2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Output-based Event-holding Control in Presence of Measurement Noise

K.J.A. Scheres, R. Postoyan, D. Nešić, W.P.M.H. Heemels

Abstract— **We present rules to stabilize the origin of a networked system, where data exchanges between the plant and the controller only occur when an output-dependent inequality has been satisfied for a given amount of time. This strategy, called Event-Holding Control (EHC), differs from time-regularized event-triggered control (ETC) techniques, which generate transmissions as soon as a triggering condition is verified and the time elapsed since the last transmission is larger than a given bound. Indeed, the clock involved in EHC is not running continuously after each transmission instant, but only when a criterion is verified. We propose an output-based design of these triggering mechanisms that are robust to additive measurement noise and ensure an input-to-state stability (ISS) property. This EHC scheme naturally has a positive lower bound on the transmission interval. Additionally, we show via an example that, in presence of measurement noise, Zeno-like behavior, where events are generated near the minimum inter-event time consistently, may occur when the system is close to the attractor. We introduce space-regularization to mitigate this issue, resulting in an input-to-state practical stability (ISpS) property rather than ISS.**

I. Introduction

In digital control systems with feedback loops, it is well known that the maximum achievable system performance depends on the rate at which information is exchanged between the plant and the controller. In some cases, this communication rate may be limited due to physical constraints, such as energy usage in battery-powered electronics or communication bandwidth in systems that are physically distributed and thus rely on low bandwidth communication to exchange information. In such cases, Event-triggered Control (ETC), see, *e.g.*, [1], [2], has been proposed as an alternative sampling strategy, whereby the communication times are not determined by a clock, but instead by a certain event occurring in the system. As these events depend on the state of the system, the underlying philosophy is that communication resources are only used when necessary. When designing the triggering mechanisms, one has to be careful to avoid so-called Zeno behavior, referring to the situation where an infinite number of transmissions are generated in finite time, which is clearly undesirable. In fact, it is desirable that the time between two consecutive events is lower-bounded by a positive number, which is often called the *minimum inter-event time* (MIET).

Koen Scheres and Maurice Heemels are with the Department of Mechanical Engineering, Eindhoven University of Technology, The Netherlands. k.j.a.scheres@tue.nl, m.heemels@tue.nl

Romain Postoyan is with the Université de Lorraine, CNRS, CRAN, F-54000 Nancy, France. romain.postoyan@univ-lorraine.fr His work is supported by the ANR under grant HANDY ANR-18-CE40-0010, and the IRP ARS from CNRS.

Dragan Nešić is with the Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville, 3010, Victoria, Australia. dnesic@unimelb.edu.au

Many ETC designs are available in the literature with these properties, although, surprisingly, few dealing formally with the presence of measurement noise [3]–[8]. It is known that Zeno behavior may occur for many existing triggering mechanisms, as the inter-event times are not robust to measurement errors. A famous example is the relative triggering mechanisms [9] for which Zeno behavior is excluded in the nominal case, but, it may occur when (arbitrary small) measurement noise is introduced, as shown in [10], [11]. To address this issue, in the previous works [7], [8], *space-regularization* was introduced, whereby the triggering condition is regularized using a tuning parameter. By selecting this tuning parameter sufficiently large, a MIET is guaranteed even in presence of noise, and the resulting stability property is an Input-to-State practical Stability (ISpS) property instead of an Input-to-State Stability (ISS) one.

It is also possible, as was shown in [7], [8], to design relative triggering mechanisms that do not exhibit Zeno in the typical sense by exploiting *time-regularization* [12] under extra conditions, whereby a MIET is enforced by design. However, in presence of measurement noise, the system may display *Zeno-like* [13] behavior when close to the attractor, whereby the triggering times are consistently close to or at the MIET, thereby reducing to (almost) periodic time-triggered control.

In [14], state feedback Event-holding Control (EHC) was proposed as an alternative to the time-regularized static event-triggered control, whereby the local clock is only run if a certain condition is violated. This strategy is motivated by several beneficial features. First, compared to time-regularized ETC, which also enforces a strictly positive minimum inter-event time, a transmission is not triggered as soon as a given amount of time has passed since the last transmission and the triggering rule is satisfied, but only when the latter has been verified for a given amount of time. This is essentially different and may help to decrease the amount of transmissions, as we will show on a numerical example. Second, it is well-suited for practical setups such as those operated by supervisory control and data acquisition (SCADA) system for instance, where a “hold time parameter” is used to adjust the maximal period that a local device holds its event before reporting to the master system. In this context, an event refers to the detection of a condition, see [15] for more details.

Thus, in this paper, we extend the ideas of event-holding control [14] to the output-feedback case, and we design triggers that are robust to measurement noise. As we will see in Section VII, EHC may still lead to Zeno-like behavior in presence of measurement noise when close to the attractor. We thus introduce space-regularization to improve the "steady-state" behavior, at the price of a practical stability property. As the time between consecutive events is lower bounded by design, the space-regularization parameter can be selected arbitrarily, contrary to [7], [8], where this parameter should be sufficiently large to ensure non-Zenoness and requires the knowledge of an upper-bound of the noise. Although a practical stability property is ensured, the added benefit of using EHC is that, by properly tuning this parameter, the average asymptotic closeness to the attractor may remain unaffected while the "steady-state" behavior in terms of average inter-event times improves significantly. This is not possible using [7], [8], due to the (positive) lower bound on the space-regularization parameter, which is used to avoid Zeno. Our contributions are hence as follows:

1) we generalize EHC to the output-feedback case;
2) we show that, with the proposed event-holding function, the system is ISS with respect to measurement noise;
3) by applying space-regularization, the behavior close to the attractor may be improved in terms of average transmission rates, however, the resulting triggering mechanism will lead to ISpS instead of ISS;
4) using numerical simulations, we show that, by tuning the space-regularization parameter, the asymptotic closeness to the attractor may remain unaffected, while the average inter-event times improve significantly.

II. Preliminaries

A. Notation

The sets of all non-negative integers are denoted $\mathbb{N}$. The field of all reals and all non-negative reals are indicated by $\mathbb{R}$ and $\mathbb{R}_{\geqslant 0}$, respectively. The vector in $\mathbb{R}^N$ whose elements are all zeros is denoted by $\mathbf{0}_N$. For any vector $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$, the stacked vector $\begin{bmatrix} u^\top & v^\top \end{bmatrix}^\top$ is denoted by (u, v). By $\langle \cdot, \cdot \rangle$ and $|\cdot|$ we denote the usual inner product of real vectors and the Euclidean norm, respectively. For any $x \in \mathbb{R}^N$, the distance to a closed non-empty set $\mathcal{A} \subseteq \mathbb{R}^N$ is denoted by $|x|_{\mathcal{A}} := \min_{y \in \mathcal{A}} |x - y|$. We use $U^\circ(x; v)$ to denote Clarke's generalized directional derivative of a locally Lipschitz function U at x in the direction v, *i.e.*, $U^\circ(x; v) := \limsup_{h \to 0^+,\, y \to x} (U(y + hv) - U(y))/h$, which reduces to the standard directional derivative when U is continuously differentiable; see [16] for more details. Given a vector $x \in \mathbb{R}^{n_x}$ and a set $\mathcal{A} \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$, $\Pi_x(\mathcal{A})$ denotes the projection of $\mathcal{A}$ onto the x-plane $\mathbb{R}^{n_x}$, *i.e.*, $\Pi_x(\mathcal{A}) = \{z \in \mathbb{R}^{n_x} \mid \exists y \in \mathbb{R}^{n_y} \text{ s.t. } (z, y) \in \mathcal{A}\}$. By $\wedge$ and $\vee$ we denote the logical *and* and *or* operators, respectively. For a measurable signal $w : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^{n_w}$, we denote by $\|w\|_\infty := \operatorname{ess\,sup}_{t \in \mathbb{R}_{\geqslant 0}} |w(t)|$ its essential $\mathcal{L}_\infty$-norm provided it is finite, in which case we write $w \in \mathcal{L}_\infty$. We use the usual definitions for comparison functions $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$, see [17].

B. Hybrid systems

Based on an extension [18] of the formalism of [17], we model hybrid systems $\mathcal{H}(F, \mathcal{C}, G, \mathcal{D}, \mathbb{X}, \mathbb{V})$ as

$$\begin{cases} \dot{\xi} \in F(\xi, \nu) & (\xi, \nu) \in \mathcal{C}, \\ \xi^+ \in G(\xi, \nu) & (\xi, \nu) \in \mathcal{D}, \end{cases} \tag{1}$$

where $\xi \in \mathbb{X} \subseteq \mathbb{R}^{n_\xi}$ denotes the state, ν an external input taking values in $\mathbb{V} \subseteq \mathbb{R}^{n_\nu}$, $\mathcal{C} \subseteq \mathbb{X} \times \mathbb{V}$ the flow set, $\mathcal{D} \subseteq \mathbb{X} \times \mathbb{V}$ the jump set, $F : \mathbb{X} \times \mathbb{V} \rightrightarrows \mathbb{R}^{n_\xi}$ the (set-valued) flow map and $G : \mathbb{X} \times \mathbb{V} \rightrightarrows \mathbb{R}^{n_\xi}$ the (set-valued) jump map. Sets $\mathcal{C}$ and $\mathcal{D}$ are assumed to be closed. We refer to [17] for notions related to (1) such as hybrid time domains or hybrid arcs. For a hybrid time domain E, $\sup_t E := \sup \{t \in \mathbb{R}_{\geqslant 0} : \exists j \in \mathbb{N} \text{ such that } (t, j) \in E\}$. Note that the disturbances ν are defined on the non-negative real line $\mathbb{R}_{\geqslant 0}$, *not* on a hybrid time domain, see [18] for a detailed motivation of this choice.

Definition 1. *Given a set of inputs $\mathcal{V} \subseteq \mathcal{L}_\infty$, a hybrid system $\mathcal{H}$ is persistently flowing if all maximal solution pairs $(\phi, v) \in \mathcal{S}_{\mathcal{H}}$ with $v \in \mathcal{V}$ are complete in the t-direction,* i.e.*, $\sup_t \operatorname{dom} \phi = \infty$.*

In this paper, we consider systems $\mathcal{H}$ that are persistently flowing, *i.e.*, systems whose maximal solutions are t-complete, and thus we focus on the stability notions below.

Definition 2. *When $\mathcal{H}$ is persistently flowing with $\mathcal{V} \subseteq \mathcal{L}_\infty$, we say that a closed, non-empty set $\mathcal{A} \subset \mathbb{R}^{n_x}$ is* input-to-state practically stable *(ISpS), if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and $d \geqslant 0$ such that for any solution pair (x, v) with $v \in \mathcal{V}$*

$$|x(t, j)|_{\mathcal{A}} \leqslant \beta(|x(0, 0)|_{\mathcal{A}}, t) + \gamma(\|v\|_\infty) + d, \tag{2}$$

for all $(t, j) \in \operatorname{dom} x$. We say that $\mathcal{A}$ is input-to-state stable *(ISS) when (2) holds with $d = 0$.*

To prove that a given non-empty closed set $\mathcal{A}$ is IS(p)S, we will use the following Lyapunov conditions.

Proposition 1. *Consider a persistently flowing system $\mathcal{H}$ with a set of inputs $\mathcal{V} \subseteq \mathcal{L}_\infty$ and let $\mathcal{A} \subset \mathbb{R}^{n_\xi}$ be a non-empty closed set. If there exist a locally Lipschitz $V : \operatorname{dom} V \to \mathbb{R}_{\geqslant 0}$, $\alpha, \underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}$ and $d \in \mathbb{R}_{\geqslant 0}$ such that*

1) $\operatorname{dom} V \supset \Pi_\xi(\mathcal{C} \cup \mathcal{D})$,
2) for any $(\xi, \nu) \in \mathcal{C} \cup \mathcal{D}$,
$$\underline{\alpha}(|\xi|_{\mathcal{A}}) \leqslant V(\xi) \leqslant \overline{\alpha}(|\xi|_{\mathcal{A}}),$$
3) for all $(\xi, \nu) \in \mathcal{C}$ and $f \in F(\xi, \nu)$,
$$V^\circ(\xi; f) \leqslant -\alpha(V(\xi)) + \gamma(|\nu|) + d,$$
4) for all $(\xi, \nu) \in \mathcal{D}$ and any $g \in G(\xi, \nu)$,
$$V(g) - V(\xi) \leqslant 0,$$

then $\mathcal{A}$ is ISpS, and $\mathcal{A}$ is ISS when item 3) holds with $d = 0$.

III. Problem formulation

We consider a continuous-time plant $\mathcal{P}$ of the form

$$\mathcal{P} : \begin{cases} \dot{x}_p = f_p(x_p, u, v), \\ y = g_p(x_p), \\ \widetilde{y} = y + w, \end{cases} \tag{3}$$

where $x_p \in \mathbb{R}^{n_{x_p}}$ is the plant state, $u \in \mathbb{R}^{n_u}$ the control input, $v \in \mathbb{R}^{n_v}$ a process disturbance, $y \in \mathbb{R}^{n_y}$ the output of the system unaffected by measurement noise, $\widetilde{y} \in \mathbb{R}^{n_y}$ the output of the system including measurement noise and

$w \in \mathbb{R}^{n_y}$ the additive measurement noise. The system is controlled by a dynamic controller $\mathcal{C}$ with dynamics

$$\mathcal{C}: \begin{cases} \dot{x}_c = f_c(x_c, \widehat{\widetilde{y}}), \\ u = g_c(x_c, \widehat{\widetilde{y}}), \end{cases} \tag{4}$$

where $x_c \in \mathbb{R}^{n_{x_c}}$ is the controller state, $\widehat{\widetilde{y}} \in \mathbb{R}^{n_y}$ the most recently received noisy output of the system $\mathcal{P}$, and $u \in \mathbb{R}^{n_u}$ the output of the controller. The functions f_p and f_c are assumed to be continuous, and g_p and g_c are assumed to be continuously differentiable and zero at zero. We proceed by emulation and assume that the controller $\mathcal{C}$ has been designed such that the origin of $\mathcal{P}$ is robustly stabilized, in the sense that the closed-loop dynamics satisfy certain properties that will be formalized in the following.

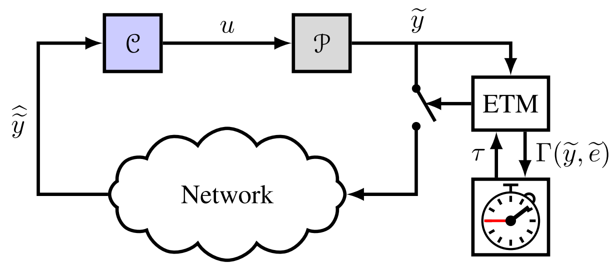

Fig. 1. Networked control setup where the controller and plant are (physically) separated and communicate via a packet-based network.

Remark 1. *We do not impose restrictions on the existence or boundedness of the derivative of w, as was required in,* e.g.*, [4], [5]. Moreover, we also do not need to know an upper bound on the $\mathcal{L}_\infty$-norm as in [7], [8]. Indeed, the only requirement on the signal w is that it is measurable in the mathematical sense.*

The output $\widetilde{y}$ is broadcast over the digital network to the controller, see Fig. 1. The corresponding transmissions occur at time instants t_k, $k \in \mathbb{N}$, which are generated by an Event-Triggering Mechanism (ETM), to be designed. Because of the packet-based communication over the network, the controller does not have continuous access to $\widetilde{y}$, but only to its latest transmitted value $\widehat{\widetilde{y}}$. When the ETM transmits the measured output of the plant over the network, $\widehat{\widetilde{y}}$ is updated according to

$$\widehat{\widetilde{y}}((t_k)^+) = \widetilde{y}(t_k), \qquad \forall t \in [t_k, t_{k+1}] \tag{5}$$

In between transmissions the estimate evolves in a Zero-Order-Hold (ZOH) fashion, *i.e.*,

$$\dot{\widehat{\widetilde{y}}} = 0. \tag{6}$$

For modeling purposes, we define $\widehat{y}$ and $\widehat{w}$, where

$$\begin{aligned} \widehat{y}((t_k)^+) &= y(t_k), \quad \dot{\widehat{y}} = 0, \quad \forall t \in [t_k, t_{k+1}], \\ \widehat{w}((t_k)^+) &= w(t_k), \quad \dot{\widehat{w}} = 0, \quad \forall t \in [t_k, t_{k+1}]. \end{aligned} \tag{7}$$

Hence, $\widehat{w}$ is the value of w at the last transmission instant of the ETM. Due to the aforementioned definitions, we obtain that $\widehat{\widetilde{y}} = \widehat{y} + \widehat{w}$.

We define the *true* network-induced error e as the difference between the sampled output $\widehat{y}$ *without* measurement noise and the current output y *without* measurement noise:

$$e := \widehat{y} - y. \tag{8}$$

Note that e is *not* known by the ETM, and therefore, cannot be used by the corresponding local triggering condition for determining t_k, $k \in \mathbb{N}$. Hence, we also define the *measured* network-induced error $\widetilde{e}$ as the difference between the estimated output $\widehat{\widetilde{y}}$ and the current measured output $\widetilde{y}$, which are both affected by noise, *i.e.*,

$$\widetilde{e} := \widehat{\widetilde{y}} - \widetilde{y} = e + \widehat{w} - w. \tag{9}$$

The ETM does have access to $\widetilde{e}$.

As mentioned above, the transmission instants are generated by the ETM, which we describe in the next section.

IV. Event-holding control

As the ETM requires a clock, we introduce the timer variable $\tau \in \mathbb{R}_{\geqslant 0}$. During flow, the timer τ satisfies the differential inclusion

$$\dot{\tau} \in \sigma(\Gamma(\widetilde{y}, \widetilde{e})), \tag{10}$$

where $\sigma : \mathbb{R} \rightrightarrows \{0, 1\}$ is a set-valued map given by

$$\sigma(s) := \begin{cases} \{1\}, & \text{if } s > 0, \\ [0, 1], & \text{if } s = 0, \\ \{0\}, & \text{if } s < 0. \end{cases} \tag{11}$$

The function Γ determines when the clock will run (*i.e.*, when $\dot{\tau} = 1$), and it will be constructed in the following. The transmission instants t_k, $k \in \mathbb{N}$, are defined by

$$t_0 = 0,\ t_{k+1} = \inf\{t > t_k \mid \Gamma(\widetilde{y}(t), \widetilde{e}(t)) \geqslant 0 \wedge \tau(t) \geqslant \tau_H\}, \tag{12}$$

where τ_H denotes the *holding time*, which is designed below.

When $\widehat{\widetilde{y}}$ is broadcast over the network, the timer τ is reset according to

$$\tau^+ = 0. \tag{13}$$

V. Hybrid model

We model the overall system as a hybrid system $\mathcal{H}$ as in Section II-B, for which a jump corresponds to the broadcasting of the noisy output $\widetilde{y}$ over the network. The full state for $\mathcal{H}$ becomes $\xi := (x, e, \widehat{w}, \tau) \in \mathbb{X}$, where $\mathbb{X} := \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}_{\geqslant 0}$, with $n_x := n_{x_p} + n_{x_c}$. We define the concatenated exogenous inputs $\nu := (v, w) \in \mathbb{V}$, where $\mathbb{V} := \mathbb{R}^{n_v} \times \mathbb{R}^{n_y}$. The flow map $F : \mathbb{X} \times \mathbb{V} \rightrightarrows \mathbb{X}$ can then be written as

$$F(\xi, \nu) := \big(f(x, e, \widehat{w}, v), g(x, e, \widehat{w}, v), \mathbf{0}_{n_y}, \sigma(\Gamma(\widetilde{y}, \widetilde{e}))\big). \tag{14}$$

Based on (3), (4) and (9), we obtain

$$f(x, e, \widehat{w}, v) := \begin{bmatrix} f_p\left(x_p, g_c(x_c, g_p(x_p) + e + \widehat{w}), v\right) \\ f_c\left(x_c, g_p(x_p) + e + \widehat{w}\right) \end{bmatrix}, \tag{15}$$

where based on (3), (6) and (9), we obtain

$$g(x, e, \widehat{w}, v) := -\frac{\partial g_p}{\partial x_p} f_p\left(x_p, g_c(x_c, g_p(x_p) + e + \widehat{w}), v\right). \tag{16}$$

According to (7) and (13), the jump map $G : \mathbb{X} \times \mathbb{V} \to \mathbb{X}$ is

$$G(\xi, \nu) := (x, \mathbf{0}, w, 0). \tag{17}$$

Using (12), the flow set $\mathcal{C} \subseteq \mathbb{X} \times \mathbb{V}$ and jump set $\mathcal{D} \subseteq \mathbb{X} \times \mathbb{V}$ are given by

$$\mathcal{C} := \{(\xi, \nu) \in \mathbb{X} \times \mathbb{V} \mid \tau \leqslant \tau_H \vee \Gamma(\widetilde{y}, \widetilde{e}) \leqslant 0\}, \tag{18}$$

$$\mathcal{D} := \{(\xi, \nu) \in \mathbb{X} \times \mathbb{V} \mid \tau \geqslant \tau_H\}. \tag{19}$$

VI. Main results

To ensure that the hybrid system has the desired stability properties, several conditions have to be satisfied. To that end, we require the following conditions to hold.

Condition 1. *There exist a locally Lipschitz function $W : \mathbb{R}^{n_y} \to \mathbb{R}_{\geqslant 0}$, a continuous function $H : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathcal{W} \to \mathbb{R}_{\geqslant 0}$, $\underline{\alpha}_W, \overline{\alpha}_W > 0$, $L_W \geqslant 0$ and $\vartheta_W \in \mathcal{K}$ such that*

(i) for any $e \in \mathbb{R}^{n_y}$,

$$\underline{\alpha}_W |e| \leqslant W(e) \leqslant \overline{\alpha}_W |e|,$$

(ii) for all $e \in \mathbb{R}^{n_y}$, $x \in \mathbb{R}^{n_x}$, $w, \widehat{w} \in \mathbb{R}^{n_y}$ and $v \in \mathbb{R}^{n_v}$,

$$W^\circ(e; g(x, e, \widehat{w}, v)) \leqslant L_W W(e) + H(x, e, \widehat{w}, v).$$

Item (ii) of Condition 1 imposes an upper bound on the growth of e between successive transmission instants along the solutions to $\mathcal{H}$. This condition can always be satisfied by taking $H(x, e, \widehat{w}, v) \geqslant |W^\circ(e; g(x, e, \widehat{w}, v)|$ with H continuous and $L_W = 0$. Next to Condition 1, we also require that the closed-loop system is robustly stable in the following sense.

Condition 2. *There exist a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$, $\mathcal{K}_\infty$-functions $\underline{\alpha}_V, \overline{\alpha}_V, \alpha_V, \alpha_W$, $\vartheta_V, \vartheta_\varrho \in \mathcal{K}$, a function $J : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_v} \to \mathbb{R}$, a locally Lipschitz function $\varrho \in \mathcal{K}_\infty$, $L_\varrho \in \mathbb{R}$ and $\gamma > 0$ such that*

(i) for all $x \in \mathbb{R}^{n_x}$,

$$\underline{\alpha}_V(|x|) \leqslant V(x) \leqslant \overline{\alpha}_V(|x|), \tag{20}$$

(ii) for all $x \in \mathbb{R}^{n_x}$, $\nu \in \mathbb{V}$ and $e, \widehat{w} \in \mathbb{R}^{n_y}$,

$$\begin{aligned} V^\circ(x; f(x, e, \widehat{w}, v)) &\leqslant -\alpha_V(V(x)) - \alpha_W(W(e)) - \overline{\varrho}(y) \\ &- J(x, e, \widehat{w}, v) - H^2(x, e, \widehat{w}, v) + \gamma^2 W^2(e) + \vartheta_V(|\nu|), \end{aligned}$$

where $\overline{\varrho} : \mathbb{R}^{n_y} \to \mathbb{R}_{\geqslant 0}$ is defined as $\overline{\varrho}(y) := \varrho(|y|)$,

(iii) for all $x \in \mathbb{R}^{n_x}$, $\nu \in \mathbb{V}$ and $e, \widehat{w} \in \mathbb{R}^{n_y}$,

$$\begin{aligned} \overline{\varrho}^\circ(y; -g(x, e, \widehat{w}, v)) &\leqslant L_\varrho \overline{\varrho}(y) + H^2(x, e, \widehat{w}, v) \\ &+ J(x, e, \widehat{w}, v) + \vartheta_\varrho(|\nu|). \end{aligned}$$

Function V is an ISS-Lyapunov function used to prove the stability of the closed-loop system $\mathcal{H}$. It is important to note that Condition 2 (and 1) is a property of the closed-loop system $\mathcal{H}$ and is independent of the communication network, as e is interpreted as a disturbance acting on the output y in Condition 2. Items (i) and (ii) of Condition 2 imply that system $\dot{x} = f(x, e, \widehat{w}, v)$ is $\mathcal{L}_2$-stable from $(W, \sqrt{\vartheta_V})$ to H. This type of condition is often used in the literature on NCS, see, *e.g.*, [5], [12], [19], where examples of systems satisfying these conditions are provided. Such a robustness assumption is natural as we proceed by emulation; the continuous-time controller (4) must satisfy some robustness property with respect to the perturbations induced by the network. The function ϱ in item (iii) will be involved in the design of the local event triggering conditions. Item (iii) of Condition 2 is a growth condition on ϱ. Similar assumptions are made in the context of periodic event-triggered control [20], where relevant case studies are provided to illustrate how Conditions 1 and 2 are verified. However, note that the scope, the model and the design are essentially different here.

Given Conditions 1 and 2, we define $\delta : [0, 1] \to \mathbb{R}$ as

$$\delta(\lambda) := \frac{\gamma\lambda}{1 - \lambda L_\varrho} \tag{21}$$

and

$$\lambda^\star := \begin{cases} 1, & \text{if } L_\varrho \leqslant -\gamma, \\ \min\left\{1, \frac{1}{L_\varrho + \gamma}\right\}, & \text{if } L_\varrho > -\gamma. \end{cases} \tag{22}$$

The next lemma states an important property of δ; its proof is given in the appendix.

Lemma 1. *For any $\lambda \in (0, \lambda^\star)$, $\delta(\lambda) \in (0, 1)$ holds.*

The following condition is required to ensure that the resulting closed-loop system is IS(p)S.

Condition 3. *There exist $\epsilon \in (0, 1)$ and $\lambda \in \left(0, \min\left\{\lambda^\star, \overline{\lambda}\right\}\right)$ such that $\delta(\lambda) < \epsilon$ with*

$$\overline{\lambda} := \begin{cases} \frac{1-\epsilon^2}{2L_W \epsilon + \gamma}, & \textit{if } L_\varrho \geqslant 0, \\ \frac{1-\epsilon^2}{2L_W \epsilon + \gamma - \epsilon^2 L_\varrho}, & \textit{if } L_\varrho < 0. \end{cases} \tag{23}$$

Note that it is always possible to satisfy Condition 3 by selecting λ sufficiently small, as $\delta(\lambda) \to 0$ when $\lambda \to 0$.

We now define the *maximum allowable event-holding time* (MAET) $T_H : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$. Fix λ, ϵ such that Condition 3 is satisfied. Select any $\overline{\mu} \in (\delta(\lambda), \epsilon/\delta(\lambda)]$. The MAET is then given by

$$T_H(\lambda, \overline{\mu}) := \begin{cases} \frac{1}{L_W r} \arctan(\theta), & \gamma > L_W, \\ \frac{1}{L_W} \frac{\overline{\mu} - \delta(\lambda)}{(\delta(\lambda)+1)(\overline{\mu}+1)}, & \gamma = L_W, \\ \frac{1}{L_W r} \operatorname{arctanh}(\theta), & \gamma < L_W, \end{cases} \tag{24}$$

where

$$\theta := \frac{r(\overline{\mu} - \delta(\lambda))}{\frac{\gamma}{L_W}(1 + \overline{\mu}\delta(\lambda)) + \overline{\mu} + \delta(\lambda)}, \quad r := \sqrt{\left|\left(\frac{\gamma}{L_W}\right)^2 - 1\right|},$$

and L_W, L_δ and γ come from Conditions 1 and 2.

We are now ready to state the main result of this paper. Its proof is omitted for space reasons.

Theorem 1. *Consider the hybrid system $\mathcal{H}$ and suppose Conditions 1-3 hold. We define for all $\xi \in \mathbb{X}$ and $\nu \in \mathbb{V}$, $\Gamma : \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \to \mathbb{R}$ as*

$$\Gamma(\widetilde{y}, \widetilde{e}) := (1 + \zeta)\gamma\overline{\alpha}_W |\widetilde{e}|^2 - \lambda\delta(\lambda)\varrho\left(\tfrac{1}{1+\kappa}|\widetilde{y}|\right) - d, \tag{25}$$

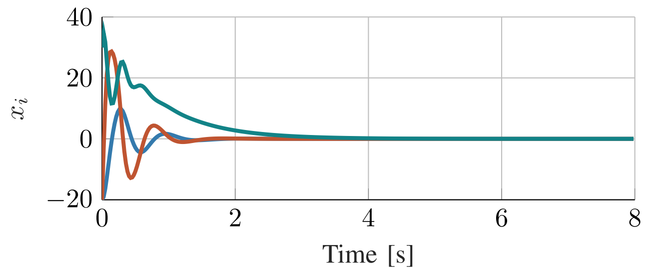

Fig. 2. Trajectories of controlled Lorenz equations under the proposed strategy with initial condition $x_0 = (-20, -20, 30)$ and no measurement noise, *i.e.*, $w(t) = 0$ for all $t \in \mathbb{R}_{\geqslant 0}$.

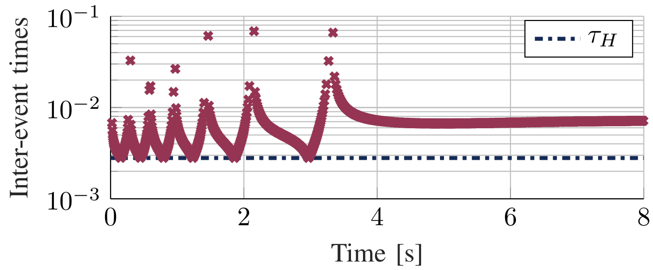

Fig. 3. Inter-event times for the event-holding control strategy and no measurement noise, *i.e.*, $w(t) = 0$ for all $t \in \mathbb{R}_{\geqslant 0}$.

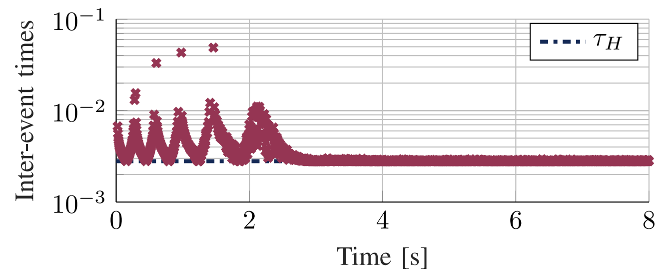

Fig. 4. Inter-event times for the event-holding control strategy and bounded noise, *i.e.*, $|w|_\infty \approx 10^{-3}$, when $d = 0$.

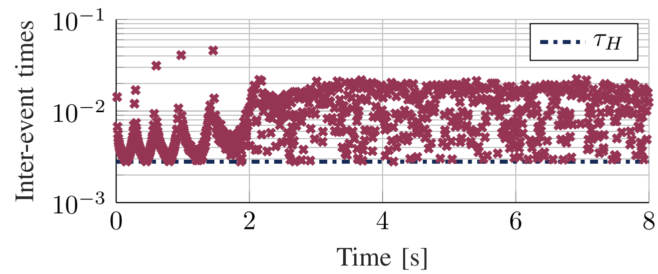

Fig. 5. Inter-event times for the event-holding control strategy and bounded noise, *i.e.*, $|w|_\infty \approx 10^{-3}$, when applying space-regularization with constant $d = 2.5 \cdot 10^{-4}$.

where $\zeta, \kappa > 0$ and $d \geqslant 0$ are tuning parameters. Then, for any $\overline{\mu} \in [\delta(\lambda), \epsilon/\delta(\lambda)]$ and any (associated) holding time $\tau_H \in (0, T_H(\lambda, \overline{\mu})]$, the system $\mathcal{H}$ with holding function (25) *and holding time τ_H is persistently flowing and the set $\mathcal{A} := \{\xi : x = 0 \wedge e = 0\}$ is ISpS w.r.t. the disturbances ν, if $d > 0$, and $\mathcal{A}$ is ISS, if $d = 0$.*

VII. Numerical case study

We consider the controlled Lorenz model of fluid convection in [21] affected by measurement noise, $\dot{x}_1 = -ax_1 + ax_2$, $\dot{x}_2 = bx_1 - x_2 - x_1x_3 + u$, $\dot{x}_3 = x_1x_2 - cx_3$ and $y = x_1$, where $a, b, c > 0$ are related to some physical constants. The static output feedback controller $u = -\big(\frac{p_1}{p_2}a + b\big)y$, where $p_1, p_2 > 0$, globally stabilizes the origin. The proposition below ensures the satisfaction of Conditions 1 and 2 for this system.

Proposition 2. *Let $a > 1$, $p_1 > \frac{2a^2+1}{2a}$ and $p_2 > 2a^2$. The controlled Lorenz system satisfies Conditions 1 and 2 with $\underline{\alpha}_W, \overline{\alpha}_W = 1$, $W(e) = |e|$, $L_W = 0$, $L_\varrho = 0$, $H(x) := a(|x_1| + |x_2|)$, $J = 0$, $\alpha_W(s) = \alpha_W s^2$, $\gamma^2 = 2p_2\big(\frac{p_1}{p_2}a + b\big)^2 + \alpha_W$ and $\varrho(s) = 2as^2$.*

Proof. We take $W(e) = |e|$. By definition, $\underline{\alpha}_W = \overline{\alpha}_W = 1$. Moreover, for all $e \in \mathbb{R}^{n_y}$, $W^\circ(e; g(x, e, \widehat{w})) \leqslant |a(x_1 - x_2)| \leqslant a(|x_1| + |x_2|) =: H(x)$. By Young's inequality, we find that $H^2(x) \leqslant 2a^2x_1^2 + 2a^2x_2^2$. We introduce the quadratic Lyapunov function $V(x) = p_1x_1^2 + p_2x_2^2 + p_2x_3^2$. We have for any $x \in \mathbb{R}^{n_x}$ and $e, \widehat{w} \in \mathbb{R}^{n_y}$

$$\langle \nabla V(x), f(x, e, \widehat{w})\rangle = -2ap_1x_1^2 - 2p_2x_2^2 - 2cp_2x_3^2 - 2p_2x_2\big(\tfrac{p_1}{p_2}a + b\big)(e + \widehat{w}).$$

Let $\varrho(|y|) = 2ay^2$ such that $\langle \nabla \overline{\varrho}(y), -g(x, e, \widehat{w})\rangle \leqslant 2a(-2ax_1^2 + 2ax_1x_2) \leqslant 2a(ax_1^2 + ax_2^2) \leqslant 2a^2x_1^2 + 2a^2x_2^2 = H^2(x)$. Thus, $L_\varrho = 0$ and $J = 0$. Hence, we find that

$$\langle \nabla V(x), f(x, e, \widehat{w})\rangle \leqslant -(2ap_1 - 2a^2 - 2a)x_1^2 - (p_2 - 2a^2)x_2^2 - 2cp_2x_3^2 + (\gamma^2 - \alpha_W)e^2 + \gamma^2\widehat{w}^2 - \varrho(|y|) - H^2(x) \quad (26)$$

with $\gamma^2 = 2p_2\big(\frac{p_1}{p_2}a + b\big)^2 + \alpha_W$, $\alpha_W > 0$ and $\varrho(s) = 2as^2$, which satisfies Condition 2 when $p_1 > a + 1$ and $p_2 > 2a^2$. □

For the parameter values we select $a = 10$, $b = 28$, $c = 8/3$ and we set $p_1 = a + 1.1$, $p_2 = 2a^2 + 1$, $\alpha_W = 0.05$ and we obtain $\gamma = 147.4331$. We select $\lambda = (2\gamma)^{-1}$ and $\epsilon = 0.6$, such that $\lambda < \overline{\lambda} < \lambda^\star$ and Condition 3 is satisfied. We select $\underline{\mu} = \delta(\lambda)$, $\overline{\mu} = \epsilon/\delta(\lambda)$, $\tau_H = T_H(\lambda, \overline{\mu}) = 2.7973 \cdot 10^{-3}$ and $\zeta = \kappa = 0.01$.

Figures 2 and 3 depict the trajectories and the inter-event times in absence of noise, respectively. The average inter-event times are generally significantly above the minimum holding time τ_H when no noise is present. We now add noise to the output signal. The noise is generated as a piecewise constant signal, where a new value of w is taken uniformly in the interval $[-10^{-3}, 10^{-3}]$ every 10^{-6} seconds. In Figure 4, the inter-event times are portrayed for the case where no space-regularization is used, *i.e.*, when $d = 0$. Observe that *Zeno-like* behavior is present due to the noise. For systems where the noise is consistently present, *i.e.*, when $w(t)$ does not vanish as $t \to \infty$, the space-regularization parameter d can be used to attain improved "steady-state" behavior in terms of average inter-event times when close to the attractor, as depicted in Figure 5, where we selected $d = 2.5 \cdot 10^{-4}$. By carefully selecting d, the average distance to the attractor in "steady-state" can be maintained, as

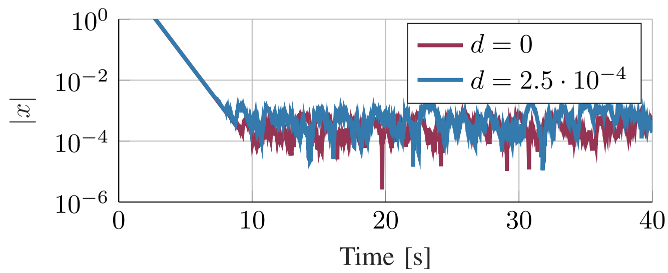

Fig. 6. Difference of $|x|$ for the event-holding control strategy and bounded noise, *i.e.*, $|w|_\infty \approx 10^{-3}$, when applying space-regularization with constant $d = 0$ (in red) or $d = 10^{-4}$ (in blue).

TABLE I

AVERAGE INTER-EVENT TIMES AND NUMBER OF EVENTS FOR EHC AND ETC OVER 50 SIMULATIONS.

	Average IET	Total # events
EHC	0.01097	1822.78
ETC	0.00286	6976.92

depicted in Figure 6. Moreover, adding space-regularization does not significantly impact the transient response, thereby not degrading performance.

To highlight the benefits of EHC, we also implement a time-regularized static ETC algorithm for the controlled Lorenz equations, see, *e.g.*, [5]. We tune the triggering condition in ETC such that the MIET of the ETC algorithm is approximately equal to the MAET τ_H, while keeping γ constant. We run 50 simulations of each triggering mechanism using random initial conditions, *i.e.*, for each pair of triggering mechanisms we select some $x(0,0) \in [0,1]^3$ according to a uniform distribution. The average inter-event times and number of events for a 20 second simulation can be found in Table I. Clearly, EHC outperforms static ETC with time regularization, which underlines the benefit of EHC with respect to time-regularized static ETC in this example.

VIII. CONCLUSIONS

We presented an output-based event-holding strategy that is robust to measurement noise. The proposed strategy guarantees a lower bound on the inter-event times by design. We obtain ISS for the closed-loop system, if no space regularization is used, however, in presence of measurement noise, it may lead to Zeno-like behavior when close to the attractor. By applying space-regularization, the inter-event times may be significantly improved as illustrated in simulations. Due to the system being Zeno-free by design, the space-regularization parameter can be tuned such that the resulting trajectories maintain similar properties in terms of (average) asymptotic closeness to the attractor, while not impacting the performance in the transient response. We showed these beneficial properties on the controlled Lorenz equations to demonstrate the efficacy of the proposed approach.

REFERENCES

[1] W. P. M. H. Heemels, K. H. Johansson, and P. Tabuada, "An introduction to event-triggered and self-triggered control," in *51st IEEE Conference on Decision and Control*, Maui, HI, USA, Dec 2012, pp. 3270–3285.

[2] K.-E. Årzén, "A simple event-based pid controller," *IFAC Proceedings Volumes*, vol. 32, no. 2, pp. 8687–8692, Jul 1999.

[3] G. Hardouin, S. Bertrand, and H. Piet-Lahanier, "Distributed event-triggered consensus of multi-agent systems with measurement noise and guaranteed interval bounds," *IFAC-PapersOnLine*, vol. 52, no. 20, pp. 7–12, 2019.

[4] S. H. Mousavi, N. Noroozi, R. Geiselhart, M. Kogel, and R. Findeisen, "On integral input-to-state stability of event-triggered control systems," in *58th IEEE Conference on Decision and Control*, Dec 2019, pp. 1674–1679.

[5] M. Abdelrahim, R. Postoyan, J. Daafouz, and D. Nešić, "Robust event-triggered output feedback controllers for nonlinear systems," *Automatica*, vol. 75, pp. 96–108, Jan 2017.

[6] A. Selivanov and E. Fridman, "A switching approach to event-triggered control," in *54th IEEE Conference on Decision and Control*, 2015, pp. 5468–5473.

[7] K. J. A. Scheres, R. Postoyan, and W. P. M. H. Heemels, "Event-triggered control in presence of measurement noise: A space-regularization approach," in *59th IEEE Conference on Decision and Control*, Jeju Island, Korea (South), Dec 2020, pp. 6234–6239.

[8] ——, "Robustifying event-triggered control to measurement noise," *Automatica*, to be published.

[9] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1680–1685, Sep 2007.

[10] D. P. Borgers and W. P. M. H. Heemels, "Event-separation properties of event-triggered control systems," *IEEE Transactions on Automatic Control*, vol. 59, no. 10, pp. 2644–2656, Oct 2014.

[11] P. Casau, R. G. Sanfelice, and C. Silvestre, "On the robustness of nominally well-posed event-triggered controllers," *IEEE Control Systems Letters*, vol. 6, pp. 415–420, 2022.

[12] V. S. Dolk, D. P. Borgers, and W. P. M. H. Heemels, "Output-based and decentralized dynamic event-triggered control with guaranteed $\mathcal{L}_p$-gain performance and Zeno-freeness," *IEEE Transactions on Automatic Control*, vol. 62, no. 1, pp. 34–49, Jan 2017.

[13] D. Yang, X. Liu, and W. Chen, "Periodic event/self-triggered consensus for general continuous-time linear multi-agent systems under general directed graphs," *IET Control Theory & Applications*, vol. 9, no. 3, pp. 428–440, Feb 2015.

[14] W. Wang, D. Nešić, R. Postoyan, I. Shames, and W. P. M. H. Heemels, "State-feedback event-holding control for nonlinear systems," in *58th IEEE Conference on Decision and Control*, Nice, France, Dec 2019, pp. 1650–1655.

[15] Control Microsystems Inc. DNP3: User and Reference Manual. [Online]. Available: https://scadahacker.com/library/Documents/ICS_Protocols/Control%20Microsystems%20-%20DNP3%20User%20and%20Reference%20Manual.pdf

[16] F. H. Clarke, *Optimization and Nonsmooth Analysis*. Classics in Applied Mathematics vol. 5, SIAM, 1990.

[17] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid dynamical systems: modeling, stability, and robustness*. Princeton University Press, 2012.

[18] W. P. M. H. Heemels, P. Bernard, K. J. A. Scheres, R. Postoyan, and R. G. Sanfelice, "Hybrid systems with continuous-time inputs: subtleties in solution concepts and existence results," in *60th IEEE Conference on Decision and Control*, Dec 2021, pp. 5368–5373.

[19] D. Carnevale, A. R. Teel, and D. Nešić, "A Lyapunov proof of an improved maximum allowable transfer interval for networked control systems," *IEEE Transactions on Automatic Control*, vol. 52, no. 5, pp. 892–897, May 2007.

[20] W. Wang, R. Postoyan, D. Nešić, and W. P. M. H. Heemels, "Periodic event-triggered control for nonlinear networked control systems," *IEEE Transactions on Automatic Control*, vol. 65, no. 2, pp. 620–635, Feb 2020.

[21] M. Abdelrahim, R. Postoyan, J. Daafouz, and D. Nešić, "Stabilization of nonlinear systems using event-triggered output feedback controllers," *IEEE Transactions on Automatic Control*, vol. 61, no. 9, pp. 2682–2687, Sep 2016.