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A novel approach to the finite frequency $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection observer design problem

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***Abstract*— In this paper, a fault detection observer design problem for a Linear Time-Invariant (LTI) system is studied. An iterative algorithm is proposed as part of the design process, effectively enabling the system to detect fault signals in a finite frequency range. Both the $\mathcal{H}_-$-index and $\mathcal{H}_\infty$-norm are introduced and combined in an observer-based design to generate a residual signal that is sensitive to the fault signal and insensitive to the disturbances over a specified finite frequency range. In particular, our approach enforces an upper bound on the $\mathcal{H}_\infty$-norm of the disturbances to residual transfer matrix and a lower bound on the $\mathcal{H}_-$-index of the faults to residual transfer matrix to ensure that the system achieves optimal performance in detecting all fault signals while limiting the impact of disturbances on the residual within this frequency range. This approach is achieved through the generalized Kalman-Yakubovich-Popov (gKYP) Lemma and uses the Projection Lemma in a novel way to reformulate the problem as a linear matrix inequality (LMI) optimization problem. To address the challenge of finding the best multiplier from the Projection Lemma, an iterative process is designed to obtain a local optimum. The initial solution of this iterative process can be selected from any existing algorithms, leading to an improved observer since each iteration yields a solution that is at least as effective as the previous one. A numerical example is provided in the last section to illustrate the effectiveness of our approach.**

I. INTRODUCTION

Fault detection is a set of algorithms that allows a designed control system to generate a residual signal [1], which is then used to identify the occurrence of fault signals as they exceed a predefined threshold. Over recent decades, previous research [2], [3] on fault detection, in both theory and applications, has received significant attention, such as multi-objective and finite frequency problems. Notably, for these fault detection problems, observer-based residual system design has emerged as a prevalent approach. With two useful indexes, $\mathcal{H}_-$ and $\mathcal{H}_\infty$, first proposed by [4] and redefined in [1], the designed observer is enabled to maximize the sensitivity of the residual signal to faults while minimizing the effect of disturbances on the residual signal. Algorithms designed with $\mathcal{H}_-$ and $\mathcal{H}_\infty$ principles effectively reframe the fault detection problem as an optimization problem, thereby fostering further research and potential extensions to this field.

Multi-objective fault detection problems, with observers designed using $\mathcal{H}_-/\mathcal{H}_\infty$ principles, have been further investigated through various methods, including co-inner-outer factorization [5], [6], [7], LMI formulation [5], [8], [9], and iterative algorithms [10]. Extensions to more complex systems, such as time-varying systems [8], Takagi-Sugeno fuzzy systems [11], uncertain systems [12], robust systems [13], and polytopic systems [14], provide researchers with insights and directions to continually enhance and refine their research.

Interest in fault detection within a finite frequency range arises because faults and disturbances typically occur in different frequency ranges. Early solutions generally involve using a weighted filter on the full frequency domain to specify the desired frequency range [1]. The $\mathcal{H}_-/\mathcal{H}_\infty$ observer design provides insights, as demonstrated in [15], where the generalized KYP lemma is first utilized, enabling an accurate observer description. However, this formulation generates a set of Bilinear Matrix Inequalities (BMIs), resulting in insufficient conditions. To address this issue, the Projection Lemma is applied, thereby reducing the original problem into LMIs. In the context of recent research, an iterative algorithm, combined with these two previous lemmas, is proposed as an improved method for resolving the $\mathcal{H}_-/\mathcal{H}_\infty$ problem. The result achieved through the iterative process depends on the different multipliers used when applying the projection lemma, as the necessary condition is no longer satisfied, making it challenging to ensure an improved result.

To address the optimality issue, we propose an iterative method to resolve the multi-objective $\mathcal{H}-/\mathcal{H}\infty$ observer design problem within a specified finite frequency range. As opposed to the direct use of the gKYP lemma, a new formulation is introduced, incorporating two constraints that facilitate the updating of the multiplier. Selecting the initial solution from existing approaches, whether for the full frequency or a finite frequency, is the first step of the process. The multiplier is then refined throughout the iterative process to ensure that the optimum result gets approached.

The paper is organized as follows. Section II presents the problem statement and preliminaries. The main results of this research are outlined in Section III, where the new algorithm, along with clear illustrations, is presented. Section IV provides an example to compare the performance of our algorithm with previous results. Concluding remarks are drawn in Section V.

The notation used in this paper is summarized here for convenience. The symbol $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ the set of n dimensional real (column) vectors and $\mathbb{R}^{n\times m}$ the set of all $n \times m$ real matrices. The set of $n \times n$ real symmetric matrices is denoted as $\mathbb{S}^n$ and A^T denotes the transpose of A. The identity matrix with dimension $n \times n$

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is denoted by I_n and the null matrix with dimension $m \times n$ is denoted by $0_{m\times n}$ with the subscripts normally omitted if they can deduced from context. If $A \in \mathbb{S}^n$, we use $A \prec 0$, $A \succ 0$ to denote that A is negative or positive definite, respectively. $\mathcal{H}(X)$ denotes $X + X^T$ for square X.

II. Problem formation and preliminaries

Consider a linear time-invariant (LTI) dynamic system, not necessarily stable [9], subject to disturbances, modeling errors and process, sensor and actuator faults modeled as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + B_f f(t) + B_d d(t), \\ y(t) &= Cx(t) + Du(t) + D_f f(t) + D_d d(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$, $y(t) \in \mathbb{R}^{n_y}$, represent the state, input and output of the system, and $f(t) \in \mathbb{R}^{n_f}$ and $d(t) \in \mathbb{R}^{n_d}$ are the fault and disturbance, respectively, and where the distribution matrices in (1) have appropriate dimensions. We assume that the pair (A, C) is detectable. To detect the fault signal, an error-state observer is used, where a residual signal is constructed to measure the difference between the actual output and the estimated output, which is sensitive to both the fault and disturbance signals. An observer gain matrix $\mathcal{L} \in \mathbb{R}^{n\times n_y}$ is therefore to be designed to discriminate between these two signals:

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) - \mathcal{L}(y(t) - \hat{y}(t)), \\ \hat{y}(t) &= C\hat{x}(t) + Du(t), \\ r(t) &= y(t) - \hat{y}(t), \end{aligned} \tag{2}$$

with $\hat{x}(t) \in \mathbb{R}^n$ and $\hat{y}(t) \in \mathbb{R}^{n_y}$ denoting the estimation vectors of the state and output, respectively, and where $r(t) \in \mathbb{R}^{n_y}$ is the residual signal.

At this point, some necessary definitions are introduced.

***Definition** 2.1:* [15] For the transfer matrix $G(s) = D + C(sI - A)^{-1}B$, the $\mathcal{H}_\infty$-norm specified up to a finite frequency is defined as

$$||G||_\infty^{[0,\bar{\omega}]} := \sup_{\omega\in[0,\bar{\omega}]} \bar{\sigma}(G(j\omega)), \tag{3}$$

where $\bar{\sigma}$ represents the maximum singular value. The $\mathcal{H}_-$-index specified up to a finite frequency is defined as

$$||G||_-^{[0,\bar{\omega}]} := \inf_{\omega\in[0,\bar{\omega}]} \underline{\sigma}(G(j\omega)), \tag{4}$$

where $\underline{\sigma}$ represents the minimum singular value.

Next, set $\tilde{x}(t) := x(t) - \hat{x}(t)$ as the state of the residual system, and combine (1) and (2) to have a representation of the residual dynamics as:

$$\begin{aligned} \dot{\tilde{x}}(t) &= \mathcal{A}^{\mathcal{L}}\tilde{x}(t) + \mathcal{B}_f^{\mathcal{L}} f(t) + \mathcal{B}_d^{\mathcal{L}} d(t), \\ r(t) &= C\tilde{x}(t) + D_f f(t) + D_d d(t), \end{aligned} \tag{5}$$

where

$$\begin{bmatrix} \mathcal{A}^{\mathcal{L}} & \mathcal{B}_f^{\mathcal{L}} & \mathcal{B}_d^{\mathcal{L}} \end{bmatrix} = \begin{bmatrix} A + \mathcal{L}C & B_f + \mathcal{L}D_f & B_d + \mathcal{L}D_d \end{bmatrix}.$$

According to [15], $\mathcal{L}$ is designed to satisfy the following conditions:

$$\begin{aligned} &\|T_{rd}\|_\infty^{[0,\bar{\omega}]} < \gamma \\ &\|T_{rf}\|_-^{[0,\bar{\omega}]} > \beta \\ &\mathcal{A}^{\mathcal{L}} \text{ is stable} \end{aligned} \tag{6}$$

where γ and β denote the maximum $\mathcal{H}_\infty$-norm and the minimum $\mathcal{H}_-$-index, respectively, and where

$$T_{rd}(s) = D_d + C(sI - \mathcal{A}^{\mathcal{L}})^{-1}\mathcal{B}_d^{\mathcal{L}}$$

and

$$T_{rf}(s) = D_f + C(sI - \mathcal{A}^{\mathcal{L}})^{-1}\mathcal{B}_f^{\mathcal{L}}$$

denote the transfer matrices from disturbances to residual and from faults to residual, respectively. The first and second conditions provide the upper and lower bound of the sensitivity to disturbances and faults, respectively, which determine the capability of the system to detect the fault signal effectively. The third condition ensures observer stability.

We will use the following version of the Projection Lemma:

***Lemma** 2.1:* (Projection Lemma [16], [17], [18]) Given $\mathcal{V} \in \mathbb{S}^n$ and $\mathcal{W} \in \mathbb{R}^{n\times m}$, assume that $\mathcal{W}$ has full column rank. Let $\mathcal{W}_\perp \in \mathbb{R}^{n\times(n-m)}$ be an orthogonal completion for $\mathcal{W}$ such that $\begin{bmatrix} \mathcal{W} & \mathcal{W}_\perp \end{bmatrix}$ is nonsingular and $\mathcal{W}^T\mathcal{W}_\perp = 0$. Then the following statements are equivalent:

1) $\mathcal{W}_\perp^T \mathcal{V} \mathcal{W}_\perp \prec 0$.
2) There exists a multiplier $X \in \mathbb{R}^{m\times n}$ such that $\mathcal{V} + \mathcal{H}(\mathcal{W}X) \prec 0$.

We will also use the following two special cases of the generalised KYP Lemma (gKYP):

***Lemma** 2.2:* (generalised KYP Lemma[19], [20]) Let matrices $A \in \mathbb{R}^{n\times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{R}^{n\times n_u}$, $C \in \mathbb{R}^{n_y\times n}$, $D \in \mathbb{R}^{n_y\times n_u}$ and a scalar $\bar{\omega} > 0$ be given and let $G(s) = D + C(sI_n - A)^{-1}B$. Then, the following statements are equivalent.

1) $\|G\|_\infty^{[0,\bar{\omega}]} < \gamma$.
2) There exist $P, Q \in \mathbb{S}^n$ such that $Q \succ 0$ and

$$\begin{aligned} &\begin{bmatrix} I_n & 0 \\ A & B \end{bmatrix}^T \begin{bmatrix} \bar{\omega}^2 Q & P \\ P & -Q \end{bmatrix} \begin{bmatrix} I_n & 0 \\ A & B \end{bmatrix} \\ &+ \begin{bmatrix} 0 & I_{n_u} \\ C & D \end{bmatrix}^T \begin{bmatrix} -\gamma^2 I_{n_u} & 0 \\ 0 & I_{n_y} \end{bmatrix} \begin{bmatrix} 0 & I_{n_u} \\ C & D \end{bmatrix} \prec 0. \end{aligned} \tag{7}$$

***Remark** 1:* Note that the original gKYP Lemma requires P and Q to be Hermitian [19]. However, [21] showed that for the frequency range $[0, \bar{\omega}]$, P and Q can be restricted to be real without loss of generality.

***Corollary** 2.3:* (gKYP Lemma for $\mathcal{H}_-$ index) [15]. Let everything be defined as in Lemma 2.2. Then the following statements are equivalent:

1) $\|G\|_-^{[0,\bar{\omega}]} > \beta$.
2) There exist $P, Q \in \mathbb{S}^n$ satisfying $Q \succ 0$ and

$$\begin{aligned} &\begin{bmatrix} I_n & 0 \\ A & B \end{bmatrix}^T \begin{bmatrix} \bar{\omega}^2 Q & P \\ P & -Q \end{bmatrix} \begin{bmatrix} I_n & 0 \\ A & B \end{bmatrix} \\ &+ \begin{bmatrix} 0 & I_{n_u} \\ C & D \end{bmatrix}^T \begin{bmatrix} \beta^2 I_{n_u} & 0 \\ 0 & -I_{n_y} \end{bmatrix} \begin{bmatrix} 0 & I_{n_u} \\ C & D \end{bmatrix} \prec 0. \end{aligned} \tag{8}$$

By virtue of the gKYP lemma, the fault detection observer design specifications in (6) can now be formulated as nonlinear matrix inequalities (NLMIs).

***Lemma** 2.4:* With everything as defined above, the following statements are equivalent:

1) $||T_{rd}||_{\infty}^{[0,\bar{\omega}]} < \gamma$.
2) There exist $P_d, Q_d \in \mathbb{S}^n$ satisfying $Q_d \succ 0$ and

$$\begin{bmatrix} I_n & 0 \\ \mathcal{A}^{\mathcal{L}} & \mathcal{B}_d^{\mathcal{L}} \end{bmatrix}^T \begin{bmatrix} \bar{\omega}^2 Q_d & P_d \\ P_d & -Q_d \end{bmatrix} \begin{bmatrix} I_n & 0 \\ \mathcal{A}^{\mathcal{L}} & \mathcal{B}_d^{\mathcal{L}} \end{bmatrix} + \begin{bmatrix} 0 & I_{n_d} \\ C & D_d \end{bmatrix}^T \begin{bmatrix} -\gamma^2 I_{n_d} & 0 \\ 0 & I_{n_y} \end{bmatrix} \begin{bmatrix} 0 & I_{n_d} \\ C & D_d \end{bmatrix} \prec 0. \tag{9}$$

3) There exist $P_d, Q_d \in \mathbb{S}^n$ satisfying $Q_d \succ 0$ and

$$T_d + \mathcal{H}(E_d \mathcal{L} F_d^T) - F_d \mathcal{L}^T Q_d \mathcal{L} F_d^T \prec 0, \tag{10}$$

where

$$T_d = \begin{bmatrix} I_n & 0 \\ A & B_d \end{bmatrix}^T \begin{bmatrix} \bar{\omega}^2 Q_d & P_d \\ P_d & -Q_d \end{bmatrix} \begin{bmatrix} I_n & 0 \\ A & B_d \end{bmatrix} + \begin{bmatrix} 0 & I_{n_d} \\ C & D_d \end{bmatrix}^T \begin{bmatrix} -\gamma^2 I_{n_d} & 0 \\ 0 & I_{n_y} \end{bmatrix} \begin{bmatrix} 0 & I_{n_d} \\ C & D_d \end{bmatrix} \in \mathbb{S}^{(n+n_d)},$$

$$F_d = \begin{bmatrix} C^T \\ D_d^T \end{bmatrix} \in \mathbb{R}^{(n+n_d) \times n_y},$$

$$E_d = \begin{bmatrix} I_n & 0 \\ A & B_d \end{bmatrix}^T \begin{bmatrix} \bar{\omega}^2 Q_d & P_d \\ P_d & -Q_d \end{bmatrix} \begin{bmatrix} 0 \\ I_n \end{bmatrix} \in \mathbb{R}^{(n+n_d) \times n}.$$

Proof: The equivalence between statements 1 and 2 follows from the gKYP Lemma by substituting the state error dynamics (5) into Lemma 2.2. It is easy to show by direct substitution that (10) is equivalent to (9). ∎

Following the same approach, the corresponding result for the $\mathcal{H}_-$-index is given next.

***Corollary** 2.5:* With everything as defined above the following statements are equivalent:

1) $||T_{rf}||_{-}^{[0,\bar{\omega}]} > \beta$.
2) There exist $P_f, Q_f \in \mathbb{S}^n$ such that $Q_f \succ 0$ and

$$\begin{bmatrix} I_n & 0 \\ \mathcal{A}^{\mathcal{L}} & \mathcal{B}_f^{\mathcal{L}} \end{bmatrix}^T \begin{bmatrix} \bar{\omega}^2 Q_f & P_f \\ P_f & -Q_f \end{bmatrix} \begin{bmatrix} I_n & 0 \\ \mathcal{A}^{\mathcal{L}} & \mathcal{B}_f^{\mathcal{L}} \end{bmatrix} + \begin{bmatrix} 0 & I_{n_f} \\ C & D_f \end{bmatrix}^T \begin{bmatrix} \beta^2 I_{n_f} & 0 \\ 0 & -I_{n_y} \end{bmatrix} \begin{bmatrix} 0 & I_{n_f} \\ C & D_f \end{bmatrix} \prec 0. \tag{11}$$

3) There exist $P_f, Q_f \in \mathbb{S}^n$ such that $Q_f \succ 0$ and

$$T_f + \mathcal{H}(E_f \mathcal{L} F_f^T) - F_f \mathcal{L}^T Q_f \mathcal{L} F_f^T \prec 0, \tag{12}$$

where

$$T_f = \begin{bmatrix} I_n & 0 \\ A & B_f \end{bmatrix}^T \begin{bmatrix} \bar{\omega}^2 Q_f & P_f \\ P_f & -Q_f \end{bmatrix} \begin{bmatrix} I_n & 0 \\ A & B_f \end{bmatrix} + \begin{bmatrix} 0 & I_{n_f} \\ C & D_f \end{bmatrix}^T \begin{bmatrix} \beta^2 I_{n_f} & 0 \\ 0 & -I_{n_y} \end{bmatrix} \begin{bmatrix} 0 & I_{n_f} \\ C & D_f \end{bmatrix} \in \mathbb{S}^{(n+n_f)},$$

$$F_f = \begin{bmatrix} C^T \\ D_f^T \end{bmatrix} \in \mathbb{R}^{(n+n_f) \times n_y},$$

$$E_f = \begin{bmatrix} I_n & 0 \\ A & B_f \end{bmatrix}^T \begin{bmatrix} \bar{\omega}^2 Q_f & P_f \\ P_f & -Q_f \end{bmatrix} \begin{bmatrix} 0 \\ I_n \end{bmatrix} \in \mathbb{R}^{(n+n_f) \times n}.$$

III. Main Result

A. Problem formulation using the Projection Lemma

While the full-frequency fault detection synthesis problem can be converted into LMIs with necessary and sufficient conditions, the finite frequency $\mathcal{H}_-/\mathcal{H}_\infty$ observer design results in nonlinear matrix inequalities. In this section, with the aid of the projection lemma, a novel way is proposed to linearize the design problem.

***Theorem** 3.1:* Let all variables be as defined in Lemma 2.4. Then the following statements are equivalent:

1) $||T_{rd}||_{\infty}^{[0,\bar{\omega}]} < \gamma$,
2) There exist $P_d, Q_d \in \mathbb{S}^n$ and multipliers $Y_d \in \mathbb{R}^{n \times (n+n_d)}$ and $Z_d \in \mathbb{R}^{n \times n}$ satisfying $Q_d \succ 0$ and

$$\begin{bmatrix} T_d & \star \\ E_d^T & -Q_d \end{bmatrix} + \mathcal{H}(\begin{bmatrix} -F_d \mathcal{L}^T \\ I_n \end{bmatrix} \begin{bmatrix} Y_d & Z_d \end{bmatrix}) \prec 0, \tag{13}$$

where $\star$ denotes terms readily deduced from symmetry.

Proof: We prove the equivalence of the statements by showing that (10) and (13) are equivalent using the Projection Lemma. Note that T_d, E_d are linear in the variables while F_d is constant. The inequality in (10) is then rearranged as

$$\overbrace{\begin{bmatrix} I_{n+n_d} & F_d \mathcal{L}^T \end{bmatrix}}^{\mathcal{W}_\perp^T} \overbrace{\begin{bmatrix} T_d & \star \\ E_d^T & -Q_d \end{bmatrix}}^{\mathcal{V}} \overbrace{\begin{bmatrix} I_{n+n_d} \\ \mathcal{L} F_d^T \end{bmatrix}}^{\mathcal{W}_\perp} \prec 0, \tag{14}$$

which has the form $\mathcal{W}_\perp^T \mathcal{V} \mathcal{W}_\perp \prec 0$ with $\mathcal{W}_\perp$ and $\mathcal{V}$ defined in (14). It follows from Lemma 2.1 that (13) is equivalent to (10) since $\mathcal{W}_\perp$ is an orthogonal complement of

$$\mathcal{W} := \begin{bmatrix} -F_d \mathcal{L}^T \\ I_n \end{bmatrix}.$$

∎

Although (13) is still nonlinear, the nonlinearities are in the products $\mathcal{L}^T Y_d$ and $\mathcal{L}^T Z_d$, since F_d is a constant, while the variables P_d, Q_d in T_d, E_d have been separated from the other variables. In the next section, we investigate the choice of the multipliers Y_d and Z_d to recover linearity and provide an initial solution.

B. Linear formulation via restricting the multipliers

To enforce linearity of (13), we restrict the multipliers Y_d and Z_d to have the form

$$\begin{bmatrix} Y_d & Z_d \end{bmatrix} = R \begin{bmatrix} Y_{d_0} & Z_{d_0} \end{bmatrix}, \tag{15}$$

where $R \in \mathbb{R}^{n \times n}$ is a variable and Y_{d_0} and Z_{d_0} are constants. This restriction linearizes the original problem by defining $\mathcal{L}^T R$ as a variable; see (13).

To obtain suitable values of the constants Y_{d_0} and Z_{d_0}, we proceed as follows. Suppose there exists an initial solution to (10), where $P_{d_0}, Q_{d_0} \in \mathbb{S}^n$, $Q_{d_0} \succ 0$ and $\mathcal{L}_0$ are given and where T_{d_0}, E_{d_0} and F_{d_0} are defined appropriately using the definitions of T_d, F_d and E_d in Lemma 2.4, so that

$$T_{d_0} + \mathcal{H}(E_{d_0} \mathcal{L}_0 F_{d_0}^T) - F_{d_0} \mathcal{L}_0^T Q_{d_0} \mathcal{L}_0 F_{d_0}^T \prec 0. \tag{16}$$

The initial solution for our analysis may be selected from either the solution derived from the standard KYP lemma, excluding the frequency constraint, or from the solution derived from the generalized KYP lemma, which incorporates the frequency constraint, using any other approach from the literature. The next step is to propose a choice of Y_{d_0} and Z_{d_0} that ensures (13), with the definitions in (15), has a feasible solution.

Substituting (15) into (13) gives

$$\begin{bmatrix} T_d & \star \\ E_d^T & -Q_d \end{bmatrix} + \mathcal{H}(\begin{bmatrix} -F_d(\mathcal{L}^T R) \\ R \end{bmatrix} \begin{bmatrix} Y_{d_0} & Z_{d_0} \end{bmatrix}) \prec 0, \quad (17)$$

or,

$$\begin{bmatrix} T_d - \mathcal{H}(F_d(R^T\mathcal{L})^T Y_{d_0}) & \star \\ E_d^T + RY_{d_0} - Z_{d_0}^T(R^T\mathcal{L})F_d{}^T & -Q_d + \mathcal{H}(RZ_{d_0}) \end{bmatrix} \prec 0, \quad (18)$$

which is linear in the variables if we define $R^T\mathcal{L}$ as a variable. Denoting the matrix in (18) by $\mathcal{K}$, then effecting the congruence $T^T\mathcal{K}T$ where

$$T = \begin{bmatrix} I_{n+n_d} & 0 \\ \mathcal{L}F_d{}^T & I_n \end{bmatrix},$$

indicates the equivalence between inequality (18) and

$$\begin{bmatrix} T_d + \mathcal{H}(E_d\mathcal{L}F_d^T) - F_d\mathcal{L}^T Q_d \mathcal{L}F_d{}^T & \star \\ E_d^T - Q_d\mathcal{L}F_d{}^T + RZ_{d_0}^T\mathcal{L}F_d{}^T + RY_{d_0} & -Q_d + \mathcal{H}(RZ_0) \end{bmatrix} \prec 0.$$

Note that the $(1,1)$ block is exactly the same as (10). Following the idea in [16] , [22] and [23], we set $R = I_n$, $P_d = P_{d_0}$, $Q_d = Q_{d_0}$, $\mathcal{L} = \mathcal{L}_0$ and $\gamma = \gamma_0$. It is clear that the $(1,1)$ entry is then equivalent to the initial solution of (16). Thus, Y_{d_0} and Z_{d_0} can be selected to make the $(2,1)$ and $(1,2)$ entries equal to zero and to make the $(2,2)$ entry negative definite. In that case, (13), with the definitions in (15), is ensured to have at least one solution which is no worse than the initial result of (16).

Next, set the (2,2) entry to be negative definite (with $R = I_n$, $P_d = P_{d_0}$, $Q_d = Q_{d_0}$, $\mathcal{L} = \mathcal{L}_0$ and $\gamma = \gamma_0$):

$$-Q_{d_0} + \mathcal{H}(Z_{d_0}) \prec 0.$$

This can be satisfied by choosing Z_{d_0} as

$$Z_{d_0} = 0.5Q_{d_0} - I_n, \quad (19)$$

the negative definiteness of the (2,2) entry is therefore enforced. Setting the (2,1) entry to zero with Z_{d_0} given in (19), Y_{d_0} is then given as

$$Y_{d_0} = -(-Q_{d_0}\mathcal{L}_0 F_d{}^T + Z_{d_0}^T\mathcal{L}_0 F_d{}^T + E_{d_0}^T). \quad (20)$$

The results are summarised in the following theorem.

***Theorem** 3.2:* Let all variables be as defined in Theorem 3.1. Suppose that (10) has an initial solution (16). Let Z_{d_0} and Y_{d_0} be as defined in (19) and (20), respectively. Then there exist $P_d, Q_d \in \mathbb{S}^n$, non-singular matrix $R \in \mathbb{R}^{n\times n}$ and $\hat{\mathcal{L}} \in \mathbb{R}^{n\times n_y}$ such that $Q_d \succ 0$ and

$$\begin{bmatrix} T_d - \mathcal{H}(F_d\hat{\mathcal{L}}^T Y_{d_0}) & \star \\ E_d^T + RY_{d_0} - Z_{d_0}^T\hat{\mathcal{L}}F_d{}^T & -Q_d + \mathcal{H}(RZ_{d_0}) \end{bmatrix} \prec 0, \quad (21)$$

with $\hat{\mathcal{L}} = R^T\mathcal{L}$. Furthermore, if (21) is satisfied then (13) (and hence (10)) is satisfied.

***Remark** 2:* Note that the initial solution may represent the most optimal solution identified in prior studies. If no solution is given, methods from [15] could be used to obtain an initial solution. This property allows for the use of iterative algorithms to enhance the solution since it is included in our solution when $R = I_n$. Note also that γ_0 need not be the same as the given γ, that is, the initial solution need not be feasible. The degrees of freedom provided in R may then be used to obtain a feasible solution.

The same approach can be applied to obtain corresponding results for the $\mathcal{H}_-$ index.

***Theorem** 3.3:* With everything as defined in Corollary 2.5, the following statements are equivalent:

1) $||T_{rf}||_-^{[0,\bar{\omega}]} > \beta$.
2) There exist $P_f, Q_f \in \mathbb{S}^n$ and multipliers $Y_f \in \mathbb{R}^{n\times(n+n_f)}$ and $Z_f \in \mathbb{R}^{n\times n}$ such that $Q_f \succ 0$ and

$$\begin{bmatrix} T_f & \star \\ E_f^T & -Q_f \end{bmatrix} + \mathcal{H}(\begin{bmatrix} -F_f\mathcal{L}^T \\ I_n \end{bmatrix} \begin{bmatrix} Y_f & Z_f \end{bmatrix}) \prec 0. \quad (22)$$

Furthermore, suppose there exist $P_{f_0}, Q_{f_0} \in \mathbb{S}^n$ with $Q_{f_0} \succ 0$ such that (12) has an initial solution

$$T_{f_0} + \mathcal{H}(E_{f_0}\mathcal{L}_0 F_{f_0}^T) - F_{f_0}\mathcal{L}_0^T Q_{f_0}\mathcal{L}_0 F_{f_0}^T \prec 0, \quad (23)$$

and let

$$\begin{bmatrix} Y_f & Z_f \end{bmatrix} = R\begin{bmatrix} Y_{f_0} & Z_{f_0} \end{bmatrix},$$

where R is a non-singular matrix variable and

$$\begin{aligned} Z_{f_0} &= 0.5Q_{f_0} - I_n, \\ Y_{f_0} &= -(-Q_{f_0}\mathcal{L}_0 F_f{}^T + Z_{f_0}^T\mathcal{L}_0 F_f{}^T + E_{f_0}^T). \end{aligned} \quad (24)$$

Then there exist $P_f, Q_f \in \mathbb{S}^n$, non-singular matrix $R \in \mathbb{R}^{n\times n}$ and $\hat{\mathcal{L}} \in \mathbb{R}^{n\times n_y}$ such that $Q_f \succ 0$ and

$$\begin{bmatrix} T_f - \mathcal{H}(F_f\hat{\mathcal{L}}^T Y_{f_0}) & \star \\ E_f^T + RY_{f_0} - Z_{f_0}^T\hat{\mathcal{L}}F_f{}^T & -Q_f + \mathcal{H}(RZ_{f_0}) \end{bmatrix} \prec 0, \quad (25)$$

with $\hat{\mathcal{L}} = R^T\mathcal{L}$. Finally, if (25) is satisfied then (22) is satisfied.

Proof: the procedures are similar to Theorem 3.1 and can be carried out by replacing all disturbance-related matrices with fault-related matrices. ■

C. Stability

The standard KYP lemma inherently guarantees closed-loop stability [24]. This is not the case for gKYP lemma due to the additional variable. Nevertheless, stability can be incorporated using the following result.

***Theorem** 3.4:* The following statements are equivalent:

1) System (5) is stable.
2) There exists $Q_s \in \mathbb{S}^n$ such that $Q_s \succ 0$ and

$$\mathcal{H}(Q_s\mathcal{A}^{\mathcal{L}}) \prec 0. \quad (26)$$

3) There exist $Q_s \in \mathbb{S}^n$ and multipliers $Y_s, Z_s \in \mathbb{R}^{n\times n}$ such that $Q_s \succ 0$ and

$$\begin{bmatrix} \mathcal{H}(Q_s A) & \star \\ Q_s & 0 \end{bmatrix} + \mathcal{H}(\begin{bmatrix} -C^T\mathcal{L}^T \\ I_n \end{bmatrix} \begin{bmatrix} Y_s & Z_s \end{bmatrix}) \prec 0. \quad (27)$$

Furthermore, there exists $Q_{s_0} \in \mathbb{S}^n$ with $Q_{s_0} \succ 0$ such that (26) has an initial solution

$$\mathcal{H}(Q_{s_0}A) + \mathcal{H}(Q_{s_0}\mathcal{L}_0 C) \prec 0. \tag{28}$$

Let

$$\begin{bmatrix} Y_s & Z_s \end{bmatrix} = R \begin{bmatrix} Y_{s_0} & Z_{s_0} \end{bmatrix},$$

where R is a matrix variable and

$$Z_{s_0} = -I_n, \qquad Y_{s_0} = \mathcal{L}_0 C - Q_{s_0}. \tag{29}$$

Then there exist $Q_s \in \mathbb{S}^n$, non-singular matrix $R \in \mathbb{R}^{n\times n}$ and $\hat{\mathcal{L}} \in \mathbb{R}^{n\times n_y}$ such that $Q_s \succ 0$ and

$$\begin{bmatrix} \mathcal{H}(A^T Q_s) - \mathcal{H}(C^T \hat{\mathcal{L}}^T Y_{s_0}) & \star \\ Q_s + RY_s - Z_{s_0}^T \hat{\mathcal{L}} C & \mathcal{H}(RZ_{s_0}) \end{bmatrix} \prec 0, \tag{30}$$

where $\hat{\mathcal{L}} = R^T \mathcal{L}$. Finally, if (30) is satisfied then (27) (and hence (26)) is satisfied.

Proof: The equivalence of statements 1 and 2 follows from a straightforward application of Lyapunov stability. The equivalence between statements 2 and 3 can be derived through similar steps of Theorem 3.1 once (26) is arranged in the following form:

$$\overbrace{\begin{bmatrix} I_n & C^T \mathcal{L}^T \end{bmatrix}}^{\mathcal{W}_\perp^T} \overbrace{\begin{bmatrix} \mathcal{H}(Q_s A) & \star \\ Q_s & 0 \end{bmatrix}}^{\mathcal{V}} \overbrace{\begin{bmatrix} I_n \\ \mathcal{L} C \end{bmatrix}}^{\mathcal{W}_\perp} \prec 0.$$

The existence of an initial stabilizing $\mathcal{L}_0$, hence Q_{s_0}, follows from the assumption that the pair (A, C) is detectable. The nonsingularity of R satisfying (30) follows from the (2,2) entry which implies that $R + R^T \succ 0$ since $Z_{s_0} = -I_n$. Finally, the proof of the sufficiency of (30) for (27) follows along the lines of the development just before Theorem 3.2. ∎

It follows that sufficient LMI conditions for (6) are formulated in Theorem 3.2, Theorem 3.3, and Theorem 3.4. In summary, given γ, the fault detection observer gain $\mathcal{L}$ can be determined through the following optimization problem: Given the initial solutions (16), (23) and (28), find

$$\begin{aligned} &max && \beta \\ &s.t. && (21),\ (25),\ (30) \\ &&& P_d, P_f, Q_d, Q_f, Q_s \in \mathbb{S}^n \\ &&& Q_d \succ 0,\ Q_f \succ 0,\ Q_s \succ 0 \\ &&& R \in \mathbb{R}^{n\times n},\ \hat{\mathcal{L}} \in \mathbb{R}^{n\times n_y} \end{aligned} \tag{31}$$

with $\hat{\mathcal{L}} = R^T \mathcal{L}$. It follows from the last part of Theorem 3.4 that R is non-singular, so $\mathcal{L}$ can be recovered from $\hat{\mathcal{L}}$.

Note that we can use the solution of (31) as the new initial solution and this will naturally define an iterative algorithm which can be run until convergence. This approach is summarised in Algorithm 1.

Algorithm 1 General procedure for $\mathcal{H}_\infty/\mathcal{H}_-$ fault detection observer design

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Result: $\mathcal{L},\ \beta$

Step 1: Prepare initial solution

Get initial solutions from Theorems 3.2, 3.3 and 3.4: let $P_{f_0} = P_{f_0}^T,\ Q_{f_0} = Q_{f_0}^T \succ 0,\ \beta_0,\ P_{d_0} = P_{d_0}^T,\ Q_{d_0} = Q_{d_0}^T \succ 0,\ Q_{s_0} = Q_{s_0}^T \succ 0,\ \gamma_0$ and $\mathcal{L}_0$ satisfy (16), (23) and (28). Define $T_{f_0},\ F_{f_0},\ E_{f_0},\ T_{d_0},\ F_{d_0}$ and E_{d_0} from the expressions in Lemma 2.4 and Corollary 2.5. Set a maximum number of iterations i_{max} and a minimum update error tolerance tol_0.

Step 2: Update 1

Evaluate $Z_{f_0},\ Y_{f_0},\ Z_{d_0},\ Y_{d_0},\ Z_{s_0}$ and Y_{s_0} using (24), (20) and (29).

Step 3: Update 2

Given $Z_{f_0},\ Y_{f_0},\ Z_{d_0},\ Y_{d_0},\ Z_{s_0}$ and Y_{s_0}, perform the optimisation in (31), record new $R,\ \mathcal{L},\ \beta$ and update $Z_f,\ Y_f,\ Z_d,\ Y_d$ and Y_s. Define update error $tol = |\beta_{new} - \beta_{old}|$.

Step 4: Stopping condition

Stop if $i > i_{max}$ or $tol < tol_0$ and retrieve $\mathcal{L},\ \beta$. Otherwise, set $P_{f_0} = P_f,\ Q_{f_0} = Q_f,\ Z_{f_0} = Z_f,\ Y_{f_0} = Y_f,\ P_{d_0} = P_d,\ Q_{d_0} = Q_d,\ Z_{d_0} = Z_d,\ Y_{d_0} = Y_d$ and $Y_{s_0} = Y_s$, subsequently update $T_f,\ E_f,\ T_d$ and E_d. **Go to Step 2**.

IV. Simulation

The simulations were implemented in MATLAB R2023b using the CVX package [25].

A well-studied linearized longitudinal dynamic of VTOL aircraft system originating from [26], and subsequently investigated in [15], is used to examine the performance of the proposed approach.

We consider the same design specification as in [15], namely, for the frequency range $[0, \bar{\omega}]$ with $\bar{\omega} = 0.1$, design a fault detection observer $\mathcal{L}$ such that the $\mathcal{H}_-$-index from fault to residual is maximized while the $\mathcal{H}_\infty$-norm from disturbance to residual are required to be smaller than 0.3017. We perform the optimisation (31) subject to γ < 0.3017, taking the observer from [15] as our initial solution. The best value of β is given as β_{opt} = 16.08 and the corresponding $\mathcal{L}$ is given as:

$$\mathcal{L} = \begin{bmatrix} -0.7093 & 1.4133 & 0.0008 & -1.8386 \\ -5.2857 & 0.5997 & -0.0039 & 1.6298 \\ 5.5274 & -11.0383 & -3.3360 & 12.0093 \\ -0.5087 & 1.0183 & -0.0001 & -1.0769 \end{bmatrix}.$$

By forming the closed-loop, it can be verified that the system is stable and the $\mathcal{H}_\infty$-norm constraint is met. It can be seen from the plot in Figure 1 that the $\mathcal{H}_\infty$ constraint is satisfied by our $\mathcal{L}$. In addition, our method increased the $\mathcal{H}_-$-index significantly from 4.17 (as given in [15]) to 16.08 within the specified frequency range. The maximized $\mathcal{H}_-$-index will greatly amplify the fault and therefore differentiate the fault from disturbance. Furthermore, it can be noticed from Figure 1 that our computed β is equal to $\|T_{rf}\|_-^{[0,\bar{\omega}]}$,

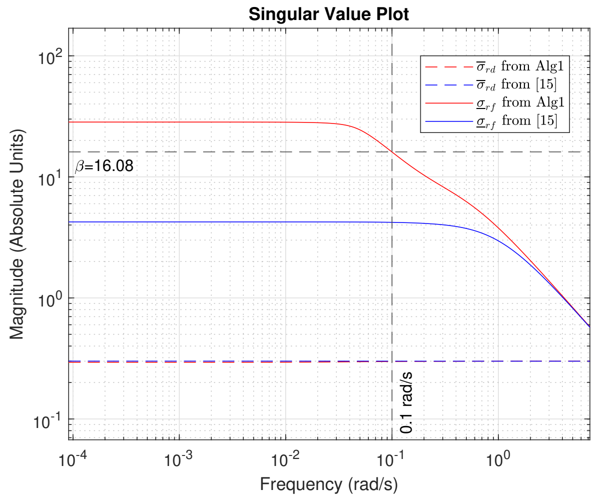

Fig. 1. Plot of the largest singular value for T_{rd} ($\bar{\sigma}_{rd}$) and the smallest singular value of T_{rf} ($\underline{\sigma}_{rf}$) for two designs

indicating the tightness of the solution.

V. Conclusion

In this paper, we presented an innovative design approach to the problem of designing fault detection observers. By employing the generalised KYP and Projection Lemmas in a novel manner, we derived sufficient conditions for the mixed $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection observer design problem over a finite frequency range. Sufficient conditions are derived to convert the synthesis problem into LMIs. We then provided iterative algorithms to optimize the solution towards a local optimum. Additionally, solution algorithms were outlined to encapsulate the contribution. Lastly, a numerical example was presented to demonstrate the effectiveness of the proposed method.

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