2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

On the monotonicity of frequency response gains

Christian Grussler and Thiago B. Burghi

***Abstract*— Linear time-invariant single-input-single-output systems with nonnegative impulse responses, commonly called externally positive systems, carry well-known monotonicity properties such as: (i) the static gain equals the H_∞-norm (peak of the Bode magnitude diagram), (ii) monotone inputs are mapped to monotone outputs, (iii) the transfer function is totally monotone on the positive reals. In this paper, we complement these properties by proving monotonicity properties of the frequency response gain with the help of variation diminishing theory. While our results give new insights into proving monotonicity properties of the gains of positive systems, they are not limited to such systems, and extend to systems that preserve the periodic monotonicity of their inputs. In particular, our results also provide an interesting sufficient condition for positive dominance.**

I. Introduction

In the past couple of decades, linear time-invariant (LTI) systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t) \\ y(t) &= cx(t) \end{aligned} \tag{1}$$

that map positive inputs to positive outputs have become an essential part of the control engineering toolkit (see, e.g., [1–3]). Such systems, which can be characterized by a nonnegative impulse response, possess a number of interesting properties that have been exploited for the development of tractable analysis methods and scalable design algorithms (see, e.g., [2, 3]). An important property of these systems is that their dominant pole has to be real. As a result, parallel, series or positive feedback interconnections of such systems behave closely to a low-dimensional first-order lag. As demonstrated in [4], very low-order approximations are often of sufficient quality. This property can also be observed in the frequency domain: it is well known that the H_∞-norm is attained at the DC gain, that is, the peak of its Bode magnitude plot occurs at zero frequency, a property called positive dominance [5].

Based on these observations, it seems natural that the frequency response gains of positive systems are often monotone in practice. Unfortunately, there is no general characterization of this property to this day. The closest related property is the characterization of positivity via the total monotonicity of the transfer function $G(s) = C(sI-A)^{-1}B$, i.e., $(-1)^k \frac{d^k G(s)}{ds} \geq 0$ for all $s \geq 0$. In this paper, we want to focus on another characterization: the preservation of monotonicity from input to output, i.e, a monotonically increasing input u is mapped by a positive system to a monotonically increasing output y. Concretely, we will investigate the periodic analogue of this property, i.e., when the system is required to preserve the *periodic* monotonicity of their inputs. Roughly, a periodically monotone signal is a periodic signal whose graph within a period can be decomposed into a monotonically increasing and a monotonically decreasing part [6]. Such systems are then called *periodic monotonicity–preserving* (PMP).

We first show that the frequency response of a PMP system displays a generalized form of the positive dominance property of positive systems. More specifically, a system that preserves the monotonicity of a T-periodic input is shown to amplify particular harmonics of such a signal in a dominant fashion: $|G(i\omega)| \geq |G(ik\omega)|$ for $\omega = 2\pi/T$ for all $k \in \mathbb{N}_{\geq 1}$. In particular, this gives the sequential property that $|G(i2^k\omega)| \geq |G(i2^{k+1}\omega)|$ for all $k \in \mathbb{N}_0$ if the system is PMP for all periods $T = \frac{\pi}{2^{k-1}\omega}$, $k \in \mathbb{Z}$.

More generally, we show that this property is also preserved if the auto-correlation function of the impulse of (1) gives rise to a periodized PMP convolution kernel via periodic summation (see, e.g., [7]). To verify this property, we derive several tractable conditions. In particular, we discuss conditions when the periodized auto-correlation is PMP for all possible periods.

Our results constitute a relaxation of related results in [8] concerning systems with a so-called cyclic variation diminishing (CVD) convolution transformation. Unfortunately, it is unknown how to verify the CVD property in most instances. As a special case of this theory, however, we show that the nonnegativity of the auto-correlation is a sufficient condition for positive dominance, which is computationally tractable through, e.g., [9–11]. Thus, our results support synthesis efforts such as [5].

II. Preliminaries

A. Periodic monotonicity and variation diminishment

At the heart of the theory used in this paper lies the concept of the variation of a signal. The two most relevant types of variation for our purposes are defined next:

Definition 1 (variation)**.** Let $u = (u_1, u_2, \ldots, u_n)$ be a vector of real numbers. We denote by $S(u)$ the number of sign changes in the sequence $u_1, u_2, \ldots, u_n$ after deleting all zero terms, with the special convention that $S(0) = -1$.

Definition 2 (cyclic variation)**.** For a bounded T-periodic function $u : \mathbb{R} \to \mathbb{R}$, the cyclic variation $S_c(u)$ is the

The first author is a Jane and Larry Sherman Fellow and was supported by the Israel Science Foundation (grant no.2406/22). The second author was supported by the Kavli Foundation (award number G118028).

C. Grussler is with the Faculty of Mechanical Engineering, Technion – Israel Institute of Technology, Haifa, Israel cgrussler@technion.ac.il

T. B. Burghi is with the Department of Engineering, University of Cambridge, Cambridge, United Kingdom tbb29@cam.ac.uk

979-8-3503-0123-6/23/$31.00 ©2023 IEEE

number of sign changes of u over the course of one period T. Formally,

$$S_c(u) := \sup_{\substack{t_1<t_2<\cdots<t_n<t_1+T \\ n\in\mathbb{N}}} S(u(t_1), u(t_2), \ldots, u(t_n), u(t_1)).$$

The idea of periodic monotonicity mentioned in the introduction can now be defined.

Definition 3 (periodic monotonicity)**.** A T-periodic function $u : \mathbb{R} \to \mathbb{R}$ is called T-periodically monotone ($u \in \text{PM}_T$) if and only if for any $\gamma \in \mathbb{R}$, we have $S_c(u - \gamma\mathbf{1}) \leq 2$, where $\mathbf{1}$ is the constant function $\mathbf{1}(t) \equiv 1$.

Roughly speaking, this implies that $u(t)$ crosses any constant function $\gamma\mathbf{1}$ at most twice within a period. T-periodic monotonicity is equivalent to the existence of numbers $t_1 \leq t_2 \leq t_1 + T$ such that u is nonincreasing for $t_1 \leq t \leq t_2$ and nondecreasing for $t_2 \leq t \leq t_1 + T$ [6].

The T-periodic monotonicity of a signal $u(t)$ may be preserved by certain operators. In this work, we are interested in the cyclic convolution

$$y(t) = (g * u)(t) := \frac{1}{T}\int_T g(t-\tau)u(\tau)d\tau,\ t \in \mathbb{R} \tag{2}$$

for T-periodic convolution kernels $g : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ such that $\int_0^T |g(t)|dt < \infty$ ($g \in \mathcal{L}_1(T)$), and bounded T-periodic inputs u.

Definition 4 (periodic monotonicity preservation)**.** The kernel g is said to be T-periodic monotonicity preserving ($g \in \text{PMP}_T$) if $y \in \text{PM}_T$ for all bounded $u \in \text{PM}_T$.

For continuously differentiable kernels, the following lemma provides a way to check the PMP_T property:

Lemma 1. *Let $g \in \mathcal{L}_1(T)$ be continously differentiable. Then, $g \in PMP_T$ if and only if all non-vanishing determinants*

$$\det\begin{pmatrix} 1 & g(t_1) & \dot{g}(t_1) \\ 1 & g(t_2) & \dot{g}(t_2) \\ 1 & g(t_3) & \dot{g}(t_3) \end{pmatrix}, \quad t_1 < t_2 < t_3 < t_1 + T$$

have the same sign.

Lemma 1 is essentially [6, Lemma 5], which states that $g \in \text{PMP}_T$ if and only if $g + i\dot{g}$ is a so-called convex curve. But it is shown in [8, p. 478] that $g + i\dot{g}$ is a convex curve if and only if the determinant condition above holds.

Lemma 1 may be difficult to use beyond simple cases. A somewhat more practical test for PMP_T is provided by the main result of [6]:

Proposition 1. *Let $g \in \mathcal{L}_1(T)$. Then $g \in PMP_T$ if and only if $g = \tilde{g}$ a.e., where $\tilde{g}$ is bounded and satisfies the following conditions:*

(i) $\tilde{g} \in PM_T$.

(ii) $\tilde{g}$ is continuous except for at most two points in a period. If $\sup_\mathbb{R} \tilde{g} = g_s$, $\inf_\mathbb{R} \tilde{g} = g_i$, and $\tilde{g}$ is not continuous at $t = t_0$, then

$$\left|\lim_{t\to t_0^+} \tilde{g}(t) - \lim_{t\to t_0^-} \tilde{g}(t)\right| = g_s - g_i$$

(iii) $\tilde{g}$ is continuously differentiable in each interval inside which $\tilde{g}$ neither assumes nor approaches g_s or g_i. Furthermore, $\log|\dot{\tilde{g}}|$ is concave in those intervals.

We also have the following useful facts:

Lemma 2. *For $g, u \in PMP_T$, the following hold:*

(i) $g * u \in PMP_T$.

(ii) $g_-(t) := g(-t)$ is in PMP_T.

(iii) $\forall \alpha \in \mathbb{R} : \alpha g \in PMP_T$.

Proof: For part i, notice that for any bounded $v \in \text{PM}_T$ we have that $y * v = g * (u * v)$ is PM_T. For parts ii and iii, notice that scaling or flipping the sign of $g(t)$ does not change any of the conditions of Proposition 1.

A closely related property to PMP is that of *cyclic variation diminishment* [8, p. 259]:

Definition 5. The convolution kernel $g(t)$ in (2) is said to be T-cyclic variation diminishing of order $2k$, or $\text{CVD}_{2k}(T)$, if $S_c(y) \leq S_c(u)$ for all T-periodic u such that $S_c(u) \leq 2k$.

If a kernel is $\text{CVD}_2(T)$, then it is PMP_T. Indeed, since for any $\gamma \in \mathbb{R}$ and $u \in \text{PM}_T$ it holds that $y - \gamma\mathbf{1} = g * (u - \beta\mathbf{1})$ some $\beta \in \mathbb{R}$, we have that $S_c(y - \gamma\mathbf{1}) \leq 2$ if g is $\text{CVD}_2(T)$. A counterexample for the converse inclusion is shown in [6].

In order to make these tools tractable, we also need the following algebraic notions. Let the i-th elements of the r-tuples in

$$\mathcal{I}_{n,r} := \{v = \{v_1, \ldots, v_r\} \subset \mathbb{N} : 1 \leq v_1 < v_2 < \cdots < v_r \leq n\}$$

be defined by *lexicographic ordering*. Then, the (i,j)-th entry of the so-called *r-th multiplicative compound matrix* $X_{[r]} \in \mathbb{R}^{\binom{n}{r}\times\binom{m}{r}}$ of $X \in \mathbb{R}^{n\times m}$ is defined by $\det(X[I, J])$, where I is the i-th and J is the j-th element in $\mathcal{I}_{n,r}$ and $\mathcal{I}_{m,r}$, respectively. For example, if $X \in \mathbb{R}^{3\times3}$, then $X_{[2]}$ reads

$$\begin{pmatrix} \det X[\{1,2\},\{1,2\}] & \det X[\{1,2\},\{1,3\}] & \det X[\{1,2\},\{2,3\}] \\ \det X[\{1,3\},\{1,2\}] & \det X[\{1,3\},\{1,3\}] & \det X[\{1,3\},\{2,3\}] \\ \det X[\{2,3\},\{1,2\}] & \det X[\{2,3\},\{1,3\}] & \det X[\{2,3\},\{2,3\}] \end{pmatrix}$$

The multiplicative compound of e^{At} can be expressed in terms of the so-called *additive compound matrix* $A^{[j]}$ [12, Section 1]: $A^{[j]} := \log(\exp(A)_{[j]}) = \frac{d}{dh} e^{Ah}_{[j]}\Big|_{h=0}$, which satisfies $(e^{Ah})_{[j]} = e^{A^{[j]}h}$.

B. LTI systems with periodic inputs

We apply the mathematical machinery of periodic monotonicity theory presented in the previous section to the cyclic convolution with *periodized system functions*, in particular the periodized impulse response and auto-correlation functions. We consider LTI systems (1) with square-integrable impulse responses $g \in \mathcal{L}_1$, bounded inputs u, and output

$$y(t) = (g * u)(t) \tag{3}$$

(with some abuse of notation, we also use $*$ to denote the non-cyclic convolution). Assume henceforth that the input $u(t)$ is T-periodic. Then the cyclic convolution and the periodized impulse response

$$g^T(t) := \sum_{k=-\infty}^{\infty} g(t - kT)$$

appear naturally in (3), since

$$\begin{aligned}(g*u)(t) &= \int_{-\infty}^{\infty} g(t-\tau)u(\tau)d\tau \\ &= \sum_{k=-\infty}^{\infty}\int_{\frac{(2k-1)T}{2}}^{\frac{(2k+1)T}{2}} g(t-\tau)u(\tau)d\tau \\ &= \int_{-\frac{T}{2}}^{\frac{T}{2}}\sum_{k=-\infty}^{\infty} g(t-kT-\tau)u(\tau+kT)d\tau \\ &= \int_{-\frac{T}{2}}^{\frac{T}{2}} g^T(t-\tau)u(\tau)d\tau, \end{aligned} \tag{4}$$

where we have used the T-periodicity of $u(t)$. Hence the cyclic convolution can be used with $g^T(t)$ to determine the periodic output $y(t)$.

The *auto-correlation function* of (3) is defined as

$$R_{gg}(t) := \int_{-\infty}^{\infty} g(t+\tau)g(\tau)d\tau = \int_{-\infty}^{\infty} g(\tau-t)g(\tau)d\tau \tag{5}$$

for $t \in \mathbb{R}$. Its convolution with $u(t) = \sin(\omega t)$ yields

$$(R_{gg}*u)(t) = \int_{-\infty}^{\infty} R_{gg}(t-\tau)u(\tau)d\tau = |G(i\omega)|^2\sin(\omega t), \tag{6}$$

because its Fourier transform computes as

$$\mathcal{F}\{R_{gg}\}(i\omega) = \int_{-\infty}^{\infty} R_{gg}(t)e^{-i\omega t}dt = |G(i\omega)|^2. \tag{7}$$

Using the same steps as in those in (4), by defining the periodized autocorrelation function

$$R_{gg}^T(t) := \sum_{k=-\infty}^{\infty} R_{gg}(t-kT)$$

one can express convolutions such as (6) as

$$(R_{gg}*u)(t) = \int_{-\frac{T}{2}}^{\frac{T}{2}} R_{gg}^T(t-\tau)u(\tau)d\tau, \tag{8}$$

whenever $u(t)$ is periodic with period $T = \frac{2\pi}{\omega}$. Furthermore, following similar steps to those in (4), it is simple to obtain the relation

$$R_{gg}^T(t) = \int_{-\frac{T}{2}}^{\frac{T}{2}} g^T(t+\tau)g^T(\tau)d\tau \tag{9}$$

between the two periodized system functions. The auto-correlation function of the (causal) impulse response of (1) can be derived from (5). Since $R_{gg}(t) = R_{gg}(-t)$, we have

$$\begin{aligned} R_{gg}(t) &= ce^{A|t|}\int_0^{\infty} e^{A\tau}bb^{\mathsf{T}}e^{A^{\mathsf{T}}\tau}d\tau c^{\mathsf{T}} = ce^{A|t|}Pc^{\mathsf{T}} && (10) \\ &= b^{\mathsf{T}}e^{A^{\mathsf{T}}|t|}\int_0^{\infty} e^{A^{\mathsf{T}}\tau}c^{\mathsf{T}}ce^{A\tau}d\tau b = b^{\mathsf{T}}Qe^{A|t|}b && (11)\end{aligned}$$

where P and Q are the (symmetric) controllability and observability Gramians, respectively.

Note that $R_{gg}^T(t)$ is indeed a periodic function of period T, which on $[-\frac{T}{2}, 0]$ can be evaluated as

$$R_{gg}^T(t) = c\left(\sum_{k=0}^{\infty}\left[e^{A(kT-t)} + e^{A(t+kT)}\right] - e^{At}\right)Pc^{\mathsf{T}}$$

and on $[0, \frac{T}{2}]$ as

$$R_{gg}^T(t) = c\left(\sum_{k=0}^{\infty}\left[e^{A(kT-t)} + e^{A(t+kT)}\right] - e^{-At}\right)Pc^{\mathsf{T}}.$$

Using the von Neumann series $\sum_{k=0}^{\infty} e^{AkT} = (I - e^{AT})^{-1}$

$$R_{gg}^T(t) = c\left[(e^{At} + e^{-At})(I - e^{AT})^{-1} - e^{-A|t|}\right]Pc^{\mathsf{T}} \tag{12}$$

on $[-\frac{T}{2}, \frac{T}{2}]$.

C. Positive systems

This work will in particular deal with externally positive systems:

Definition 6. An LTI system is said to be *externally positive* if its impulse response $g(t)$ is nonnegative for all $t \geq 0$.

Such systems are known to satisfy the following property (see e.g. [5]):

Definition 7. An LTI system is said to be *positively dominated* if its frequency response satisfies $G(0) \geq G(i\omega)$ for all $\omega \geq 0$.

III. Main Results

The main objective of this paper is to exploit the properties of $g(t)$ and $R_{gg}(t)$ in order to characterize monotonicity properties of the system gain $|G(i\omega)|$. In this section, we often omit the period T from the notation of the PM and PMP properties, as they will be clear from context.

A. PMP and the frequency response gain

We begin with a technical lemma:

Lemma 3. *For $\omega > 0$ and $k \in \mathbb{N}_{>0}$, there exists an $a > 0$ such that $u = \sin(\omega t) - a\sin(k\omega t)$ is PMP$_{2\pi/\omega}$.*

Proof: It suffices to show the claim for $\omega = 1$ as scaling of t leaves the claim invariant. By Lemma 1, we need to show then that there exists an $a > 0$ such that

$$M := \det\begin{pmatrix} 1 & \sin(t_1) - a\sin(kt_1) & \cos(t_1) - ak\cos(kt_1) \\ 1 & \sin(t_2) - a\sin(kt_2) & \cos(t_2) - ak\cos(kt_2) \\ 1 & \sin(t_3) - a\sin(kt_3) & \cos(t_3) - ak\cos(kt_3)\end{pmatrix}$$

has the same non-zero sign for all $t_1 < t_2 < t_3 < t_1 + 2\pi$. To verify this, we note that

$$\begin{aligned} M &= \begin{pmatrix} 1 & \sin(t_1) & \cos(t_1) \\ 1 & \sin(t_2) & \cos(t_2) \\ 1 & \sin(t_3) & \cos(t_3)\end{pmatrix} + \mathcal{O}(a) \\ &= 4\sin\left(\frac{t_1 - t_2}{2}\right)\sin\left(\frac{t_2 - t_3}{2}\right)\sin\left(\frac{t_1 - t_3}{2}\right) + \mathcal{O}(a)\end{aligned}$$

with

$$0 > \frac{t_1 - t_2}{2} > -\pi, \quad 0 > \frac{t_1 - t_3}{2} > -\pi,$$
$$0 > \frac{t_2 - t_3}{2} > \frac{t_1 - t_3}{2} > -\pi.$$

Thus, $M < 0$ for sufficiently small $a > 0$.

Our first main result is the following:

Theorem 1. *Let $\omega > 0$ be such that $R_{gg}^{\frac{2\pi}{\omega}}$ is PMP. Then, $|G(i\omega)| \geq |G(ik\omega)|$ for all $k \in \mathbb{N}_{\geq 1}$.*

Proof: By Lemma 3, there exists an $a > 0$ such that $u_a(t) = \sin(\omega t) - a\sin(k\omega t)$ is PMP. Let $\bar{a}$ be the largest such a. It is easy to see that $\bar{a} < \infty$, because for sufficiently large a, $u_a(t)$ will be dominated by $\sin(k\omega t)$ and, then u is no longer PM. It follows from (6) that

$$(R_{gg} * u_{\bar{a}})(t) = |G(i\omega)|^2 \sin(\omega t) - \bar{a}|G(ik\omega)|^2 \sin(k\omega t).$$

Since $R_{gg}^{\frac{2\pi}{\omega}}$ is PMP by assumption, and $u_{\bar{a}}$ is PMP by Lemma 3, then Lemma 2 implies that

$$u_b = |G(i\omega)|^{-2}(R_{gg} * u_{\bar{a}}), \quad b = \bar{a}\frac{|G(ik\omega)|^2}{|G(i\omega)|^2}$$

is PMP. But since we have chosen $\bar{a}$ to be the largest a such that u_a is PMP, u_b can only be PMP if $\frac{|G(ik\omega)|}{|G(i\omega)|} \leq 1$.

If R_{gg}^T is PMP for all $T = \frac{\pi}{2^{k-1}\omega} > 0$, $k \in \mathbb{Z}$, then Theorem 1 indicates a monotonic gain decrease:

$$|G(i2^k\omega)| \geq |G(i2^{k+1}\omega)|, \ k \in \mathbb{Z}.$$

It can be shown that the above also applies to LTI systems that preserve the PM_T property of their inputs:

Proposition 2. *Let g^T be PMP_T. Then, R_{gg}^T is PMP_T.*

Proof: Analogously to what is done in [8, p. 264], we use (9) to write the cyclic convolution (8) as the composition of two cyclic convolutions involving g^T. Namely, a few simple manipulations show that $R_{gg}^T * u = (g^T * g_-^T) * u$, where $g_-^T(t) := g^T(-t)$. By Lemma 2, it follows that both $g_-^T \in \text{PMP}_T$ and $R_{gg}^T \in \text{PMP}_T$.

Unfortunately, at present it is not known whether the converse of Proposition 2 holds. For this reason, in the rest of the section we focus on analyzing the periodized autocorrelation, which provides the most general way of obtaining the sampled frequency response monotonicity property above.

Next, we would like to provide conditions that allow us to apply Theorem 1. We begin by the PM condition on $R_{gg}^{\frac{2\pi}{\omega}}$.

Lemma 4. *The following relationships between R_{gg}^T and R_{gg} hold:*

i. $R_{gg}^T(t)$ is convex on $[0, T)$ for all $T > 0$ if and only if R_{gg} is convex on $\mathbb{R}_{\geq 0}$.
ii. $R_{gg}^T(t)$ is log-concave on $[0, T)$ for all $T > 0$ if and only if R_{gg} is log-concave on $\mathbb{R}_{\geq 0}$.

In particular, $R_{gg}^T(t)$ is PM in both cases.

Proof: We only need to show the sufficiency part, as the properties of R_{gg} follow from those of R_{gg}^T by taking $T \to \infty$.

In the first claim, if R_{gg} is convex on $\mathbb{R}_{\geq 0}$, then $\sum_{k=0}^{\infty} R_{gg}(t + kT)$ is convex on $[0, T)$ since it is an infinite sum of convex functions on that interval [13, Section 3.2.1]. Since R_{gg} is even, it is convex on $\mathbb{R}_{\leq 0}$, and hence $\sum_{k=1}^{\infty} R_{gg}(t - kT)$ is also convex on $[0, T)$. Hence $R_{gg}^T(t) = \sum_{k=1}^{\infty} R_{gg}(t - kT) + \sum_{k=0}^{\infty} R_{gg}(t + kT)$ is convex on $[0, T)$.

For the second claim, assume that R_{gg} is log-concave on $\mathbb{R}_{\geq 0}$. Then by Lemma 7 in appendix A we have

$$\det\begin{pmatrix} R_{gg}(t + iT) & R_{gg}(t - h + jT) \\ R_{gg}(t + h + iT) & R_{gg}(t + jT) \end{pmatrix} \geq 0, i \leq j$$

and

$$\det\begin{pmatrix} R_{gg}(t + jT) & R_{gg}(t - h + jT) \\ R_{gg}(t + h + iT) & R_{gg}(t + iT) \end{pmatrix} \geq 0, i > j$$

for all $h > 0$ and t. Thus,

$$\det\begin{pmatrix} \sum_{k=-\infty}^{\infty} R_{gg}(t + kT) & \sum_{k=-\infty}^{\infty} R_{gg}(t - h + kT) \\ \sum_{k=-\infty}^{\infty} R_{gg}(t + h + kT) & \sum_{k=-\infty}^{\infty} R_{gg}(t + kT) \end{pmatrix}$$
$$= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \det\begin{pmatrix} R_{gg}(t + iT) & R_{gg}(t - h + jT) \\ R_{gg}(t + h + iT) & R_{gg}(t + jT) \end{pmatrix}$$

is nonnegative for all $h > 0$ and $t + h \in [0, T)$. This is equivalent to R_{gg}^T being log-concave on $[0, T)$ by Lemma 7 in appendix A.

In particular, as convex and log-concave functions are unimodal, $R_{gg}^T(t)$ is PM for both cases.

Note that while both cases of Lemma 4 deal with a unimodal R_{gg}, unimodality is in general not sufficient to claim PM of R_{gg}^T.

Analogously to the second claim in Lemma 4, the following relationship for $\dot{R}_{gg}^T$ will allow us to verify the third item in Proposition 1.

Lemma 5. *$|\dot{R}_{gg}^T|$ is log-concave on $(0, T)$ for all $T > 0$ if and only if $|\dot{R}_{gg}|$ is log-concave on $\mathbb{R}_{>0}$.*

Proof: The necessity follows from taking $T \to \infty$. To see the sufficiency, first notice that if $|\dot{R}_{gg}|$ is log-concave on $\mathbb{R}_{>0}$, then $\dot{R}_{gg}$ cannot change sign, since in that case $|\dot{R}_{gg}|$ would not even be unimodal on $\mathbb{R}_{>0}$. Hence $\dot{R}_{gg} \geq 0$ or $\dot{R}_{gg} \leq 0$ for all $t \geq 0$. But since $R_{gg}(0) = \|g\|_{\mathcal{L}_2}^2 > 0$, it is the latter that must hold. Hence, the log-concavity of $|\dot{R}_{gg}|$ on $\mathbb{R}_{>0}$ is equivalent to log-concavity of $-\dot{R}_{gg}$ on that interval. Hence,

$$\det\begin{pmatrix} \dot{R}_{gg}(t + iT) & \dot{R}_{gg}(t - h + jT) \\ \dot{R}_{gg}(t + h + iT) & \dot{R}_{gg}(t + jT) \end{pmatrix} \geq 0, i \leq j$$

and

$$\det\begin{pmatrix} \dot{R}_{gg}(t + jT) & \dot{R}_{gg}(t - h + jT) \\ \dot{R}_{gg}(t + h + iT) & \dot{R}_{gg}(t + iT) \end{pmatrix} \geq 0, i > j$$

for all $h > 0$, $t + h \in (0, T)$ and $i, j \in \mathbb{N}_0$. Further, these inequalities are also true for $-i, -j \in \mathbb{N}$ by the evenness of R_{gg}. Analogously to the proof of the second claim in Lemma 4, it follows then that $|\dot{R}_{gg}^T|$ is log-concave on $(0, T)$.

Note that R_{gg}^T is continuous on $[0, T]$ and smooth on $(0, T)$. For the cases of Lemma 4, the evenness of R_{gg}^T then shows that $\sup_t R_{gg}^T(t) = R_{gg}^T(0)$. Thus, in conjunction with

Lemmas 4 and 5, all items of Proposition 1 are fulfilled. This yields our next main result.

Theorem 2. *Let (A, b, c) be such that*

$$\ddot{R}_{gg}^2(t) \geq \dot{R}_{gg}(t)\dddot{R}_{gg}(t) \text{ for } t \in \mathbb{R}_{>0} \tag{13}$$

and either of the following two conditions hold:

i. $\ddot{R}_{gg}(t) \geq 0$ *for all* $t \in \mathbb{R}_{\geq 0}$

ii. $R_{gg}(t) > 0$ *and* $\dot{R}_{gg}^2(t) \geq R_{gg}(t)\ddot{R}_{gg}(t)$ *for all* $t \in \mathbb{R}_{\geq 0}$

Then, R_{gg}^T *is PMP for all* $T > 0$.

In the next section, we will illustrate Theorem 2 by analytical examples and discuss why its conditions are often computationally tractable.

B. Variation diminishment and the frequency response gain

In order to close the gap between the frequencies $2^j\omega$ and $2^{j+1}\omega$, we would require a generalization of the PMP property to signals with $\max_\gamma \mathrm{S}_c(u - p\gamma\mathbf{1}) \leq 2k$, $k > 1$. At the present moment, we are not aware of such a generalization.

An alternative is given by the CVD_{2k} property (see Definition 5). It has been shown by Karlin that a CVD_{2k} kernel satisfies a different type of frequency response gain monotonicity with regards to that of Theorem 1; see [8, Lemma 5.7.2]. Unfortunately, working with CVD_{2k} comes with two major drawbacks:

1) CVD_2 is more restrictive than PMP [6, pp.131-132].
2) Computationally verifiable conditions such as Proposition 1 seem to be unknown at this point.

An interesting special case of CVD_0 is the following sufficient condition for positive domination based on R_{gg}.

Proposition 3. *Let (A, b, c) be such that (A, Pc^T, c) is externally positive, where $AP + PA^\mathsf{T} = -bb^\mathsf{T}$. Then (A, b, c) is positively dominated.*

Proof: For $u(t) = 1 - \sin(\omega t)$, it follows by (6) and (10) that $(R_{gg} * u)(t) = |G(0)|^2 - |G(i\omega)|^2 \sin(\omega t)$. Then, if (A, Pc^T, c) is externally positive, $(R_{gg} * u)(t)$ is nonnegative, because u and R_{gg} are nonnegative. Taking $t = \frac{\pi}{2\omega}$ proves that G is positively dominated.

IV. Verifiability & Examples

The conditions in Theorem 2 can be verified in several ways. First of all, they are equivalent to external positivity of the following associated systems:

- (13) holds if and only if

$$\begin{aligned} 0 &\leq \det \begin{pmatrix} \ddot{R}_{gg}(t) & \dot{R}_{gg}(t) \\ \dddot{R}_{gg}(t) & \ddot{R}_{gg}(t) \end{pmatrix} \\ &= \det \begin{pmatrix} cAe^{At}APc^\mathsf{T} & cAe^{At}Pc^\mathsf{T} \\ cA^2e^{At}APc^\mathsf{T} & cA^2e^{At}Pc^\mathsf{T} \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} cA \\ cA^2 \end{pmatrix}_{[2]}}_{=:\bar{c}} e^{At}_{[2]} \underbrace{\begin{pmatrix} APc^\mathsf{T} & Pc^\mathsf{T} \end{pmatrix}_{[2]}}_{\bar{b}}, \end{aligned}$$

  i.e., $(A^{[2]}, \bar{b}, \bar{c})$ is externally positive.
- Item i in Theorem 2 holds if and only if (A, Pc^T, cA) is externally positive.
- Item ii in Theorem 2 holds if and only if

$$\begin{aligned} 0 &\leq \det \begin{pmatrix} \ddot{R}_{gg}(t) & \dot{R}_{gg}(t) \\ \dddot{R}_{gg}(t) & \ddot{R}_{gg}(t) \end{pmatrix} \\ &= \det \begin{pmatrix} ce^{At}APc^\mathsf{T} & cAe^{At}Pc^\mathsf{T} \\ cAe^{At}APc^\mathsf{T} & cA^2e^{At}Pc^\mathsf{T} \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} c \\ cA \end{pmatrix}_{[2]}}_{=:\tilde{c}} e^{At}_{[2]} \underbrace{\begin{pmatrix} APc^\mathsf{T} & Pc^\mathsf{T} \end{pmatrix}_{[2]}}_{\tilde{b}}, \end{aligned}$$

  and $R_{gg}(t) > 0$ for $t \geq 0$, i.e., $(A^{[2]}, \tilde{b}, \tilde{c})$ is externally positive and (A, Pc^T, c) is strictly externally positive.

This can be checked efficiently by computational tools such as [9–11]. Alternatively, one can also use the following composition rules.

Lemma 6. *Let $g_1, g_2 \in \mathcal{L}_1$ be impulse responses, $h := g_1 * g_2$ and $T > 0$. Then, the following hold:*

i. If $|g_1|$ is convex on $\mathbb{R}_{\geq 0}$, then $R_{g_1g_1}$ is convex on $\mathbb{R}_{\geq 0}$

ii. If $|g_1|$ is log-concave, then $R_{g_1g_1}$ is log-concave.

iii. If $R_{g_1g_1}$ and $R_{g_2g_2}$ are log-concave, then $R_{hh} = R_{g_1g_1} * R_{g_2g_2}$ is log-concave.

iv. If $R_{g_1g_1}$ and $R_{g_2g_2}$ are convex on $\mathbb{R}_{\geq 0}$, then R_{hh} is convex on $\mathbb{R}_{\geq 0}$.

v. If $|g_1|$ and $|\dot{g}_1|$ are log-concave on $\mathbb{R}_{>0}$, then $|\dot{R}_{g_1g_1}|$ is log-concave on $\mathbb{R}_{>0}$.

vi. If $|g_1|$, $|g_2|$, $|\dot{g}_1|$ and $|\dot{g}_1|$ are log-concave on $\mathbb{R}_{>0}$, then $|\dot{R}_{hh}|$ is log-concave on $\mathbb{R}_{>0}$.

Proof: Item i: If $|g_1|$ is convex on $\mathbb{R}_{\geq 0}$, either g_1 or $-g_1$ has to be a nonnegative function due to $\lim_{t\to\infty} g_1(t) = 0$. Hence, $R_{g_1g_1} = R_{|g_1||g_1|}$ is convex as an integral over the product of a convex function and a nonnegative function (see, e.g., [13]).

Item ii: If $|g_1|$ is log-concave and, thus, unimodal, either g_1 or $-g_1$ has to be a nonnegative function. Hence $R_{g_1g_1} = R_{|g_1||g_1|}$ is log-concave as marginals of log-concave functions remain log-concave [14].

Items iii and iv: Since $\mathcal{F}\{R_{g_1g_1} * R_{g_2g_2}\} = |G_1(i\omega)|^2|G_2(i\omega)|^2 = G_1(i\omega)G_2(i\omega)G_1(-i\omega)G_2(-i\omega) = \mathcal{F}\{R_{hh}\}$, it holds that $R_{hh} = R_{g_1g_1} * R_{g_2g_2}$. Thus, the claims follows by the same argument as in the end ofItems i and ii.

Item v: Since $\dot{R}_{g_1g_1}(t) = \int_{-\infty}^{\infty} \dot{g}_1(t+\tau)g_1(\tau)d\tau$, it follows by the same arugments as in Item ii that either $-\dot{R}_{g_1g_1}$ or $\dot{R}_{g_1g_1}$ has to be log-concave.

Item vi: Since $\dot{R}_{hh} = \dot{R}_{g_1g_1} * R_{g_2g_2}$, the claim follows as in Item v.

A. Example: First Order Systems

In the following, we would like illustrate Theorems 1 and 2 and Lemma 6 based on a first order lag, i.e., $g(t) = \beta e^{-\alpha t}$ for all $t \geq 0$ with $\alpha > 0$ and $\beta \in \mathbb{R}$.

Since $|g|$ is log-concave on $\mathbb{R}$ and convex on $\mathbb{R}_{\geq 0}$, the same applies to $R_{gg}(t) = \frac{\beta^2}{2\alpha}e^{-\alpha|t|}$ by Lemma 6. Thus, both items i and ii in Theorem 2 are verified. Further, as $|\dot{g}|$ is log-concave, Lemma 6 also implies that $|\dot{R}_{gg}|$ is log-concave, which verifies (13). Alternatively, one can use Proposition 1 to verify that $g^T = \frac{\beta}{1-e^{-\alpha T}} e^{-\alpha t}$ is PMP, which by Proposition 2 implies that R_{gg} is PMP.

Thus, by Theorems 1 and 2, it holds that $|G(i2^k\omega)| \geq |G(i2^{k+1}\omega)|$ for all $k \in \mathbb{Z}$ and all $\omega > 0$. For first order systems as well as their series interconnection (see Lemma 6), we observe that our result partially recovers the well-known monotonic behaviour of $|G(i\omega)|$. Unfortunately, for sums of first order externally positive systems, R_{gg}^T is not PMP for all $T > 0$.

B. Example: 2-Positive Systems

Systems with a 2-positive Toeplitz operator, i.e., systems that map unimodal inputs to unimodal outputs have been studied, e.g., in [15–17]. By Lemma 7, such systems are characterized by a log-concave g. By Lemma 6 and Theorem 2, in appendix A, such systems are prototypical for systems where R_{gg}^T is PMP for all $T > 0$.

An illustrative example for such systems is the difference of two externally positive first order lags such as

$$g(t) = 3e^{-t} - 2e^{-2t}.$$

Indeed, as

$$\forall t \geq 0 : g(t) > 0,\ \dot{g}^2 - g\ddot{g} = 6e^{-3t} > 0,$$

it follows that g is log-concave. Thus, by Lemma 6 also

$$R_{gg}(t) = \frac{5}{2}e^{-|t|} - \frac{7}{6}e^{-2|t|}$$

is log-concave with

$$\ddot{R}_{gg}^2(t) - \dot{R}_{gg}(t)\dddot{R}_{gg}(t) = \frac{35}{12}e^{-3t} > 0,\ t > 0.$$

Form Theorem 2 it follows then that R_{gg}^T is PMP for all $T > 0$.

C. Example: Positively dominated system

As mentioned earlier, positive dominance, i.e., $|G(0)| \geq |G(i\omega)|$ for all $\omega > 0$, is a well-known characteristic of externally positive systems, which is not exclusive to systems with nonnegative impulse responses. In Proposition 3, we have provided a sufficient condition, which recovers the externally positive case. Next, we would like to give an example that illustrates that Proposition 3 also covers systems with an indefinite impulse response. To this end, consider the impulse response $g(t) = 2e^{-t} - 3e^{-2t}$, which has negative $g(0) = -1$ and is positive for sufficiently large $t > 0$. Since, $R_{gg}(t) = \frac{1}{4}e^{-2|t|} > 0,\ t \geq 0$, it follows as in Proposition 3 that G is positively dominated.

V. Conclusion

In this work, we have derived monotonicity properties of frequency response gains through the framework of periodic monotonicity preservation. We have shown that if the auto-correlation of an impulse response gives rise to a PMP kernel, then a sampled sequence of the frequency response gain is monotonically decreasing. As a consequence, our results have outlined a path for showing that this property holds for many, if not all, log-concave impulse responses. This is interesting as externally positive systems appear to have monotone frequency response gains in practice, but a proof of this property is lacking till this day.

Finally, we would like to note that systems with monotone gain and phase diagrams play a vital role in PID-autotuning. In the future, it would be interesting to see if the PMP property also gives rise to monotone phase shifts.

Appendix

A. Auxiliary Results

The following is a well-known characterization of log-concavity [8, Proposition 7.1.2]:

Lemma 7. *Let $g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a Lebesgue-measurable function. Then g is log-concave on $\mathbb{R}$ if and only if for all real $t_1 < t_2$ and all real $\tau_1 < \tau_2$, we have*

$$\det \begin{bmatrix} g(t_1 - \tau_1) & g(t_1 - \tau_2) \\ g(t_2 - \tau_1) & g(t_2 - \tau_2) \end{bmatrix} \geq 0.$$

References

[1] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*. John Wiley & Sons, 2011.

[2] T. Tanaka and C. Langbort, "The bounded real lemma for internally positive systems and h-infinity structured static state feedback," *IEEE Transactions on Automatic Control*, vol. 56, no. 9, pp. 2218–2223, 2011.

[3] A. Rantzer and M. E. Valcher, "A tutorial on positive systems and large scale control," in *2018 IEEE Conference on Decision and Control (CDC)*, 2018, pp. 3686–3697.

[4] C. Grussler and T. Damm, "A symmetry approach for balanced truncation of positive linear systems," in *51st IEEE Conference on Decision and Control (CDC)*, 2012, pp. 4308–4313.

[5] A. Rantzer, "Optimizing positively dominated systems," in *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*, 2012, pp. 272–277.

[6] S. Ruscheweyh and L. C. Salinas, "On the preservation of periodic monotonicity," *Constructive Approximation*, vol. 8, no. 2, pp. 129–140, Jun. 1992.

[7] M. A. Pinsky, *Introduction to Fourier analysis and wavelets*. American Mathematical Soc., 2008, vol. 102.

[8] S. Karlin, *Total positivity*. Stanford University Press, 1968, vol. 1.

[9] C. Grussler and A. Rantzer, "On second-order cone positive systems," *SIAM Journal on Control and Optimization*, vol. 59, no. 4, pp. 2717–2739, 2021.

[10] H. Taghavian and M. Johansson, "External positivity of discrete-time linear systems: transfer function conditions and output feedback," *IEEE Transactions on Automatic Control*, pp. 1–15, 2023.

[11] L. Benvenuti and L. Farina, "A tutorial on the positive realization problem," *IEEE Transactions on Automatic Control*, vol. 49, no. 5, pp. 651–664, 2004.

[12] J. S. Muldowney, "Compound matrices and ordinary differential equations," *Rocky Mountain J. Math.*, vol. 20, no. 4, pp. 857–872, 12 1990. [Online]. Available: https://doi.org/10.1216/rmjm/1181073047

[13] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

[14] A. Saumard and J. A. Wellner, "Log-concavity and strong log-concavity: a review," *Statistics surveys*, vol. 8, p. 45, 2014.

[15] C. Grussler and R. Sepulchre, "Strongly unimodal systems," in *2019 18th European Control Conference (ECC)*, 2019, pp. 3273–3278.

[16] ——, "Variation diminishing linear time-invariant systems," *Automatica*, vol. 136, p. 109985, 2022.

[17] S. R. Weller and J. H. Martin, "On strongly unimodal third-order siso linear systems with applications to pharmacokinetics," *IFAC-PapersOnLine*, vol. 53, no. 2, pp. 4654–4661, 2020, 21st IFAC World Congress.