude control of the VT-690 AUV is impossible at forward speeds below a minimum threshold, called bare steerageway, and at attitudes where pitch and roll near or exceed $\pm 90\,\mathrm{deg}$. Therefore, we restrict our pitch commands to $\pm 20\,\mathrm{deg}$ and verify that measured vehicle forward speed exceeds the bare steerageway of the vehicle prior to exercising our controller.

As discussed in II-B, using (3), we approximate the uncertain function f using a set of N kernel centers. Learning the approximate function, $\hat{f}_N$ amounts to learning the vector of coefficients $\alpha = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{3N} \end{bmatrix}^\mathsf{T} \in \mathbb{R}^{3N}$ with each set of three coefficients associated with a kernel centered at one of the N centers. The set of kernel functions is given by $\bigcup_{i=1}^{N} \{\mathbb{K}_{\xi_i} e_1, \mathbb{K}_{\xi_i} e_2, \mathbb{K}_{\xi_i} e_3\}$ where $e_j \in \mathbb{R}^3$ is the standard basis vector with a one at entry j and zeros elsewhere. To develop an update law for α from (12), following a similar process to that described in [5], we take the inner product of $\dot{\hat{f}}_N$ with each basis function

$$\langle \mathbb{K}_{\xi_i} y_i, \dot{\hat{f}}_N \rangle_{\mathbb{H}} = \langle \mathbb{K}_{\xi_i} y_i, \sum_{k=1}^{3N} \dot{\hat{\alpha}}_k(t) \mathbb{K}_{\xi_k} y_k \rangle_{\mathbb{H}}$$

Using the reproducing property and the linearity properties of the inner product results in an update law with the form

$$\dot{\hat{\alpha}} = \gamma_f G^{-1} \Phi(x) \tag{28}$$

where $\{G\}_{jk} = y_j^{\mathsf{T}} \mathbb{K}(\xi_j, \xi_k) y_k \in \mathbb{R}^{3N \times 3N}$ and

$$\Phi(x) = \begin{bmatrix} e^{\mathsf{T}} PB\mathbb{K}(\xi_1, x) e_1 \\ e^{\mathsf{T}} PB\mathbb{K}(\xi_1, x) e_2 \\ \vdots \\ e^{\mathsf{T}} PB\mathbb{K}(\xi_N, x) e_3 \end{bmatrix}.$$

For the purposes of the numerical illustration in this paper we placed the kernel centers in a uniformly spaced grid throughout a subset of the function's input space ($\Omega \subset \mathbb{R}^6 \times \mathbb{R}^3$) and we use a simple kernel function:

$$\mathbb{K}(x_1, x_2) = k_{5/2}(x_1, x_2) \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} \tag{29}$$

where $k_{5/2}(x_1, x_2)$ is the scalar 5/2 Matern kernel parameterized by amplitude $a = 0.1$ and length scale $l = 0.3$ (see [22]). This kernel was chosen to capture the closely coupled nature of the roll and yaw behavior of the vehicle model.

We designed the controller reference model to have the desired transient characteristics consistent with the vehicle maximum turning and pitch rates based on the control surface actuation limits. The control surfaces are each limited to a maximum deflection of $\pm 20\,\mathrm{deg}$, which in turn places practical limits on the rise time of the step response when designing the reference model.

The adaptation rate matrices and gains M_x, M_r, γ_f were set empirically based on observations of the adaptive parameter behaviors.

C. Experiment Description and Results

We first simulate a AUV traveling at a constant speed of 1.5 m/sec, performing repeated step changes in pitch and yaw. The AUV is modeled using the 6-DOF dynamics of [18]. The controller is designed according to the RKHS MIMO MRAC formulation described in the previous sections. The pitch commands alternate between $\pm 5\,\mathrm{deg}$ and the yaw commands alternate between $\pm 10\,\mathrm{deg}$ as shown by the dotted lines in the figures below.

We performed the first experiment multiple times with increasing numbers of kernel centers and recorded the average function error norm, $\|\tilde{f}(t)\|_2$ and the average state error norm, $\|e(t)\|_2$. Figure 1 shows the improvement in performance as kernel centers are added and demonstrates that the magnitude of performance improvement is additionally dependent on kernel location. Above 1458 kernel centers the computational intensity became prohibitive. Figures 2-4 show the simulation with 1458 kernel centers.

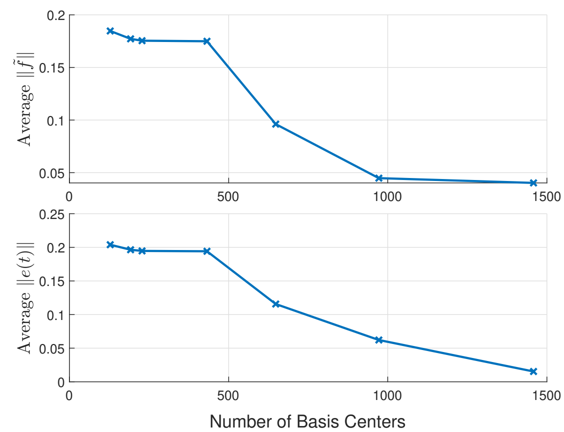

Fig. 1. Experiment 1: Comparison of error norm and functional error norm versus number of kernel centers

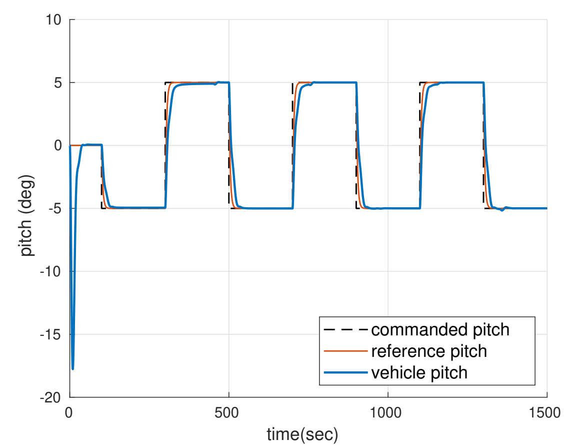

Fig. 2. Experiment 1: Vehicle Pitch

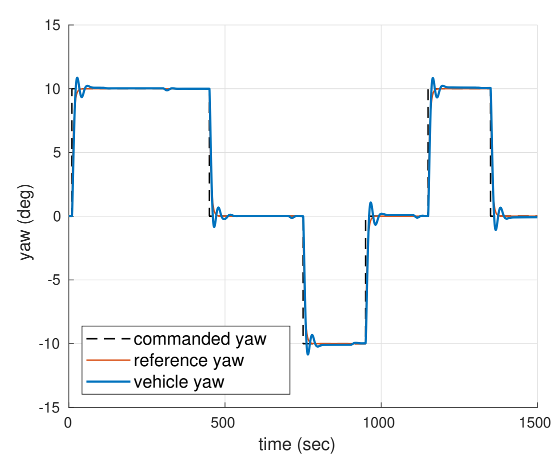

Fig. 3. Experiment 1: Vehicle Yaw

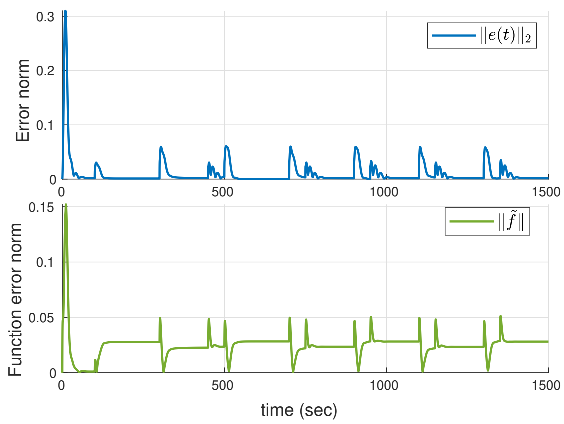

Fig. 4. Experiment 1: Norm of State Error and Function Error

In the second experiment we repeat the steps of the first simulation but add an instantaneous change in vehicle buoyancy from 1% positively buoyant to 1% negatively buoyant at $t = 650$ sec. Figures 5-7 show the attitude results of experiment 2.

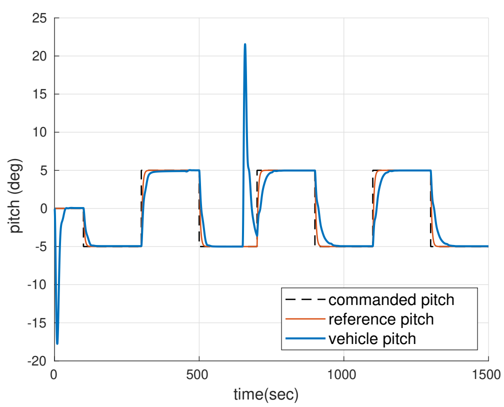

Fig. 5. Experiment 2: Vehicle Pitch

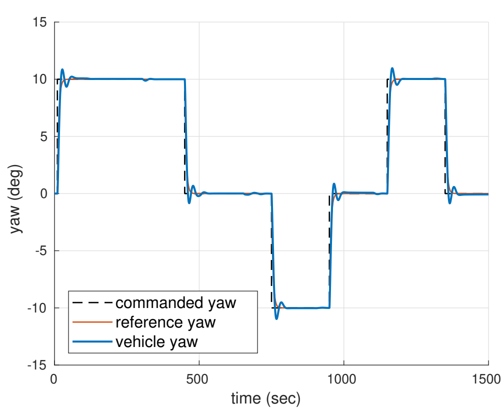

Fig. 6. Experiment 2: Vehicle Yaw

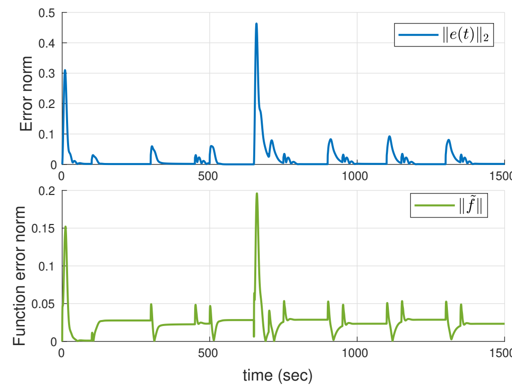

Fig. 7. Experiment 2: Norm of State Error and Function Error

V. Conclusion

This paper presents an MRAC controller based on vector-valued RKHS embedding methods. Our adaptive update laws yield ultimate boundedness of the reference signal tracking error under that the dead-zone modification. We further illustrate implementation of this controller for attitude control of an underwater vehicle. A benefit of non-parametric function estimation for adaptive control in an RKHS setting, rather than parametric function estimation that arises is conventional adaptive control, is that we are able to influence the estimation performance (e.g., error) by choice of kernel, location of kernel centers, and the number of kernel centers. For the attitude control example, we explicitly demonstrate that increasing the number of kernels and kernel density yields reduced function estimation error.

References

[1] N. Aronszajn. Theory of reproducing kernels. *Transactions of the American Mathematical Society*, 68:337–404, 1950.

[2] Christopher M. Bishop. *Pattern Recognition and Machine Learning*. Springer, 2006.

[3] Parag Bobade, Suprotim Majumdar, Savio Pereira, Andrew J. Kurdila, and John B. Ferris. Adaptive estimation in reproducing kernel hilbert spaces. In *2017 American Control Conference (ACC)*, pages 5678–5683, 2017.

[4] John A. Burns, Jia Guo, Andrew J. Kurdila, Sai Tej Paruchuri, and Haoran Wang. Error bounds for native space embedding observers with operator-valued kernels. *2023 American Control Conference (ACC), to appear*, 2023.

[5] John A. Burns, Andrew J. Kurdila, Derek Oesterheld, Daniel Stilwell, and Haoran Wang. Learning theory convergence rates for observers and controllers in native space embedding. *2023 American Control Conference (ACC), to appear*, 2023.

[6] Girish Chowdhary, Hassan A Kingravi, Jonathan P How, and Patricio A Vela. Bayesian nonparametric adaptive control using gaussian processes. *IEEE transactions on neural networks and learning systems*, 26(3):537–550, 2014.

[7] Jay Farrell and Marios Polycarpou. *Adaptive Approximation Based Control: Unifying Neural, Fuzzy and Traditional Adaptive Approximation Approaches*. John Wiley & Sons, 2006.

[8] Thor I. Fossen. *Guidance and Control of Ocean Vehicles*. John Wiley, 1999.

[9] Aditya Gahlawat, Pan Zhao, Andrew Patterson, Naira Hovakimyan, and Evangelos Theodorou. L1-gp: L1 adaptive control with bayesian learning. In *Learning for Dynamics and Control*, pages 826–837. PMLR, 2020.

[10] Tingran Gao, Shahar Z Kovalsky, and Ingrid Daubechies. Gaussian process landmarking on manifolds. *SIAM Journal on Mathematics of Data Science*, 1(1):208–236, 2019.
[11] J. Guo. *Convergence of kernel methods for modeling and estimation of dynamical systems*. PhD thesis, Virginia Tech, 2021.
[12] Jia Guo, Michael E. Kepler, Sai Tej Paruchuri, Hoaran Wang, Andrew J. Kurdila, and Daniel J. Stilwell. Adaptive estimation of external fields in reproducing kernel hilbert spaces. *International Journal of Adaptive Control and Signal Processing*, 36:1931–1957, 2022.
[13] Jia Guo, Sai Tej Paruchuri, and Andrew J. Kurdila. Persistence of excitation in uniformly embedded reproducing kernel hilbert (rkh) spaces. In *2020 American Control Conference (ACC)*, pages 4539–4544, 2020.
[14] Jia Guo, Sai Tej Paruchuri, and Andrew J. Kurdila. Partial persistence of excitation in RKHS embedded adaptive estimation. *IEEE Transactions on Automatic Control*, pages 1–13, 2022.
[15] Andreas Krause, Ajit Singh, and Carlos Guestrin. Near-optimal sensor placements in gaussian processes: Theory, efficient algorithms and empirical studies. *Journal of Machine Learning Research*, 9(2), 2008.
[16] E. Lavretsky and K. Wise. *Robust and Adaptive Control: With Aerospace Applications*. Advanced Textbooks in Control and Signal Processing. Springer London, 2012.
[17] Charles A. Micchelli and Massimiliano Pontil. On learning vector-valued functions. *Neural Computation*, 17(1):177–204, 2005.
[18] Taylor Njaka, Stefano Brizzolara, and Daniel J Stilwell. Technical and research bulletin 1-45: Guide for CFD-informed AUV maneuvering models, Jan 2022.
[19] Sai Tej Paruchuri, Jia Guo, and Andrew J. Kurdila. Kernel center adaptation in the reproducing kernel hilbert space embedding method. *International Journal of Adaptive Control and Signal Processing*, 36:1562–1583, 2022.
[20] Vern I. Paulsen and Mrinal Raghupathi. *An Introduction to the Theory of Reproducing Kernel Hilbert Spaces*. Cambridge University Press, 1 edition, Apr 2016.
[21] Gianluigi Pillonetto, Francesco Dinuzzo, Tianshi Chen, Giuseppe De Nicolao, and Lennart Ljung. Kernel methods in system identification, machine learning and function estimation: A survey. *Automatica*, 50(3):657–682, 2014.
[22] Carl Edward Rasmussen and Christopher K. I. Williams. *Gaussian Processes for Machine Learning*. The MIT Press, 11 2005.
[23] Adam J Thorpe, Cyrus Neary, Franck Djeumou, Meeko MK Oishi, and Ufuk Topcu. Physics-informed kernel embeddings: Integrating prior system knowledge with data-driven control. *arXiv preprint arXiv:2301.03565*, 2023.
[24] Dominik Wittwar, Bernard Haasdonk, and Gabriele Santin. Interpolation with uncoupled separable matrix-valued kernels. *Dolomites Research Notes on Approximation*, 11(11/2018):23–39, 2018.