2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Sparse Approximate Hamilton-Jacobi Solutions for Optimal Feedback Control with Terminal Constraints

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Abstract— **A semi-analytic method is proposed to solve a class of optimal control problems while exploiting its underlying Hamiltonian structure. Optimal control problems with a fixed final state at a fixed terminal time are considered. The solution methodology proposed in this work solves the Hamilton-Jacobi equation over a predefined domain of states and co-states. The advantage over traditional methods is that an approximate generating function (analogous to the value function of HJB theory) is obtained as a function of time, which allows for the computation of co-states for any final time and final state specified. Numerical experiments are conducted to demonstrate the efficacy of developed method while considering benchmark problems including spin stabilization.**

I. INTRODUCTION

The problem of controlling the dynamics of engineering systems from an arbitrary initial state to a desired target state within a predefined time range is of fundamental interest to engineers. Hamilton's principle states that the flow between two specified states at two specified times is obtained as the extremum of the action integral. This observation that the dynamical flow is an optimization problem motivates finding the solution to general optimization problems using appropriate Hamiltonian formulations. Further, Hamilton-Jacobi (HJ) theory can be utilized to obtain an analytic solution to Hamilton's equations by prescribing canonical transformation of variables through a generating function [1], [2]. Particularly, the transformation that maps the state and co-state at *any time* instant to its value at the *initial time* is useful in obtaining the feedback control law as a function of the initial, final, and current state.

Since feedback control utilizes the current state information as opposed to the state information from an earlier epoch, the resulting closed-loop system is modestly robust to model errors and state uncertainties. Traditionally, the feedback solution is obtained by solving the Hamilton Jacobi Bellman (HJB) equation. The HJB equation is a partial differential equation (PDE) whose analytic solution is primarily limited to linear systems. Since the HJB equation generally lacks closed-form solutions, different computational methods have been formulated to approximate the solution of the HJB equation, also known as the value function over an apriori defined domain [3]–[7]. The main challenge in the solution of HJB equation is the *curse of dimensionality,* as the dimension of spatial variables is equal to the state dimension. More recently, sparse approximation methods in conjunction with non-product quadrature methods are employed to derive a computationally efficient method to solve the HJB equation [8]–[11]. However, one has to solve the HJB equation again if the specified final time and/or boundary condition is changed like any other computational method.

The methodology proposed in this work is fundamentally different from the traditional computational methods to derive optimal feedback solutions via the HJB equation. The proposed approach utilizes the HJ equation to solve for generating functions for the underlying Hamiltonian system in state and co-states. The generating functions provide a map between the current state and co-state to its value at the initial time. Numerous strategies have been proposed for solving the HJ equation within the context of optimal control [12]–[15]. The special nature of the relationship between the value function and the generating function is clarified in [13], [14], where a family of value functions can be derived from a single generating function. Thus, solving the HJ equation for a generating function achieves a family of optimal feedback control profiles as an explicit function of the boundary conditions. Eapen et al. [16], [17] have investigated this property of Hamiltonian dynamical systems in the context of the optimal feedback control problem. By connecting the value function to the Hamilton-Jacobi generating function, a systematic way to evaluate the optimal feedback control and cost function while still satisfying the general boundary conditions was obtained. These findings are expanded upon in the current work. The Hamilton-Jacobi equation will be solved locally using a collocation scheme for approximating the generating function using a non-product sampling method called Conjugate Unscented Transformation (CUT). Additionally, the required polynomial basis function set for the collocation-based approximation is automatically generated from an overcomplete dictionary of basis functions using current developments in sparse approximation. It is shown that a form for the feedback control law, which is typically unknown, is automatically identified by the solution process using the basis function selection method.

The remainder of the paper is structured as follows: First, it is briefly shown how the generating functions are obtained from the variational principal and provides the necessary background for the method developed. Next, the Two-Point Boundary Value Problem (TPBVP) is posed, and its connections to the Hamilton-Jacobi theory is discussed. The general solution process is delineated for solving the HJ equation using a collocation-based scheme. Finally, the utility of the developed method is demonstrated using two examples, the

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van der Pol oscillator and the attitude stabilization example.

II. Problem Formulation and Hamilton-Jacobi Preliminaries

The primary objective of this research is to develop a numerical framework for solving the optimal control problem with terminal constraints:

$$\min_{\mathbf{u}(t)} \quad J=\phi\left(\mathbf{x}\left(t_f\right), t_f\right)+\int_0^{t_f} L(\mathbf{x}(t), \mathbf{u}(\mathbf{x}, \tau)) d \tau \quad (1)$$

$$\text{subject to: } \dot{\mathbf{x}}=\mathbf{f}(\mathbf{x}, \mathbf{u}, t), \quad \mathbf{x}(0) \in \mathscr{X}_0, \quad \mathbf{x}(t_f) \in \mathscr{X}_f \quad (2)$$

$$\psi\left[\mathbf{x}\left(t_f\right), t_f\right]=0 \quad (3)$$

where $\mathbf{x} \in \Re^n$ is the state, $\mathbf{u} \in \Re^m$ is the control, $(\mathscr{X}_0, \mathscr{X}_f) \subset \Re^n$ is the domain of initial and final states, and $\psi\left[\mathbf{x}\left(t_f\right), t_f\right] \in \Re^q$ $(q \leq n)$ is the terminal state constraint. $\phi\left(\mathbf{x}\left(t_f\right), t_f\right)$ is a penalty on the terminal state, and $L(\mathbf{x}(\tau), \mathbf{u}(\tau), t)$ is a problem-dependent penalty on the state and control variables and is assumed as convex in both $\mathbf{x}(t)$ and $\mathbf{u}(t)$. Furthermore, only fixed final time problems are considered in this work. The general procedure of solving the optimal control problem is done by defining an augmented cost function of the Hamiltonian, which is given as:

$$H(\mathbf{x}, \lambda, \mathbf{u}, t)=L(\mathbf{x}, \mathbf{u}, t)+\lambda^T \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \quad (4)$$

The Hamiltonian H in Eq. (4) governs the evolution of the state and co-state variables. The paths of these variables produce a certain stationary functional:

$$\delta \int \lambda \cdot d \mathbf{x}-H d t=0 \quad (5)$$

The first-order necessary condition for optimality ($\frac{\partial H(\mathbf{x}, \lambda, \mathbf{u}, t)}{\partial \mathbf{u}}=0$) is obtained by the variation of Eq. (5) and is related to Pontryagin's Minimum Principle (PMP), which states that the optimal control is the one that minimizes the Hamiltonian. Therefore, the optimal feedback control law can be computed as:

$$\mathbf{u}(\mathbf{x}, \lambda, t)=\arg \min_{\overline{\mathbf{u}} \in \mathscr{U}}\{H(\mathbf{x}, \lambda, \overline{\mathbf{u}}, t)\} \quad (6)$$

where $\mathscr{U}$ is the set of admissible control laws.

It is easy to verify that the addition of a path-independent term to Eq. (5) does not affect the dynamics. If one were to introduce a new state and co-state $(\mathbf{x_0}, \lambda_0)$ that are expressed in terms of the old state and co-state $(\mathbf{x}, \lambda)$ in such a way that they can be represented as:

$$\mathbf{x}_0=\mathbf{x}_0(\mathbf{x}(t), \lambda(t), t, t_0), \quad \lambda_0=\lambda_0(\mathbf{x}(t), \lambda(t), t, t_0) \quad (7)$$

The path-independent term ensures that the transformation above is canonical and called the generating function. It is noted that Eq. (7) is not an explicit function of the control because it is assumed that the cost function in Eq. (1) is in control-affine form, and therefore the control can be expressed as a function of the co-states. Since the generating function is path-independent, it should satisfy:

$$\sum_{i=1}^n \lambda^i \dot{x}^i-H(\mathbf{x}, \lambda, t)=\sum_{i=1}^n \lambda_0{}^i \dot{x}_0^i-K(\mathbf{x_0}, \lambda_0, t)+\frac{d F}{d t} \quad (8)$$

Therefore, for a new set of variables $(\mathbf{x_0}, \lambda_0)$ and a new Hamiltonian, K, the dynamics are the same if the integrand differs by a path-independent term, dF, such that

$$\lambda^T \dot{\mathbf{x}}-H(\mathbf{x}, \lambda, t)=\lambda_0{}^T \dot{\mathbf{x}}_0-K(\mathbf{x_0}, \lambda_0, t)+\frac{d F}{d t} \quad (9)$$

Eq. (9) gives a functional form of the differential criterion that leads to $[\mathbf{x_0}, \lambda_0]$ satisfying the canonical differential equations of Hamilton. The variable change that preserves $\oint \lambda \cdot d\mathbf{x}$ also preserves dynamics (with appropriately modified Hamiltonian). In this work, the freedom to add path-independent terms to specify canonical transformations is leveraged to relate the states and co-states at any current time with its value at the initial time, thereby providing the optimal feedback control at any time instant as a function of the generating function, F. There exist four possible forms of the generating function which can generate the given canonical transformation to the new variable space $(\mathbf{x}_0, \lambda_0)$ and satisfy the HJ equation as shown in Table I.

$F_1(\mathbf{x}, \mathbf{x_0}, t, t_0)$	$\lambda=\frac{\partial F_1}{\partial \mathbf{x}}$	$\lambda_0=-\frac{\partial F_1}{\partial \mathbf{x}_0}$	$\frac{\partial F_1}{\partial t}+H\left(\mathbf{x}, \frac{\partial F_1}{\partial \mathbf{x}}, t\right)=0$
$F_2(\mathbf{x}, \lambda_0, t, t_0)$	$\lambda=\frac{\partial F_2}{\partial \mathbf{x}}$	$\mathbf{x}_0=\frac{\partial F_2}{\partial \lambda_0}$	$\frac{\partial F_2}{\partial t}+H\left(\mathbf{x}, \frac{\partial F_2}{\partial \mathbf{x}}, t\right)=0$
$F_3(\lambda, \mathbf{x_0}, t, t_0)$	$\mathbf{x}=-\frac{\partial F_3}{\partial \lambda}$	$\lambda_0=-\frac{\partial F_3}{\partial \mathbf{x}_0}$	$\frac{\partial F_3}{\partial t}+H\left(-\frac{\partial F_3}{\partial \lambda}, \lambda, t\right)=0$
$F_4(\lambda, \lambda_0, t, t_0)$	$\mathbf{x}=\frac{\partial F_4}{\partial \lambda}$	$\mathbf{x}_0=-\frac{\partial F_4}{\partial \lambda_0}$	$\frac{\partial F_4}{\partial t}+H\left(-\frac{\partial F_4}{\partial \lambda}, \lambda, t\right)=0$

TABLE I: Types of Generating Functions

Notice that the above relationship of type-1 generating function with the fixed final state (instead of the fixed initial state) and the state at time t can be derived $(F_1\left(\mathbf{x}_f, \mathbf{x}, t_\mathrm{f}, t\right))$:

$$\lambda=-\frac{\partial F_1}{\partial \mathbf{x}}, \lambda_\mathrm{f}=\frac{\partial F_1}{\partial \mathbf{x}_f},-\frac{\partial F_1}{\partial t}+H\Big(\mathbf{x},-\frac{\partial F_1}{\partial \mathbf{x}}, t\Big)=0 \quad (10)$$

Since any time instant t $(t<t_f)$ can be the initial time, the above equation holds for arbitrary initial conditions. Thereby, using $\lambda=-\frac{\partial F_1}{\partial \mathbf{x}}$ in Eq. (6):

$$\mathbf{u}=\arg \min_{\overline{\mathbf{u}}} H\Big(\mathbf{x},-\frac{\partial F_1}{\partial \mathbf{x}}, \overline{\mathbf{u}}, t\Big) \quad (11)$$

Thus, Eq. (11) provides the optimal feedback control using the type-1 generating function.

An appropriate generating function must be utilized to find the solution to these transformations. The selection of the generating function depends upon the type of initial and final boundary conditions specified for the optimal control problem. First, the relationship of these generating functions in the HJ equation with the value function corresponding to the HJB equation is obtained.

A. *Relationship between the value function and the generating function*

To obtain the relationship between the optimal feedback control using the value function and the generating function, the control profile is compared using both methods. Dynamic programming yields the relationship between the co-states and the optimal value function of the HJB equation, $\lambda=\frac{\partial V(\mathbf{x}, t)}{\partial \mathbf{x}}$. Thus, the optimal feedback control can be obtained by substituting $\lambda=\frac{\partial V(\mathbf{x}, t)}{\partial \mathbf{x}}$ into Eq. (6) as:

$$\mathbf{u}(\mathbf{x}, \lambda, t)=\arg \min_{\overline{\mathbf{u}} \in \mathscr{U}}\left\{H\Big(\mathbf{x}, \frac{\partial V(\mathbf{x}, t)}{\partial \mathbf{x}}, \overline{\mathbf{u}}, t\Big)\right\} \quad (12)$$

Equations (12) and (11) imply the existence of a relationship between the value function computed from the HJB equation and the generating function computed through the HJ equation. This relationship between the value function and the type-1 generating function is expressed as [14], [18]:

$$V(\mathbf{x}, t)=-F_1(\mathbf{x}_f, \mathbf{x}, t_f, t)+\phi\left(\mathbf{x}(t_f), t_f\right) \quad (13)$$

This can easily be proved for the hard constraint boundary value problem where the final state is fixed, i.e., $\phi\left(\mathbf{x}(t_f), t_f\right) = 0$. This results in $V(\mathbf{x},t) = -F_1$ and substituting this in HJ equation results in:

$$\frac{\partial V(\mathbf{x},t)}{\partial t} + H\left(\mathbf{x}, \frac{\partial V(\mathbf{x},t)}{\partial \mathbf{x}}, t\right) = 0 \tag{14}$$

which is the HJB equation for value function V. In fact, other generating functions can also be used to compute the value function by exploiting the Legendre transformations. For that purpose, however, using F_2, F_3, or F_4 requires one to solve a set of implicit equations as well as to take partial differentiations, whereas employing F_1 only necessitates taking partial differentiations [14]. For the terminal constraint given by a hyperplane $\psi\left[\mathbf{x}\left(t_f\right), t_f\right] = 0$ in $\mathscr{R}^{q\le n}$, mixed terminal conditions for both states and co-states in general is obtained. In this case, a more generalized kind of generating function is required, which would mix all 4 kinds of variables (initial and terminal states and co-states).

If one uses the type-2 generating function, it is observed that at the terminal time, $F_2 = \mathbf{x}_f^T\lambda$ generated: $\mathbf{x} = \frac{\partial F_2}{\partial \lambda}\mid_{\mathbf{x}=\mathbf{x}_f}$ and same with λ_0. The generating function F_1 can be obtained from F_2 using the Legendre transformation, $F_1\left(\mathbf{x}, \mathbf{x}_f, t, t_f\right) = F_2\left(\mathbf{x}, \lambda_f, t, t_f\right) - \lambda^T\mathbf{x}$, which at the final time evaluates to:

$$F_1\left(\mathbf{x}_f, \mathbf{x}_f, t_f, t_f\right) = F_2\left(\mathbf{x}_f, \lambda_f, t_f, t_f\right) - \lambda_f^T\mathbf{x}_f = 0 \tag{15}$$

That is, at the final time, F_1 satisfies the HJB equation and is equal to the value function $V\left(\mathbf{x}_f, t_f\right) = 0$. Thus, one can observe that there may be drawbacks to using the generating function of the first kind: F_1 becomes singular as it loses the independence of its arguments. In fact, this is an equivalent statement that the optimal cost function becomes singular at the terminal time for the hard constraint problem. Additionally, mapping the states from any arbitrary time instant, t, to the final time is counterproductive because it will require multiple HJ equations to be solved for different terminal constraints. Contrastingly, mapping to the initial time provides the advantage of solving the HJE only once.

For these reasons, this paper will utilize the generating function of type 2, with the canonical mapping from any arbitrary time instant to the initial time. One of the main reasons for this is that, unlike F_1, F_2 can generate identity transformations. This means that the generating function at initial time is known exactly as $F_2 = \lambda_0^T\mathbf{x}$. Furthermore, assuming that the implicit relations arising from F_2 generating function can be solved, the terminal time and terminal state can be kept in a functional form, and the HJ PDE can be solved in forward time (as opposed to HJB, which is solved in backward time). This freedom allows for the development of a systematic methodology to solve OCPs in functional form and obtain the solution in a semi-analytic way.

III. Solution Methodology for OCP solution using type-2 generating function

As mentioned in the previous section, a methodology to solve the Hamilton-Jacobi equation stemming from the optimal control problem is presented in this work. Recall that the generating function enables the canonical transformation to map the space variables to initial condition space variables, i.e., $(\mathbf{x}(t), \lambda(t)) \to (\mathbf{x}_0, \lambda_0)\,\forall t \in \left[t_0, t_f\right]$. Considering a type-2 generating function, the HJ equation and the corresponding relation between co-state and state are given as:

$$\frac{\partial F_2(\mathbf{x}, \lambda_0, t, t_0)}{\partial t} + H\left(\mathbf{x}, \frac{\partial F_2(\mathbf{x}, \lambda_0, t, t_0)}{\partial \mathbf{x}}, t\right) = 0 \tag{16a}$$

$$\lambda = \frac{\partial F_2(\mathbf{x}, \lambda_0, t, t_0)}{\partial \mathbf{x}}, \quad \mathbf{x}_0 = \frac{\partial F_2(\mathbf{x}, \lambda_0, t, t_0)}{\partial \lambda_0} \tag{16b}$$

In the case where the initial and terminal states are explicitly given, the generating function $F_2(\mathbf{x}, \lambda_0, t)$ can be directly used to find the initial state and final co-states from the above relationship. The key observation is that solving for λ_0 solves the boundary-value problem and, hence, the optimal control problem. Suppose there exists an analytical form for $F_2(\mathbf{x}, \lambda_0, t)$. Then, by taking its partial derivatives and specifying $\mathbf{x}_0$ and $\mathbf{x}_f$, the appropriate co-states to generate the optimal control profile can be found. The major advantage is that the solution process for solving the HJ equation is an initial value problem as opposed to a boundary value problem that the HJB equation gives. This exclusive advantage allows the HJ equation to be solved semi-analytically.

Since the existence of a generating function is not guaranteed for nonlinear dynamical systems, a numerical method is used to approximate the solution in a pre-defined neighborhood. To do so, a sparse-collocation method will be employed to precisely determine the minimal number of coefficients required to approximate the generating function from an over-complete dictionary of basis functions. In addition, the methodology uses a low number of collocation points that can accurately represent the entire domain and alleviate the curse of dimensionality.

A. Development of Collocation Equations

Consider a dynamical system with state $\mathbf{x}$ and co-state λ. Following the definition of the approximate generating function, the solution to the HJ equation is assumed to be:

$$F_2(\mathbf{x}, \lambda_0, t) = \sum_{j=1}^{m} c_j(t)\phi_j(\mathbf{x}, \lambda_0) = \mathbf{c}(t)\Phi(\mathbf{x}, \lambda_0) \tag{17}$$

where $\mathbf{c}(t) \in \mathbb{R}^m$ is a vector of time-varying coefficients, while $\phi(\mathbf{x}, \lambda_0) \in \mathbb{R}^m$ is a vector of the basis function and are assumed to have at least continuous first-order derivatives. There are infinitely many choices for basis functions, such as polynomials, radial basis functions, wavelets, and so on. A key difficulty in choosing an optimal basis function set is due to the unknown characteristics of the generating function. It should be noted that a good approximation can lead to a large basis function matrix, $\Phi(\mathbf{x}, \lambda_0)$, which can consequently increase computational costs. An ingenious choice of basis function should create sparse matrices and result in a parsimonious model, i.e., very few $\mathbf{c}(t)$ are non-zero while meeting the approximation requirements. A series of predefined continuous functions always exists on a compact interval [19] according to the Stone-Weierstrass theorem. This theorem allows for the accurate approximation of any continuous function over a compact interval using polynomial variables with a sufficient number of terms. Due to these reasons, along with the ease of differentiation and integration, polynomial functions are commonly employed as basis functions in approximation theory.

Note here that since the Hamiltonian H is non-autonomous, the approximate generating function is written in a way that the basis functions are in the spatial variables, and its coefficients are temporal. Substituting Eq. (17) in Eq. (16a), the HJ equation can be written as:

$$\dot{\mathbf{c}}_i(t)\Phi(\mathbf{x},\lambda_0) = -H\Big(\mathbf{x},\frac{\partial F_2}{\partial \mathbf{x}},t\Big) \tag{18}$$

Expanding the time derivatives as finite differences, the above equation can be written as:

$$\mathbf{A}\delta\mathbf{c}_{k+1} = \mathbf{b} \tag{19}$$

where $\delta\mathbf{c}_{k+1} = \mathbf{c}_{k+1} - \mathbf{c}_k$. Here $\mathbf{c}_k$ are the coefficients at time t_k and $\delta\mathbf{c}_{k+1}$ are the departure coefficients at time t_{k+1}. The residual error from Eq. (19) is given as:

$$e(\zeta) = \mathbf{A}\delta\mathbf{c}_{k+1} - \mathbf{b}, \quad \mathbf{A}_i^T = \Phi^i(\zeta), \mathbf{b}_i = -H\Big(\mathbf{x},\frac{\partial F_2}{\partial \mathbf{x}},t_k\Big)dt \tag{20}$$

where $\zeta = [\mathbf{x},\lambda_0]$. In the collocation method, the error is projected onto a series of delta functions centered at chosen collocation points, resulting in a residual error being zero at the collocation points. Assuming there are total N collocation points, leading to a system of N equations in m unknowns to exactly solve the HJ equation at prescribed points, ζ_i:

$$\int e(\zeta)\delta(\zeta-\zeta_i)\,d\zeta = 0 \rightarrow e(\zeta_i) = 0, \quad i = 1,2,\ldots N \tag{21}$$

where ζ_i are the chosen collocation points. The selection of the collocation points is crucial in obtaining a well-conditioned system of equations for the unknown coefficients. Therefore, an efficient numerical sampling method known as the Conjugate Unscented Transformation (CUT) method is used to generate collocation points in n-dimensional space. The CUT method avoids the tensor product of 1-D points and hence provides a lower number of points as compared to traditional quadrature methods like Gauss quadrature and sparse grid [20]. Further simplifications can be made by examining the structure of the Hamiltonian.

Theorem 1: If Hamiltonian is an even function, the type-2 generating function will only be a function of even coefficients.

Proof: Since H is an even function, it can be written as: $H(\mathbf{z}) = \sum_{i=1}^{p}\mathbf{a}_i\Phi(\mathbf{z}_i^2)$. Assuming a generating function with the unknown coefficients, $F_2(\mathbf{z}) = \sum_{i=1}^{q}\mathbf{c}_i\Phi(\mathbf{z}_i)$. By substituting $F_2(\mathbf{z})$ in the HJ equation (Eq. 16a), the following relation isobtained: $\sum_{i=1}^{q}\dot{\mathbf{c}}_\mathbf{i}\Phi(\mathbf{z}_i) = \sum_{i=1}^{p}\mathbf{a}_i\Phi(\mathbf{z}_i^2)$. Equating the odd coefficients on both sides results in the following: $\dot{\mathbf{c}}_{odd} = \mathbf{0}$. Using the finite-difference method, the following is obtained: $\mathbf{c}_{k+1_{odd}} = \mathbf{c}_{k_{odd}}$ which means the odd coefficients have the same value at each time step. Since the initial coefficients are even ($F_2 = \mathbf{x}_0^T\lambda$), the odd coefficients will never be able to get excited due to the even Hamiltonian. ∎

Due to the fact that standard polynomials are used to approximate the generating function, numerical challenges may develop for larger domains with high-order polynomials. The system dynamics are thereby mapped to a local domain within a unit hypercube. This assures that the polynomial basis functions are well-conditioned numerically. One crucial point to remember is that, for a general nonlinear system, the co-states have no physical meaning. Therefore, in order to determine the domain of discretization of the co-states, an open-loop control problem is solved for obtaining the domain's edge values. Assuming the global domain ($\pm\mathbf{a}$) is known apriori, a constant linear transformation can be applied to scale the global space to within $\pm\mathbf{1}$ in all dimensions.

$$\mathbf{y}_\zeta = \mathbf{T}\zeta \quad \Rightarrow \quad \begin{bmatrix} \mathbf{y}_\mathbf{x} \\ \mathbf{y}_{\lambda_0} \end{bmatrix} = \begin{bmatrix} \mathbf{T}_1 & 0 \\ 0 & \mathbf{T}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \lambda_0 \end{bmatrix} \tag{22}$$

Thus, Eq. (20) can be written as:

$$\mathbf{A}_i^T = \Phi^i(\mathbf{y}_\zeta), \quad \mathbf{b}_i = -H^i(\mathbf{x},\mathbf{T}_1\frac{\partial F_2}{\partial \mathbf{y}_\mathbf{x}})dt, \quad i = 1,2,\ldots,N \tag{23}$$

and Eq. (16b) can be written as:

$$\lambda = \frac{\partial F_2\left(\mathbf{y}_\zeta,t_k\right)}{\partial \mathbf{y}_\mathbf{x}}\mathbf{T}_1, \qquad \mathbf{x}_0 = \frac{\partial F_2\left(\mathbf{y}_\zeta,t_k\right)}{\partial \mathbf{y}_{\lambda_0}}\mathbf{T}_2 \tag{24}$$

Notice that the numerical solution of Eq. (20) can be computed by minimizing l_2-norm of error, which determines the best-fit solution for the given N collocation points resulting in M coefficients. These M coefficients obtained by the l_2-norm solution utilize most of the M coefficients from the complete dictionary of basis functions and thus provide the smallest possible two-norm error. As a result, this method tends to overfit the training data (collocation points) and can yield a high norm error on the testing data (interpolation). Therefore, an alternative method utilizing l_1-norm approximation is implemented to provide the minimum possible number of coefficients required to accurately describe the full domain while maintaining the optimization convex.

B. Optimal Selection of Basis Functions

As described in the section III-A, the solution of Eq. (20) can be calculated by minimizing the weighted two-norm error, which aims to find the best-fit solution for the given CUT collocation points:

$$\delta\mathbf{c}_{k+1_{l_2}} = \min_{\delta\mathbf{c}_{k+1}} \|\mathbf{W}(\mathbf{A}\delta\mathbf{c}_{k+1} - \mathbf{b})\|_2 \tag{25}$$

where $\mathbf{W}$ is the weight matrix. As previously stated, $\delta\mathbf{c}_{k+1_{l_2}}$ is known to pick all possible coefficients from the dictionary of basis functions and is therefore not sparse. This research seeks a minimal polynomial expansion that guarantees sparsity for the generating function. Therefore, a weighted l_1-norm optimization problem is proposed to select the minimum possible coefficients from the extensive dictionary of basis functions. In lieu of the equality constraint of Eq. (19), this optimization problem is considered as a bounded two-norm error using ε as a soft inequality constraint. This allows sparse coefficients $\delta\mathbf{c}_{k+1_s}$ to trade sparsity for approximation error, providing a more flexible option. The complete optimization problem to select the minimal polynomial expansion of log-pdf is illustrated in Algorithm 1 and more details can be found in Ref. [21], [22].

After obtaining the sparse coefficients $\delta\mathbf{c}_{k+1_s}$, it is possible to separate the dominant coefficients from the non-dominant coefficients by choosing a user-defined coefficient threshold δ_{rs}. In addition, the non-dominant coefficients can be ignored

Algorithm 1 Iterative weighted l_1-norm optimization: $\delta\mathbf{c}_{k+1_s}$= WeightedOpt$(\mathbf{A},\mathbf{b},\mathbf{W},\Delta_s,\alpha,\varepsilon,\eta)$

Input: $\mathbf{A},\mathbf{b},\mathbf{W},\Delta_s,\alpha,\varepsilon,\eta$
Output: $\delta\mathbf{c}_{k+1_s}$
Initialization $\mathbf{K} \propto \mathscr{O}(basis), \delta = 1$
compute $\delta\mathbf{c}^-_{k+1} = \min\limits_{\delta\mathbf{c}_{k+1}} \|\mathbf{K}\delta\mathbf{c}_{k+1}\|_1$
  subject to: $\|\mathbf{W}(\mathbf{A}\delta\mathbf{c}_{k+1}-\mathbf{b})\|_2 \le \varepsilon$
while $\delta \ge \Delta_s$ **do**
  Update $\mathbf{K} = \frac{1}{(\delta\mathbf{c}^-_{k+1}+\eta)}$, to find $\delta\mathbf{c}^+_{k+1} = \min\limits_{\delta\mathbf{c}_{k+1}} \|\mathbf{K}\delta\mathbf{c}_{k+1}\|_1$
    subject to: $\|\mathbf{W}(\mathbf{A}\delta\mathbf{c}_{k+1}-\mathbf{b})\|_2 \le \varepsilon$
  Compute $\delta = \|\delta\mathbf{c}^+_{k+1} - \delta\mathbf{c}^-_{k+1}\|_2$
  $\delta\mathbf{c}^-_{k+1} = \delta\mathbf{c}^+_{k+1}$
end while
$\delta\mathbf{c}_{k+1_s} = \delta\mathbf{c}^-_{k+1}$

for computational purposes by substituting zero. A new minimal representation of the basis functions $\mathbf{A}_{RS} \in \Re^{N\times m_r}$ corresponding to the m_r non-zero coefficients can be constructed. Therefore, the reduced sparse (RS) coefficients $\delta\mathbf{c}_{k+1} \in \Re^{m_r}$ can be calculated using the l_2-norm minimization:

$$\delta\mathbf{c}_{k+1} = \mathbf{A}^{\dagger}_{RS}\mathbf{b} \tag{26}$$

where $\mathbf{A}^{\dagger}_{RS}$ is the pseudo-inverse of the dominant basis functions. Moreover, $\mathbf{c}_{k+1}$ can then be computed as:

$$\mathbf{c}_{k+1} = \mathbf{c}_k + \delta\mathbf{c}_{k+1} \tag{27}$$

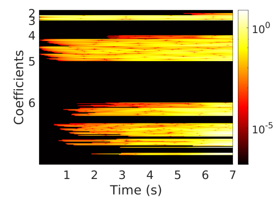
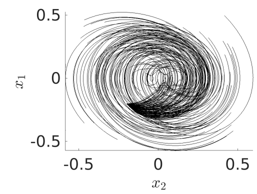
This minimal representation $\mathbf{c}_{k+1}$ is then employed to compute the generating function at time t_{k+1} and the procedure can be repeated till the final time t_f. Now to obtain the solution of the OCP, given any initial state $\mathbf{x}_0$ and final state $\mathbf{x}_f$, the initial co-state can be computed using coefficients at final time ($\mathbf{c}_{t_f}$) using Eq. (24) as:

$$\mathbf{x}_0 = \frac{\partial F_2\left(\mathbf{y}_\zeta, t_f\right)}{\partial \mathbf{y}_{\lambda_0}}\mathbf{T}_2 = \frac{\partial \Phi(\mathbf{x}_f,\lambda_0)\mathbf{c}_{t_f}}{\partial \mathbf{y}_{\lambda_0}}\mathbf{T}_2 \tag{28}$$

Once the value of the initial co-state corresponding to an initial and final state is obtained, the TPBVP is solved. The problem is now modified to an initial value problem where the initial conditions on state and co-state is solved using the necessary conditions.

IV. Numerical Validations

The current section presents two motivating examples. The first example considered is the TPBVP for the van der Pol dynamical system. The reason for introducing this example is to emphasize that the HJ theory approach to TPBVP is not limited to the optimal control of Hamiltonian systems only. The van der Pol system is non-conservative, as evidenced by the existence of a limit cycle. The second example considered is the attitude stabilization of a rigid body. This example is chosen to demonstrate that an exact analytical solution can be obtained through the approximation of the generating function, provided the correct set of basis functions is chosen. The rigid body stabilization has a closed-form solution for the infinite horizon problem, which is retrieved using the solution methodology presented in this paper.

OCP	Parameters
$\min\limits_{\mathbf{u}(t)} \quad J = \frac{1}{2}\int_{t_0}^{t_f} u^2 dt$ subject to: $\dot{x}_1 = x_2$ $\dot{x}_2 = -x_1 + \varepsilon(1-x_1^2)x_2 + u$ $\psi[\mathbf{x}(t_f),t_f] = [-0.2,-0.2]$	$\varepsilon = 0.1$ $t_f = 7s, dt = 0.01$ $\alpha = 10^{-7}, \eta = 10^{-5}, \delta_{rs} = 10^{-4}$ $(\mathbf{x},\lambda) \in [-0.5, 0.5]$, $N = 161$ Basis$(\mathscr{O}(10)) \to m = 1001$
Coefficient Variation	State Evolution
	

	x	u	J
RMS Error	4.2611e-03	4.0059e-03	4.0274e-06

TABLE II: Van der Pol Oscillator

A. Van der Pol Oscillator

The optimal control problem for the Van der Pol oscillator is considered through the minimization of the total control energy as shown in Table II. Using these parameters mentioned, the heatmap of RS coefficients up to 6^{th} order is shown in the second row as the higher order coefficients ($\mathscr{O}(7-10)$) come out to be zero. It can be noticed that all odd-order coefficients are inactive in approximating the generating function. This is due to the fact that the Hamiltonian of Van der Pol oscillator comes out to be an even function in terms of states and co-states, as $H(\mathbf{x},\lambda) = H(-\mathbf{x},-\lambda)$.

With the acquired RS coefficients, the functional form of the generating function can be determined. Thus, given the initial and final boundary condition ($\mathbf{x}_0$, $\mathbf{x}_f$), the initial co-state (λ_0) can be obtained directly from generating function using the relations in Eq. (24).

$$\mathbf{x}_0 = \frac{\partial F_2\left(\mathbf{y}_\zeta, t_f\right)}{\partial \mathbf{y}_{\lambda_0}}\mathbf{T}_2 = \kappa(\mathbf{x}_f,\lambda_0,\mathbf{c}_{t_f}) \Rightarrow \lambda_0 = \bar{\kappa}(\mathbf{x}_0,\mathbf{x}_f,\mathbf{c}_{t_f})$$

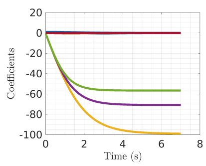
Utilizing these relations, one can find closed-form solutions for the co-state given any initial and final state. To further demonstrate the effectiveness of the developed approach in computing the optimal solution, 100 random initial condition samples within the domain are considered to reach the final target state. The state evolution between x_1 and x_2 is also shown in the Table II. Moreover, the obtained results RS solution are then compared with the bvp4c solution in MATLAB for the given final target. The RMS error over time for the state, control, and cost function corresponding to the 100 initial states is shown in the last row.

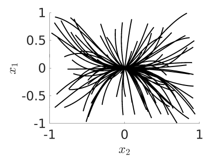
B. Spin Stabilization

The second example examines the spin stabilization of a spacecraft in a torque-free environment. Here, an appropriate feedback control is employed to prevent the rotational motion of a tumbling rigid body. A performance index that minimizes the state and control energy describes the optimal control problem as shown in Table III.

The evolution of the RS coefficients in approximating the generating function is also shown. The stationary value of the type-2 generating function at $t_f = 7s$ is shown in (29a) along with the type-1 generating function in (29b).

OCP	Parameters
$\min_{\mathbf{u}(t)} \quad J = \int_0^{t_f} \left(\omega^T \mathbf{Q}\omega + \mathbf{u}^T \mathbf{R}\mathbf{u}\right) dt$ s.t.: $\quad \dot{\omega} = -\mathbf{J}^{-1}\omega \times \mathbf{J}\omega + \mathbf{g}\mathbf{u}(\mathbf{x}, t)$ $\psi[\omega(t_f), t_f] = [0, 0, 0]$ $\mathbf{g} = \mathbf{J}^{-1}, \mathbf{Q} = q_0 \mathbf{I}_{3x3}, \mathbf{R} = r_0 \mathbf{I}_{3x3}$	$\mathbf{J} = \mathrm{diag}[7, 10, 12]$ $r_0 = 1, q_0 = 50$ $t_f = 7s, dt = 0.01$ $\alpha = 10^{-5}$, $\eta = 10^{-4}$, $\delta_{rs} = 10^{-4}$ $\mathbf{x} \in [-1, 1], N = 301$ Basis$(\mathscr{O}(8)) \rightarrow m = 3003$
Coefficient Variation	State Evolution





	ω	$\mathbf{u}$	J
RMS Error	3.3247-03	2.7813e-02	2.8861e-03

TABLE III: Spin Stabilization

$$F_2(\omega, \lambda, t_f) = -98.90\omega_1^2 - 70.64\omega_2^2 - 56.51\omega_3^2 \tag{29a}$$

$$F_1(\omega_f, \omega_f, t_f) = -\alpha\left(J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2\right) \tag{29b}$$

where $\alpha \approx \sqrt{50}$. This demonstrates that the stationary generating function is directly proportional to the spacecraft's kinetic energy. In addition, the stationary generating function comes out to be negative of the stationary value function of the HJB equation, as shown in Ref. [23]. Moreover, the coefficient variation of the generating function is the opposite to that of the value function. The coefficients of the generating function become stationary as time progresses, while the coefficients of the value function exhibit transient behavior. This clearly demonstrates that the HJB equation is solved backward in time, unlike the HJ equation, which is solved in forward time.

Moreover, the state variation for 100 random initial states to the final state is shown in Table III. It can be noticed that these initial states are brought to zero in the chosen final time, thereby achieving detumbling of the spacecraft. Finally, the RMS error over time of the state, control, and cost function between the RS solution and the bvp4c solution is shown in Table III.

V. CONCLUSIONS

This paper proposes a semi-analytic method for solving a class of optimal control problems with fixed final state in a given final time by leveraging the underlying Hamiltonian structure. The Hamilton-Jacobi equation is utilized to solve the TPBVP by mapping the state and co-state at any time instant to the initial time by exploiting the type-2 generating function. In addition, the CUT-based sparse-collocation method provides a minimal number of collocation points and coefficients in accurately approximating the generating function. The utility of the developed method is demonstrated using two examples, the van der Pol oscillator and the spin stabilization problem.

VI. ACKNOLWEDGEMENT

This material is based upon work supported through United States Air Force Office of Scientific Research (AFOSR) grant FA9550-20-1-0176.

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