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Design of a Nonlinear Observer for a Class of Locally Lipschitz Systems by Using Input-to-State Stability: An LMI Approach

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Abstract— **This note addresses the problem of state estimation for a class of local Lipschitz nonlinear systems under the effect of noise/disturbances. An observer structure based on Hilbert projection is used to handle the same class of nonlinearities. The boundedness of the state estimation error of the proposed observer is guaranteed by deploying the input-to-state stability (ISS) property. Based on ISS stability criterion, the new LMI condition is derived by combining a well-known LPV approach with a variant of Young inequality and novel matrix multipliers. The proposed LMIs have more decision variables than the methodologies presented in the literature, which provides extra degrees of freedom, and hence enhances LMI feasibility. Further, the effectiveness of the developed approach is highlighted through a numerical example. The performance of the observer is validated through the application of slip angle estimation in a nonlinear autonomous vehicle.**

Index Terms— **Nonlinear observer design, Input-to-state stability (ISS) property, Linear Matrix Inequalities (LMI), Local Lipschitz nonlinearities**

I. INTRODUCTION

Designing observers to estimate system states in the presence of noise and disturbances is one of the challenging tasks in control theory. Due to noisy measurements and external disturbances, it is difficult to obtain the plant states precisely. In such circumstances, it is essential to develop observers that perform within specified performance thresholds. Thus, the observers become an indispensable component in modern-day applications, for example, the state-of-charge (SoC) estimation of the lithium-ion battery model [1], the control of glucose levels in type-2 diabetic [2], and so on.

The development of state estimation tools for nonlinear systems is a complex task. Authors of [3] had deployed high-gain observer methodology, while the sliding-mode observer approach is used in [4] for the state estimation. LMI-based nonlinear observers have recently received a lot of attention in the control domain [5], [6], [7]. All these LMI-based methods remain conservative and subject to improvement.

In [5] and [6], an essential condition for the observer design is that the nonlinearities are assumed to be globally Lipschitz. However, in many practical cases, it can be restrictive. A solution for tackling the locally Lipschitz nonlinearities was proposed in [8]. Due to the judicious use of Hilbert transformation, the proposed nonlinear observer performs efficiently for the same class of nonlinearities. The authors of [5] and [6] used an $\mathcal{H}_\infty$ criterion in the observer design for the estimation of the state in the presence of noise. An alternative for an $\mathcal{H}_\infty$ criterion is the use of the input-to-state stability (ISS) property. In [9], an ISS notion was introduced in the control system domain. Further, the authors of [10] had proposed an ISS-Lyapunov function to use the ISS property for the stability of systems. An observer based on the ISS-Lyapunov function was proposed in [11] and [12]. All these cited papers provide efficient state estimation. Along with this, an ISS-Lyapunov function aids to obtain an LMI condition.Hence, in this paper, the ISS property along with the ISS-Lyapunov function are used to obtain a novel LMI condition. The proposed LMI is based on reformulated Lipschitz property, newly defined matrix multipliers and a variant of Young inequality. Thus, it contains additional decision variables than the existing LMI methods. These variables add extra degrees of freedom and improve LMI feasibility. Further, the effectiveness of the proposed matrix multipliers is investigated in MATLAB using a numerical example.In addition to this, the developed observer is implemented for slip angle estimation in the case of a nonlinear autonomous vehicle model.

The remainder of the letter is organized as follows: Section II includes some notations and mathematical tools which are used throughout the paper. The system description and observer structure are illustrated in Section III. Further, Section IV is devoted to the formulation of the LMI condition. The efficiency of the proposed methodology is highlighted in Section V through a numerical example and an application. Section VI draws some concluding remarks.

II. NOTATIONS AND PRELIMINARIES

A. Notations

Throughout the article, the following notations are used: $||e||$ and $||e||_{\mathcal{L}_2}$ indicate the euclidean norm and the $\mathcal{L}_2$ norm of a vector e, respectively. For any $\mathcal{L}_\infty$ bounded function $e(t)$, we can define $||e||_\infty = ||e||_{(0,+\infty)} = \text{esssup}_{t\in(0,+\infty)}||e(t)||$. The symbol $(\star)$ represents the blocks inside a symmetrical matrix. The transpose of matrix A is expressed as $A^\top$. For a matrix $A \in \mathbb{R}^{n\times n}$, $A > 0$ $(A < 0)$ indicates that A is a positive definite matrix (a negative definite matrix). Similarly, a positive semi-definite matrix (a negative semi-definite matrix) is given by $A \geq 0$ $(A \leq 0)$. $A = \text{block-diag}(A_1,\ldots,A_n)$ is a diagonal matrix having elements $A_1,\ldots,A_n$ in the diagonal. $\mathbb{I}$ denotes an identity matrix. $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ describe the minimum and maximum eigenvalues of matrix A, respectively. $e_s(i) = (\underbrace{0,\ldots,0,\overbrace{1}^{i^{\text{th}}},0,\ldots,0}_{s\text{ components}})^\top \in \mathbb{R}^s$, $s \geq 1$ is a vector of the canonical basis of $\mathbb{R}^s$. For any two vectors

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$X = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}^\top$, and $Y = \begin{pmatrix} y_1 & \dots & y_n \end{pmatrix}^\top$, an auxiliary vector $X^{Y_i} \in \mathbb{R}^n, \forall i = \{1, \dots, n\}$ corresponding to X and Y is defined as

$$X^{Y_i} = \begin{cases} \begin{pmatrix} y_1 & \dots & y_i & x_{i+1} & \dots & x_n \end{pmatrix}^\top, & \text{for } i = 1, \dots, n \\ X, & \text{for } i = 0. \end{cases}$$

B. Preliminaries

Definition 1 (Input-to-State stability [10]): Let us consider a generalized class of nonlinear systems:

$$\dot{\zeta} = f(\zeta, u). \tag{1}$$

The system (1) is input-to-state stable if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for any initial state $\zeta(0)$, and any bounded input $u(t)$, solution $\zeta(t)$ exists for all $t \geq 0$ and satisfies:

$$||\zeta(t)|| \leq \beta(||\zeta(0)||, t) + \gamma(||u||_\infty), \forall t \geq 0. \tag{2}$$

Definition 2 (ISS-Lyapunov function [10]): A smooth function $V(\zeta) : \mathbb{R}^n \to \mathbb{R}$ is called as an ISS-Lyapunov function for the system (1) if there exist class $\mathcal{K}_\infty$ functions $\alpha_i, i \in \{1, \dots, 4\}$ such that it holds

$$\alpha_1(||\zeta||) \leq V(\zeta) \leq \alpha_2(||\zeta||), \tag{3}$$

$$\dot{V}(\zeta, u) \leq -\alpha_3(||\zeta||) + \alpha_4(||u||). \tag{4}$$

Definition 3 (Hilbert Projection [8]): Let $\Omega \subset \mathbb{R}^n$ be a convex closed and nonempty set. Let us define the linear application $\pi_\Omega : \mathbb{R}^n \to \Omega$ such that:

$$\pi_\Omega(x) = \underset{y \in \Omega}{\operatorname{argmin}} ||x - y||.$$

Such an application is called the projection on convex set Ω.

The authors of [8] have mentioned the following important properties of projection π_Ω, which will be useful for the observer design:

1) The projection π_Ω is idempotent, i.e., $\pi_\Omega \circ \pi_\Omega = \pi_\Omega$.
2) π_Ω is a 1-Lipschitz function:

$$\forall x, u \in \mathbb{R}^n, ||\pi_\Omega(x) - \pi_\Omega(y)|| \leq ||x - y||. \tag{5}$$

Lemma 1 ([8]): Let us consider a nonlinear function $\gamma : \mathbb{R}^n \to \mathbb{R}^n$. Assume that γ is Lipschitz on a convex closed and non-empty set Ω. Then, there exist functions $\gamma_{ij}^\Omega : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and constants $\gamma_{ij_{\min}}^\Omega$ and $\gamma_{ij_{\max}}^\Omega$ such that $\forall X, Y \in \mathbb{R}^n, X \neq Y$,

$$\gamma(X) - \gamma(Y) = \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij}^\Omega \mathcal{H}_{ij}(X - Y), \tag{6}$$

where $\mathcal{H}_{ij} = e_n(i) e_n^\top(j)$, and $\gamma_{ij}^\Omega \triangleq \gamma_{ij}^\Omega(X^{Y_{j-1}}, X^{Y_j})$. The functions $\gamma_{ij}^\Omega(.)$ are globally bounded as follows:

$$\gamma_{ij_{\min}}^\Omega \leq \gamma_{ij}^\Omega \leq \gamma_{ij_{\max}}^\Omega. \tag{7}$$

Lemma 2 ([5]): If there exist two vectors $X, Y \in \mathbb{R}^n$ and a matrix $Z = Z^\top > 0 \in \mathbb{R}^{n \times n}$, then the following matrix inequalities hold:

$$X^\top Y + Y^\top X \leq X^\top Z^{-1} X + Y^\top Z Y, \tag{8}$$

$$X^\top Y + Y^\top X \leq \frac{1}{2}(X + ZY)^\top Z^{-1}(X + ZY). \tag{9}$$

III. Problem formulation

Consider the class of nonlinear systems described by the following equations:

$$\begin{aligned} \dot{x} &= Ax + Gf(x) + Bu + E\omega, \\ y &= Cx + Hh(x) + D\omega, \end{aligned} \tag{10}$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$ are the system states and outputs respectively. $u \in \mathbb{R}^s$ is the system input. $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times s}$, $G \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $H \in \mathbb{R}^{p \times r}$, $E \in \mathbb{R}^{n \times q}$ and $D \in \mathbb{R}^{p \times q}$ are known constant matrices. $\omega \in \mathbb{R}^q$ is $\mathcal{L}_2$ bounded disturbances affecting the process dynamics and the measurements.

Nonlinear functions $f(\cdot)$ and $h(\cdot)$ are written in the following detailed form:

$$f(x) = \begin{bmatrix} f_1(F_1 x) \\ \vdots \\ f_m(F_m x) \end{bmatrix}; \; h(x) = \begin{bmatrix} h_1(H_1 x) \\ \vdots \\ h_r(H_r x) \end{bmatrix}, \tag{11}$$

where $F_i \in \mathbb{R}^{\bar{n} \times n}$, $F_i x \in \Omega_{f_i} \forall i \in \{1, \dots, m\}$ and $H_i \in \mathbb{R}^{\bar{p} \times n}$, $H_i x \in \Omega_{h_i} \forall i \in \{1, \dots, r\}$. $f_i(\cdot)$ and $h_i(\cdot)$ are locally Lipschitz-continuous functions in Ω_{f_i} and Ω_{h_i}, respectively. It is assumed that both sets Ω_{f_i} and Ω_{h_i} are convex and positively invariant in a compact non-empty sets ($\Omega \neq \varnothing$).

For the state estimation purpose, the following standard Luenberger observer is extensively used in the literature:

$$\dot{\hat{x}} = A\hat{x} + Bu + Gf(\hat{x}) + L(y - (C\hat{x} + Hh(\hat{x}))), \tag{12}$$

where $\hat{x}$, $\hat{y}$ are estimated states and outputs of the observer, respectively, and L is the gain matrix. One of the major drawbacks of this observer is that the nonlinearities present in the systems are assumed to be globally Lipschitz. Therefore, it can not be used for the system (10).

By using the properties of Hilbert projection, locally Lipschitz functions can be extended to globally Lipschitz in $\mathbb{R}^n$. For more details, one can refer to [8]. Hence, by using the definition of Hilbert projection, we obtain:

$$f \circ \pi_{\Omega_f}(\hat{x}) = \begin{bmatrix} f_1 \circ \pi_{\Omega_{f_1}}(F_1 \hat{x}) \\ \vdots \\ f_m \circ \pi_{\Omega_{f_m}}(F_m \hat{x}) \end{bmatrix}, \; h \circ \pi_{\Omega_h}(\hat{x}) = \begin{bmatrix} h_1 \circ \pi_{\Omega_{h_1}}(H_1 \hat{x}) \\ \vdots \\ h_r \circ \pi_{\Omega_{h_r}}(H_r \hat{x}) \end{bmatrix}, \tag{13}$$

such that each $f_i \circ \pi_{\Omega_{f_i}}(\cdot)$ and $h_i \circ \pi_{\Omega_{h_i}}(\cdot)$ is globally Lipschitz in $\mathbb{R}^n$.

Let us now consider the following observer form:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + G(f \circ \pi_{\Omega_f}(\hat{x})) + Bu + L(y - \hat{y}), \\ \hat{y} &= C\hat{x} + H(h \circ \pi_{\Omega_h}(\hat{x})), \end{aligned} \tag{14}$$

where $f \circ \pi_{\Omega_f}(\hat{x})$ and $h \circ \pi_{\Omega_h}(\hat{x})$ are defined in (13).

From (10) and (14), the estimation error dynamic ($\tilde{x} = x - \hat{x}$) is given by

$$\dot{\tilde{x}} = \underbrace{(A - LC)}_{\mathbb{A}} \tilde{x} + G\Delta_f - LH\Delta_h + \underbrace{(E - LD)}_{\mathbb{E}} \omega, \tag{15}$$

where $\Delta_f = f(x) - f \circ \pi_{\Omega_f}(\hat{x})$ and $\Delta_h = h(x) - h \circ \pi_{\Omega_h}(\hat{x})$.

As system state $x(t) \in \Omega$, $\forall t \geq 0$, we have $f(x) = f(\pi_\Omega(x))$ and $h(x) = h(\pi_\Omega(x))$. It means that applying Hilbert projection $\pi_\Omega(\cdot)$ does not modify the system behaviour. Further, implementing Lemma 1 on the nonlinearities f and h leads to: There exist functions $f_{ij}^{\Omega_{f_i}} : \mathbb{R}^{\bar{n}} \times \mathbb{R}^{\bar{n}} \to \mathbb{R}$, $h_{ij}^{\Omega_{h_i}} : \mathbb{R}^{\bar{p}} \times \mathbb{R}^{\bar{p}} \to \mathbb{R}$ and constants $f_{ij\,\min}^{\Omega}, f_{ij\,\max}^{\Omega}, h_{ij\,\min}^{\Omega}, h_{ij\,\max}^{\Omega}$ such that for all $F_i x \in \Omega_{F_i}$, $H_i x \in \Omega_{H_i}$ and $\hat{x} \in \mathbb{R}^n$, $x \neq \hat{x}$, we have:

$$\Delta_f = \sum_{i,j=1}^{m,\bar{n}} f_{ij}^{\Omega_{f_i}} \mathcal{F}_{ij} F_i \tilde{x}, \text{ and } \Delta_h = \sum_{i,j=1}^{r,\bar{p}} h_{ij}^{\Omega_{h_i}} \mathcal{H}_{ij} H_i \tilde{x}, \tag{16}$$

where $f_{ij}^{\Omega_{f_i}} \triangleq f_{ij}^{\Omega}(X_i^{\hat{X}_{j-1}}, X_i^{\hat{X}_j})$, $h_{ij}^{\Omega_{h_i}} \triangleq h_{ij}^{\Omega}(X_i^{\hat{X}_{j-1}}, X_i^{\hat{X}_j})$, $\mathcal{F}_{ij} = e_m(i) e_{\bar{n}}^{\top}(j)$ and $\mathcal{H}_{ij} = e_r(i) e_{\bar{p}}^{\top}(j)$. The functions $f_{ij}^{\Omega_{f_i}}$, $h_{ij}^{\Omega_{h_i}}$ hold $f_{ij\,\min}^{\Omega} \leq f_{ij}^{\Omega_{f_i}} \leq f_{ij\,\max}^{\Omega}$ and $h_{ij\,\min}^{\Omega} \leq h_{ij}^{\Omega_{h_i}} \leq h_{ij\,\max}^{\Omega}$, respectively.

For simplicity, f_{ij}^{Ω} and h_{ij}^{Ω} are used to denote $f_{ij}^{\Omega_{f_i}}$ and $h_{ij}^{\Omega_{h_i}}$, respectively. Without loss of generality, let us assume that $f_{ij\,\min}^{\Omega} = 0$ and $h_{ij\,\min}^{\Omega} = 0$, i.e.,

$$0 \leq f_{ij}^{\Omega} \leq f_{ij\,\max}^{\Omega}, \text{ and } \ 0 \leq h^{\Omega_{ij}} \leq h_{ij\,\max}^{\Omega}. \tag{17}$$

For more details, one can refer to [5].

Therefore, the error dynamic is obtained as:

$$\dot{\tilde{x}} = \mathbb{A}\tilde{x} + \sum_{i,j=1}^{m,\bar{n}} f_{ij}^{\Omega} G \mathcal{F}_{ij} \tilde{x} - \sum_{i,j=1}^{r,\bar{p}} h_{ij}^{\Omega} L H \mathcal{H}_{ij} \tilde{x} + \mathbb{E}\omega. \tag{18}$$

The principal objective is to compute the observer gain L such that the system (18) is ISS with respect to ω.

IV. Observer design

For the simplicity of presentation, this section is divided into two parts: First, we enumerate the conditions which guarantee the ISS stability of the system (18) with respect to ω. Further, an LMI approach is derived based on these conditions.

A. ISS stability

The following theorem establishes necessary conditions for the ISS stability of system (18).

Theorem 1: The system (18) is ISS with respect to ω if it admits an ISS-Lyapunov function

$$V(\tilde{x}) = \tilde{x}^{\top} P \tilde{x}, \ P = P^{\top} > 0 \in \mathbb{R}^{n \times n}. \tag{19}$$

In addition, the trajectories of the system (18) satisfies the following bound:

$$||\tilde{x}(t)|| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} e^{-\frac{\sigma}{2}t} ||\tilde{x}(0)|| + \sqrt{\frac{\delta}{\sigma \lambda_{\min}(P)}} ||\omega(.)||_{\infty}, \tag{20}$$

for any $\mathcal{L}_{\infty}$ bounded $\omega \in \mathbb{R}^q$. Further, $\tilde{x}(t)$ is bounded when $t \to \infty$, i.e.,

$$||\tilde{x}(\infty)|| \leq \sqrt{\frac{\delta}{\sigma \lambda_{\min}(P)}} ||\omega(.)||_{\infty}. \tag{21}$$

Proof: From (19), it is easy to note that function $V(\tilde{x})$ fulfils:

$$\lambda_{\min}(P) ||\tilde{x}||^2 \leq V(\tilde{x}) \leq \lambda_{\max}(P) ||\tilde{x}||^2. \tag{22}$$

Further, the derivative of the Lyapunov function $V(\tilde{x})$ is computed as:

$$\begin{aligned} \dot{V}(\tilde{x}) = \tilde{x}^{\top} \Bigg[& P\mathbb{A} + \mathbb{A}^{\top} P + \sum_{i,j=1}^{m,\bar{n}} \Big(f_{ij}^{\Omega} P G \mathcal{F}_{ij} F_i + (f_{ij}^{\Omega} G \mathcal{F}_{ij} F_i)^{\top} P \Big) \\ & - \sum_{i,j=1}^{r,\bar{p}} \Big(h_{ij}^{\Omega} P L H \mathcal{H}_{ij} H_i + (h_{ij}^{\Omega} L H \mathcal{H}_{ij} H_i)^{\top} P \Big) \Bigg] \tilde{x} + \tilde{x}^{\top} P \mathbb{E} \omega \\ & + \omega^{\top} \mathbb{E}^{\top} P \tilde{x}. \end{aligned}$$

By using inequality (8),

$$\tilde{x}^{\top} P \mathbb{E} \omega + \omega^{\top} \mathbb{E}^{\top} P \tilde{x} \leq \delta \tilde{x}^{\top} (P\mathbb{E})^{\top} (P\mathbb{E}) \tilde{x} + \delta \omega^{\top} \omega,$$

where $\delta > 0$.

Hence,

$$\begin{aligned} \dot{V}(\tilde{x}) \leq & \ \tilde{x}^{\top} \Big[P\mathbb{A} + \mathbb{A}^{\top} P + \delta (P\mathbb{E})^{\top} (P\mathbb{E}) \Big] \tilde{x} + \delta \omega^{\top} \omega \\ & + \tilde{x}^{\top} \Bigg[\sum_{i,j=1}^{m,\bar{n}} \Big(f_{ij}^{\Omega} P G \mathcal{F}_{ij} F_i + (f_{ij}^{\Omega} G \mathcal{F}_{ij} F_i)^{\top} P \Big) - \\ & \sum_{i,j=1}^{r,\bar{p}} \Big(h_{ij}^{\Omega} P L H \mathcal{H}_{ij} H_i + (h_{ij}^{\Omega} L H \mathcal{H}_{ij} H_i)^{\top} P \Big) \Bigg] \tilde{x}. \end{aligned} \tag{23}$$

Then, we can write $\dot{V}(\tilde{x})$ as:

$$\dot{V}(\tilde{x}) \leq \tilde{x}^{\top} Q \tilde{x} + \delta \omega^{\top} \omega, \tag{24}$$

where

$$\begin{aligned} Q = & \ P\mathbb{A} + \mathbb{A}^{\top} P + \delta (P\mathbb{E})^{\top} (P\mathbb{E}) \\ & + \sum_{i,j=1}^{m,\bar{n}} \Big(f_{ij}^{\Omega} P G \mathcal{F}_{ij} F_i + (f_{ij}^{\Omega} G \mathcal{F}_{ij} F_i)^{\top} P \Big) \\ & - \sum_{i,j=1}^{r,\bar{p}} \Big(h_{ij}^{\Omega} P L H \mathcal{H}_{ij} H_i + (h_{ij}^{\Omega} L H \mathcal{H}_{ij} H_i)^{\top} P \Big). \end{aligned} \tag{25}$$

Let us consider that Q satisfies $Q \leq -\sigma P, \ \sigma > 0$.

Then, (24) is rewritten as:

$$\dot{V}(\tilde{x}) \leq -\sigma V(\tilde{x}) + \delta ||\omega||^2, \tag{26}$$

and it provides the following inequality:

$$\dot{V}(\tilde{x}) \leq -\sigma \lambda_{\max}(P) ||\tilde{x}||^2 + \delta ||\omega||^2. \tag{27}$$

From (22) and (27), $V(\tilde{x})$ fulfils the conditions (3) and (4). Hence, the Lyapunov function (19) is an ISS-Lyapunov function with $\alpha_1(\tilde{x}) = \lambda_{\min}(P) ||\tilde{x}||^2$, $\alpha_2(\tilde{x}) = \lambda_{\max}(P) ||\tilde{x}||^2$, $\alpha_3(\tilde{x}) = -\sigma \lambda_{\max}(P) ||\tilde{x}||^2$ and $\alpha_4(\omega) = \delta ||\omega||^2$.

Further, from (26), the trajectories of $V(\tilde{x})$ satisfies:

$$V(\tilde{x}(t)) \leq V(\tilde{x}_0) \mathrm{e}^{-\sigma t} + \frac{\delta}{\sigma} \big(1 - \mathrm{e}^{-\sigma t} \big) \sup_{s \in [0,t]} \|\omega(s)\|_2^2. \tag{28}$$

Since $0 \leq 1 - \mathrm{e}^{-\sigma t} \leq 1, \ \sup_{s \in [0,t]} \|\omega(s)\|_2^2 \leq \|\omega\|_{\mathcal{L}_{\infty}}^2$,

$$V(\tilde{x}(t)) \leq V(\tilde{x}_0) \mathrm{e}^{-\sigma t} + \frac{\delta}{\sigma} \|\omega\|_{\mathcal{L}_{\infty}}^2. \tag{29}$$

By using (22), we obtain:

$$||\tilde{x}(t)||^2 \leq \frac{V(\tilde{x},t)}{\lambda_{\min}(P)} \leq \frac{e^{-\sigma t} \lambda_{\max}(P) ||\tilde{x}(0)||^2 + \delta \sigma^{-1} ||\omega(.)||_{\infty}^2}{\lambda_{\min}(P)}. \tag{30}$$

From (30), we get the inequality (20) which is in the form of (2).

Since the system (18) admits an ISS-Lyapunov function (19), it is ISS with respect to ω. It is easy to obtain (21) from (20) by considering $t \to \infty$. Hence, $\tilde{x}(t)$ is bounded when $t \to \infty$ for any bounded ω.
This ends the proof. ■

Remark 1: In the absence of disturbances, i.e., $\omega = 0$, inequality (26) becomes $\dot{V}(\tilde{x}) \leq -\sigma V(\tilde{x})$, which ensures the exponential stability of $\tilde{x}$.

B. LMI design

This subsection is devoted to the development of an LMI formulation. Let us define $R = L^\top P$. From (25), the condition $Q \leq -\sigma P$ is written as:

$$\begin{aligned}
&\overbrace{\begin{bmatrix} A^\top P + PA - R^\top C - C^\top R + \sigma P & PE - R^\top D \\ (\star) & -\delta \mathbb{I} \end{bmatrix}}^{\mathbb{L}_1} + \\
&\Big[\sum_{i,j=1}^{m,\bar{n}} \Big(\underbrace{\begin{bmatrix} PG\mathcal{F}_{ij} \\ 0 \end{bmatrix}}_{\mathbb{U}_{ij}^\top} \overbrace{f_{ij}^{\Omega} \underbrace{\begin{bmatrix} F_i & 0 \end{bmatrix}}_{\mathbb{F}_i}}^{\mathbb{V}_{ij}} + \mathbb{V}_{ij}^\top \mathbb{U}_{ij} \Big)\Big] + \\
&\Big[\sum_{i,j=1}^{r,\bar{p}} \Big(\underbrace{\begin{bmatrix} -R^\top H \mathcal{H}_{ij} \\ 0 \end{bmatrix}}_{\mathbb{M}_{ij}^\top} \overbrace{h_{ij}^{\Omega} \underbrace{\begin{bmatrix} H_i & 0 \end{bmatrix}}_{\mathbb{H}_i}}^{\mathbb{N}_{ij}} + \mathbb{N}_{ij}^\top \mathbb{M}_{ij} \Big)\Big] \leq 0.
\end{aligned} \quad (33)$$

For simplicity of representation, let us consider the following notations:

$$\mathbb{U} = \begin{bmatrix} \mathbb{U}_{11}^\top & \dots & \mathbb{U}_{1\bar{n}}^\top & \dots & \mathbb{U}_{m1}^\top & \dots & \mathbb{U}_{m\bar{n}}^\top \end{bmatrix}^\top, \quad (34)$$

$$\mathbb{V} = \begin{bmatrix} \mathbb{V}_{11}^\top & \dots & \mathbb{V}_{1\bar{n}}^\top & \dots & \mathbb{V}_{m1}^\top & \dots & \mathbb{V}_{m\bar{n}}^\top \end{bmatrix}^\top, \quad (35)$$

$$\mathbb{M} = \begin{bmatrix} \mathbb{M}_{11}^\top & \dots & \mathbb{M}_{1\bar{p}}^\top & \dots & \mathbb{M}_{r1}^\top & \dots & \mathbb{M}_{r\bar{p}}^\top \end{bmatrix}^\top, \quad (36)$$

$$\mathbb{N} = \begin{bmatrix} \mathbb{N}_{11}^\top & \dots & \mathbb{N}_{1\bar{p}}^\top & \dots & \mathbb{N}_{r1}^\top & \dots & \mathbb{N}_{r\bar{p}}^\top \end{bmatrix}^\top, \quad (37)$$

where $\mathbb{U}_{ij}$, $\mathbb{V}_{ij}$, $\mathbb{M}_{ij}$ and $\mathbb{N}_{ij}$ are described in (33).
With these notations,

$$\sum_{i,j=1}^{m,\bar{n}} \big(\mathbb{U}_{ij}^\top \mathbb{V}_{ij} + \mathbb{V}_{ij}^\top \mathbb{U}_{ij} \big) = \mathbb{U}^\top \mathbb{V} + \mathbb{V}^\top \mathbb{U}, \quad (38)$$

$$\sum_{i,j=1}^{r,\bar{p}} \big(\mathbb{M}_{ij}^\top \mathbb{N}_{ij} + \mathbb{N}_{ij}^\top \mathbb{M}_{ij} \big) = \mathbb{M}^\top \mathbb{N} + \mathbb{N}^\top \mathbb{M}. \quad (39)$$

The following inequalities are derived by applying inequality (9) on (38) and (39):

$$\mathbb{U}^\top \mathbb{V} + (\mathbb{V})^\top \mathbb{U} \leq \frac{1}{2}\big[(\mathbb{U} + \mathbb{Z}\mathbb{V})^\top \mathbb{Z}^{-1} (\mathbb{U} + \mathbb{Z}\mathbb{V}) \big], \quad (40)$$

$$\mathbb{M}^\top \mathbb{N} + \mathbb{N}^\top \mathbb{M} \leq \frac{1}{2}\big[(\mathbb{M} + \mathbb{S}\mathbb{N})^\top \mathbb{S}^{-1} (\mathbb{M} + \mathbb{S}\mathbb{N}) \big], \quad (41)$$

where $\mathbb{Z}$ and $\mathbb{S}$ are defined in (31) and (32), respectively.

Therefore, inequality (33) holds if

$$\begin{aligned}
&\mathbb{L}_1 + \frac{1}{2}\big[(\mathbb{U} + \mathbb{Z}\mathbb{V})^\top \mathbb{Z}^{-1} (\mathbb{U} + \mathbb{Z}\mathbb{V}) \big] \\
&\quad + \frac{1}{2}\big[(\mathbb{M} + \mathbb{S}\mathbb{N})^\top \mathbb{S}^{-1} (\mathbb{M} + \mathbb{S}\mathbb{N}) \big] \leq 0.
\end{aligned} \quad (42)$$

From (17), each element inside $\mathbb{U}$ and $\mathbb{N}$ is bounded and belongs to convex sets, and their sets of vertices are given by

$$\mathcal{V}_{\mathcal{F}_m} = \big\{ \{F_{11}, \dots, F_{1\bar{n}}, \dots, F_{m1}, \dots, F_{m\bar{n}}\} : F_{ij} \in [0, f_{ij_{\max}}^{\Omega}] \big\},$$
$$\mathcal{V}_{\mathcal{H}_r} = \big\{ \{H_{11}, \dots, H_{1\bar{p}}, \dots, H_{r1}, \dots, H_{r\bar{p}}\} : H_{ij} \in [0, h_{ij_{\max}}^{\Omega}] \big\}.$$

Hence, (42) is rewritten as:

$$\begin{aligned}
&\mathbb{L}_1 + \frac{1}{2}\big[(\mathbb{U} + \mathbb{Z}\mathbb{V})^\top \mathbb{Z}^{-1} (\mathbb{U} + \mathbb{Z}\mathbb{V}) \big]_{\forall \mathbb{V} \in \mathcal{F}_m} \\
&\quad + \big[\frac{1}{2} (\mathbb{M} + \mathbb{S}\mathbb{N})^\top \mathbb{S}^{-1} (\mathbb{M} + \mathbb{S}\mathbb{N}) \big]_{\forall \mathbb{N} \in \mathcal{H}_r} \leq 0.
\end{aligned} \quad (43)$$

Theorem 2: The system (18) is ISS with respect to ω. if it holds the following optimization problem:
minimize δ subject to,

$$\begin{bmatrix} \mathbb{L}_1 & (\mathbb{U} + \mathbb{Z}\mathbb{V})^\top & (\mathbb{M} + \mathbb{S}\mathbb{N})^\top \\ \star & -2\mathbb{Z} & 0 \\ \star & \star & -2\mathbb{S} \end{bmatrix}_{<0, \forall \mathbb{V} \in \mathcal{F}_m, \forall \mathbb{N} \in \mathcal{H}_r} \quad (44)$$

where $P = P^\top > 0 \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{p \times n}$, $\mathbb{Z}$ under the form of (31), $\mathbb{S}$ described in (32), and $\sigma, \delta > 0$. The matrices $\mathbb{L}_1$, $\mathbb{U}$, $\mathbb{V}$, $\mathbb{M}$ and $\mathbb{N}$ are defined in (33), (34), (35), (36) and (37), respectively. The observer gain is computed as $L = P^{-1} R^\top$.

Proof: The Schur's compliment of (43) yields the LMI (44). From convexity principal [13], the system (18) holds (26) if the LMI (44) is solved for all $\mathbb{V} \in \mathcal{F}_m$ and $\mathbb{N} \in \mathcal{G}_r$. Further from Theorem 1, the system (18) is ISS with respect to ω. ■

V. IMPLEMENTATION

In this section, a numerical example is provided to demonstrate the effectiveness of the proposed methodology. Further, observer performances are evaluated by implementing it to the slip angle estimation for the nonlinear autonomous vehicle model.

A. Example 1: Numerical example

Consider the system under the form of (10) with:
$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -5 \end{bmatrix}, G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, E = \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix},$
$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $f_1(x) = x_2^2$, $f_2(x) = x_2 x_3$, $h_1(x) = sin(x_1)$ and $h_2(x) = sin(x_3)$. Hence, $m = 2, r = 2$. Let us consider $\bar{n} = 3, \bar{p} = 3$.

For initial condition $x(0) = \begin{bmatrix} -1 & -1 & -1 \end{bmatrix}^\top$ and input $u = square(2t)$, the states are bounded in the following sets:

$$\begin{aligned}
x \in \{ &-10.2929 \leq x_1 \leq -1, -1.0739 \leq x_2 \leq 0.2935, \\
&\text{and } -1 \leq x_3 \leq 0.4686 \} \ \forall t \in [0, 30].
\end{aligned}$$

Let us consider that system dynamics and measurements are corrupted with the Gaussian noise ($\omega \rightsquigarrow (0,1)$). The feasibility of LMI (44) is tested in each of the following cases:

$$\mathbb{Z}=\begin{bmatrix}\mathbb{Z}_1 & \mathbb{Z}_{b_2^1} & \dots & \mathbb{Z}_{b_m^1}\\ \star & \mathbb{Z}_2 & \dots & \mathbb{Z}_{b_m^2}\\ & & \ddots & \vdots\\ \star & \star & & \\ \star & \star & \dots & \mathbb{Z}_m\end{bmatrix},\text{where } \mathbb{Z}_i=\begin{bmatrix}Z_{i1} & Z_{a_{i2}^1} & \dots & Z_{a_{i\bar{n}}^1}\\ \star & Z_{i2} & \dots & Z_{a_{i\bar{n}}^2}\\ & & \ddots & \vdots\\ \star & \star & & \\ \star & \star & \dots & Z_{i\bar{n}}\end{bmatrix}\ \forall i\in\{1,\dots,m\},\ \mathbb{Z}_{b_i^j}=\begin{bmatrix}Z_{b_{i1}^{j1}} & Z_{b_{i2}^{j1}} & \dots & Z_{b_{i\bar{n}}^{j1}}\\ Z_{b_{i1}^{j2}} & Z_{b_{i2}^{j2}} & \dots & Z_{b_{i\bar{n}}^{j2}}\\ \vdots & \vdots & \ddots & \vdots\\ Z_{b_{i1}^{j\bar{n}}} & Z_{b_{i2}^{j\bar{n}}} & \dots & Z_{b_{i\bar{n}}^{j\bar{n}}}\end{bmatrix}\ \forall i\in\{2,\dots,m\}, j\in\{1,\dots,m-1\}, \tag{31}$$

such that $\mathbb{Z}=\mathbb{Z}^\top>0$ and $Z_{ij}=Z_{ij}^\top\in\mathbb{R}^{\bar{n}\times\bar{n}},\ Z_{a_{ij}^k}=Z_{a_{ij}^k}^\top\in\mathbb{R}^{\bar{n}\times\bar{n}}\forall i,k\in\{1,\dots,m\},\&j\in\{1,\dots,\bar{n}\};\ Z_{b_{ij}^{kj}}=Z_{b_{ij}^{kj}}^\top\in\mathbb{R}^{\bar{n}\times\bar{n}},\forall i\in\{2,\dots,m\},k\in\{1,\dots,m-1\},\&j\in\{1,\dots,\bar{n}\}$

$$\mathbb{S}=\begin{bmatrix}\mathbb{S}_1 & \mathbb{S}_{b_2^1} & \dots & \mathbb{S}_{b_r^1}\\ \star & \mathbb{S}_2 & \dots & \mathbb{S}_{b_r^2}\\ & & \ddots & \vdots\\ \star & \star & & \\ \star & \star & \dots & \mathbb{S}_r\end{bmatrix},\text{where } \mathbb{S}_i=\begin{bmatrix}S_{i1} & S_{a_{i2}^1} & \dots & S_{a_{i\bar{p}}^1}\\ \star & S_{i2} & \dots & S_{a_{i\bar{p}}^2}\\ & & \ddots & \vdots\\ \star & \star & & \\ \star & \star & \dots & S_{i\bar{p}}\end{bmatrix}\ \forall i\in\{1,\dots,r\},\ \mathbb{S}_{b_i^j}=\begin{bmatrix}S_{b_{i1}^{j1}} & S_{b_{i2}^{j1}} & \dots & S_{b_{i\bar{p}}^{j1}}\\ S_{b_{i1}^{j2}} & S_{b_{i2}^{j2}} & \dots & S_{b_{i\bar{p}}^{j2}}\\ \vdots & \vdots & \ddots & \vdots\\ S_{b_{i1}^{j\bar{p}}} & S_{b_{i2}^{j\bar{p}}} & \dots & S_{b_{i\bar{n}}^{j\bar{p}}}\end{bmatrix}\ \forall i\in\{2,\dots,r\}, j\in\{1,\dots,r-1\}, \tag{32}$$

such that $\mathbb{S}=\mathbb{S}^\top>0$ and $S_{ij}=S_{ij}^\top\in\mathbb{R}^{\bar{p}\times\bar{p}},\ S_{a_{ij}^k}=S_{a_{ij}^k}^\top\in\mathbb{R}^{\bar{p}\times\bar{p}}\forall i,k\in\{1,\dots,r\},\&j\in\{1,\dots,\bar{p}\};\ S_{b_{ij}^{kj}}=S_{b_{ij}^{kj}}^\top\in\mathbb{R}^{\bar{p}\times\bar{p}},\forall i\in\{2,\dots,r\},k\in\{1,\dots,r-1\},\&j\in\{1,\dots,\bar{p}\}$

TABLE I: Optimal values of γ,δ and gain matrix L for different cases

	Case 1	**Case 2**	**Case 3**
δ	5.5499×10^{-18}	0.6160	0.4237
$\gamma=\sqrt{\frac{\delta}{\lambda_{\min}(P)}}$	3.0326×10^{-12}	2.4819	2.0585
Gain matrix L	$\begin{bmatrix}260.7229 & 0.0000\\ 149.7926 & 0\\ -17.8321 & 0.5000\end{bmatrix}$	$\begin{bmatrix}0.8643 & 1.4132\\ 0.9302 & 1.7084\\ -0.2009 & -0.3696\end{bmatrix}$	$\begin{bmatrix}4.4501 & 1.1749\\ 2.4572 & 0.7608\\ -0.6503 & 0.1732\end{bmatrix}$

1) **Case 1:** LMI (44) with proposed matrix multipliers,

$$\mathbb{Z}=\begin{bmatrix}Z_{11} & Z_{b_{21}} & Z_{b_{22}}\\ Z_{b_{21}} & Z_{21} & Z_{a_{22}}\\ Z_{b_{22}} & Z_{a_{22}} & Z_{22}\end{bmatrix};\mathbb{S}=\begin{bmatrix}S_{11} & S_{b_{21}}\\ S_{b_{21}} & S_{21}\end{bmatrix}, \tag{45}$$

where $Z_{ij},Z_{b_{ij}},Z_{a_{ij}}\in\mathbb{R}^{\bar{n}\times\bar{n}}, S_{ij},S_{b_{ij}}\in\mathbb{R}^{\bar{p}\times\bar{p}},\forall i,j\in\{1,2\}$ are symmetric matrices such that $\mathbb{Z}>0$ and $\mathbb{S}>0$.

2) **Case 2:** LMI (44) with the following matrices which are similar to [7]:

$$\mathbb{Z}=\begin{bmatrix}Z_{11} & \alpha Z_{21} & \alpha Z_{22}\\ \alpha Z_{21} & Z_{21} & \alpha Z_{22}\\ \alpha Z_{22} & \alpha Z_{22} & Z_{22}\end{bmatrix};\mathbb{S}=\begin{bmatrix}S_{11} & \beta S_{21}\\ \beta S_{21} & S_{21}\end{bmatrix}, \tag{46}$$

where $\alpha,\beta=0.5$, and all other matrices are described in (45).

3) **Case 3:** LMI (44) with block-diagonal matrix (same as in [5]), i.e.,

$$\mathbb{Z}=\text{block-diag}(Z_{11},Z_{21},Z_{22});\mathbb{S}=\text{block-diag}(S_{11},S_{21}), \tag{47}$$

where all matrices are defined in (45).

All LMIs defined in the above cases are solved by using MATLAB toolbox. The estimated optimal values of δ, γ and gain matrix L are summarized in Table I. It emphasizes that the LMI (44) provides the best ISS gain with the proposed matrix multipliers as compared to other cases. The proposed matrix multipliers contain more decision variables than the other multipliers described in [5], [6] and [7]. Hence, it is obvious that the solution provided by the matrices of (45) is more general than other cases. Therefore, the introduction of general matrix multipliers relaxes the existing LMI conditions from a feasibility point of view.

TABLE II: RMSE values of $\tilde{x}$ for different cases

Time		Case 1	Case 2	Case 3
$15\le t\le30$	$\tilde{x}_1$	2.1450×10^{-9}	0.0454	0.0252
	$\tilde{x}_2$	5.5216×10^{-7}	0.0254	0.0126
	$\tilde{x}_3$	5.5216×10^{-7}	0.0254	0.0126

Further, the proposed observer (14) is simulated in MATLAB. For each case, RMSE values of state estimation errors are calculated in steady state and presented in Table II. It interprets that the designed observer performs more efficiently in Case 1 as compared to the other cases.

B. Example 2: Application to slip angle estimation

In this subsection, the proposed observer is implemented for the estimation of the slip angle of a nonlinear autonomous vehicle model. The nonlinear vehicle model is represented in the form of (10) with the following parameters: $x=\begin{bmatrix}\alpha_f\\ \alpha_r\end{bmatrix}$,
$A=\begin{bmatrix}-(\frac{u_x}{a+b}+\frac{a^2c_{1f}}{I_zu_x}) & (\frac{u_x}{a+b}+\frac{abc_{1r}}{I_zu_x})\\ -(\frac{u_x}{a+b}-\frac{abc_{1f}}{I_zu_x}) & (\frac{u_x}{a+b}-\frac{b^2c_{1r}}{I_zu_x})\end{bmatrix}$, $B=\begin{bmatrix}\frac{u_x}{a+b} & 1 & \frac{-1}{u_x}\\ \frac{u_x}{a+b} & 0 & \frac{-1}{u_x}\end{bmatrix}$,
$G=\begin{bmatrix}\frac{a^2}{I_zu_x} & -\frac{ab}{I_zu_x}\\ -\frac{ab}{I_zu_x} & \frac{b^2}{I_zu_x}\end{bmatrix}$, $C=\begin{bmatrix}-\frac{u_x}{a+b} & \frac{u_x}{a+b}\\ \frac{c_{1f}}{m} & \frac{c_{1r}}{m}\end{bmatrix}$, $H=\begin{bmatrix}0 & 0\\ \frac{-1}{m} & \frac{-1}{m}\end{bmatrix}$,
$E=\begin{bmatrix}1\\ 1\end{bmatrix}$ and $D=\begin{bmatrix}1\\ 1\end{bmatrix}$. $f(x)=\begin{bmatrix}-\eta(\alpha_f)\\ -\eta(\alpha_r)\end{bmatrix}$ and $h(x)=\begin{bmatrix}-\eta(\alpha_f)\\ -\eta(\alpha_r)\end{bmatrix}$ are the nonlinearities present in the dynamics and outputs, respectively, where, $\eta(\zeta)=-c_2\zeta^2\,\text{sgn}(\zeta)+c_3\zeta^3$. The state vectors α_f and α_r denote the front slip angle and rear slip angle of the tires of a vehicle. $u=\begin{bmatrix}\delta & \dot{\delta} & \dot{a}_y\end{bmatrix}^\top$ is input of the system, where δ is the steering

angle and a_y is the lateral acceleration of the vehicle. The slip angle β of the vehicle is computed as follows: $\beta = \frac{rb}{u_x} - \alpha_r$. The values of the parameters of the model are illustrated as m (mass of the vehicle): $1573\,\mathrm{kg}$; I_z (Inertia of the vehicle): $2873\,\mathrm{kg}\cdot\mathrm{m}^2$; a (distance of front tires from c.g.): $1.1\,\mathrm{m}$; b (distance of rear tires from c.g.): $1.58\,\mathrm{m}$; u_x (longitudinal velocity): $10\ \mathrm{m}\cdot\mathrm{s}^{-1}$; r (Yaw rate) : $0.8\,\mathrm{rad}\cdot\mathrm{s}^{-1}$; $c_2, c_3 = 8000$; $c_{1f}, c_{1r} = 80000$.

Let us consider that both dynamics and measurements are contaminated by the Gaussian noise ($\omega \rightsquigarrow (0,1)$). The proposed LMI (44) is solved in MATLAB toolbox for the above system, and we obtain: $\delta = 1.969\times 10^{-20}$, $\gamma = 9.7954\times 10^{-15}$ and $L = \begin{bmatrix} 0.0535 & 0.9465 \\ 0.3580 & 0.6420 \end{bmatrix}$.

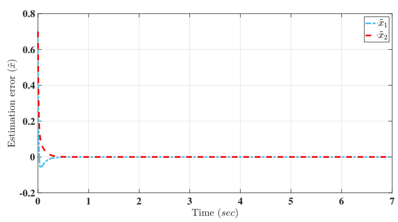
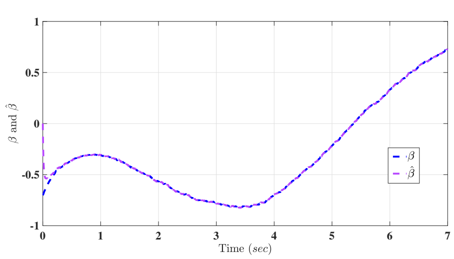
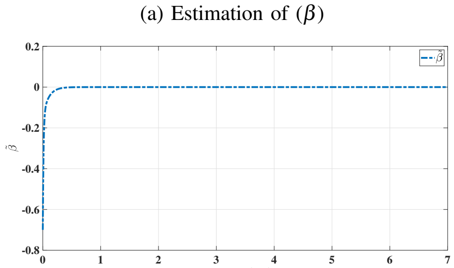


Fig. 1: Behaviour of estimation error ($\tilde{x}$)

In MATLAB, the observer (14) is simulated for the vehicle model. The graph of the estimation error of vehicle states $\tilde{x}$ is presented in Figure 1. It emphasises the efficient noise attenuation which is achieved by the proposed observer. Further, the plots of the estimation of slip angle β and its error are shown in Figure 2. The estimation accuracy is illustrated in Figure 2a and 2b. All these figures highlight the precise estimation of the slip angle of the vehicle with efficient noise compensation.



(a) Estimation of (β)



(b) Estimation error of slip angle ($\tilde{\beta} = \beta - \hat{\beta}$)

Fig. 2: Estimation of slip angle (β)

VI. Conclusion

A new LMI-based observer approach for the class of a locally Lipschitz nonlinear system is proposed in this paper for the state estimation purpose. A recently proposed Hilbert transformation is used in the observer structure to handle the same class of nonlinearities. An ISS notion is employed to ensure the boundedness of the estimation error. Further, the LPV approach is combined with the reformulated Lipschitz property, a variant of Young inequality, and newly defined matrix multipliers to derive a new LMI condition. The use of newly defined matrix multipliers allows the inclusion of additional decision variables as compared to the methods proposed in the literature. The effectiveness of the developed LMI is highlighted through a numerical example. The proposed methodology is evaluated by implementing it for slip angle estimation in the autonomous vehicle model.

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