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Characterizing Compositionality of LQR from the Categorical Perspective

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***Abstract*—Composing systems is a fundamental concept in modern control systems, yet it remains challenging to formally analyze how controllers designed for individual subsystems can differ from controllers designed for the composition of those subsystems. To address this challenge, we propose a novel approach to composing control systems based on *resource sharing machines*, a concept from applied category theory. We use resource sharing machines to investigate the differences between (i) the linear-quadratic regulator (LQR) designed directly for a composite system and (ii) the LQR that is attained through the composition of LQRs designed for each subsystem. We first establish novel formalisms to compose LQR control designs using resource sharing machines. Then we develop new sufficient conditions to guarantee that the LQR designed for a composite system is equal to the LQR attained through composition of LQRs for its subsystems. In addition, we reduce the developed condition to that of checking the controllability and observability of a certain linear, time-invariant system, which provides a simple, computationally efficient procedure for evaluating the equivalence of controllers for composed systems.**

I. Introduction

Modern technology has made it simple to design and construct large-scale dynamical systems through the composition of many subsystems. For example, smart communities may combine control systems such as networks of smart grids, water distribution systems, and transportation systems to provide rich, complex capabilities. It is therefore crucial to understand how to perform control design for subsystems and compose these controllers for the composite system[1].

In the control community, it has been observed that many systems can be analyzed by analyzing their subsystems [1], where some properties of the subsystems are inherited by the composite system. For example, the stability of composite feedback systems can be analyzed based on their subsystems using the small-gain theorem and composed Lyapunov functions [2], and parallel interconnections of passive systems will induce a composite passive system [3]. Additionally, [4]–[6] have explored how controllability of networks can be explored through their sub-networks. However, the lack of formal analytical language in composing control systems makes it challenging to further advance and unify this work.

Recent advances in hybrid systems have used *applied category theory* [7] as a formal language for describing the composition of continuous- and discrete-time systems [8]–[11]. Applied category theory is also being used in mathematical modeling [12], scientific computing [13], data science [14], and more. Roughly speaking, applied category theory focuses on the abstract connections between objects, rather than studying the objects themselves [15], [16]. Although category theory has found success in various areas of engineering, there has been relatively limited work on using it to study compositions of control systems; a representative sample of existing works includes [17]–[19].

We focus on exploring a specific way of composing dynamical systems known as *resource sharing machines* (RSMs) [20], [21], which are defined in the language of applied category theory. RSMs present a way of composing dynamical systems in which, the states of the composite system can preserve the states of the subsystems, and vice versa. Thus, the use of RSMs can enable new analyses for a composite system through combining analyses originally designed for its subsystems [20], [21].

In this paper, we study the relationship between (i) the composition of the control designs of a collection of subsystems and (ii) control designs made directly for the composition of those subsystems. In both cases, the composition of subsystems is formalized by RSMs. In particular, we will focus on the linear-quadratic regulator (LQR) because it is a cornerstone of control theory [22] and the focus of ongoing research [23], [24]. Additionally, LQR has a closed-form solution, which lets us study the *compositionality* of the LQR, which we define as the property that the composition of LQR controllers designed for subsystems is equivalent to the LQR controller designed for the composition of those subsystems[2].

To summarize, our contributions are:

- We present a novel approach to composing control systems using resource sharing machines (RSMs). Specifically, we compose LQR control designs using RSMs.
- We show that LQR is *not* always compositional.

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[1]Throughout this paper, we use the term *composite system* to refer to a system that is formed through the composition of many subsystems.

[2]A formal mathematical definition of "compositionality" in this context is given in category-theoretic terms in Section II-A.

- We introduce a new, generalized Riccati equation, and we use it to derive new sufficient conditions under which the compositionality of LQR is guaranteed.
- We reduce the problem of validating compositionality of LQR designs to that of checking the controllability and observability of a certain linear, time-invariant system.

The rest of the paper is structured as follows. Section II provides background and problem statements. Section III provides results on composing linear control systems through resource sharing machines (RSMs). Section IV examines the compositionality of LQR via RSMs. Section V concludes.

Notation: We use $\mathbb{R}$ and $\mathbb{N}$ to denote the real and natural numbers, respectively. All vectors are columns. We use δ_i to denote a basis vector with i^{th} entry 1 and all others 0. We define $\underline{n} := \{1, 2, 3, \ldots, n\}$. For a matrix A, we use $A_{:,j}$ and $A_{i,:}$ to denote its j^{th} column and i^{th} row, respectively. We use $x(t)$ and x interchangeably to denote a state vector at time t. We use the terms "variable" and "state" interchangeably. We use $\mathbf{0}$ to denote a vector or matrix of zeroes, and its dimension will be clear from context. Note that due to page limit, we include all proofs in [25].

II. Preliminaries and Problem Formulation

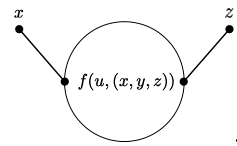
In this section, we first provide background from resource sharing machines and the composition of linear systems. Then we state the problems that are the focus of this paper.

A. Applied Category Theory and Resource Sharing Machines

In this paper, we focus on the compositional properties of LQR control design for coupled dynamical systems. In this section, we present a graphical calculus for specifying such coupled systems based on "resource sharing" as defined in [20]. We extend it to closed-loop control designs and define what it means for an LQR controller to be *compositional*.

Definition 1. *Let $X = \mathbb{R}^n$ and $U = \mathbb{R}^m$ for some $m, n \in \mathbb{N}$. The symbol $TX \cong X \times X$ denotes the tangent bundle of X. An* ***open-loop control system*** *is a U-parameterized vector field on X, i.e., a smooth function $f\colon U \times X \to TX$, where $f(u, x) = (x, d)$ for all $x \in X$ and $u \in U$ [26]. Here $d \in X$ is the tangent vector at the point x. The set X is called the* ***state space*** *of the system and U is called the* ***control surface****. Such a vector field defines a system of ordinary differential equations $\dot{x} = f(x, u)$, which we call the* ***dynamics*** *of the system. In the case of a linear open-loop control system, the ODE system has the form $\dot{x} = Ax + Bu$ for some $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$.*

We use the term *subsystem* to refer to a system that is not a composition of other systems. To model coupled dynamical systems, each open-loop subsystem must specify its *boundary*, a subspace of its state space designating which variables can couple to other systems. The intuition is that given some open-loop control subsystems with boundary variables, we can "glue" the subsystems together along shared boundary variables to form a composite system. We can represent an open-loop control subsystem with a boundary graphically. For example, consider a system S_1 with state space $\mathbb{R}^3$, control surface $\mathbb{R}$, and dynamics $(\dot{x}, \dot{y}, \dot{z}) = f(u, (x, y, z))$, where x and z are boundary variables. This is represented by the following *undirected wiring diagram* (UWD):



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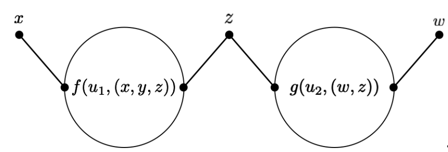
The circle represents the open-loop subsystem defined by f, and the wires represent its boundary variables, x and z.

B. Composing Open-Loop Control Subsystems

To compose open-loop control subsystems diagrammatically, we connect the wires for the boundary variables that are coupled. For example, suppose we want to compose S_1 with another subsystem S_2 that has state space $\mathbb{R}^2$, control surface $\mathbb{R}^2$, and dynamics $(\dot{w}, \dot{z}) = g(u_2, (w, z))$ with both of its state variables in its boundary. Consider coupling the third state of S_1 (i.e., z in f) with the second state of S_2 (i.e., z in g), we have the resultant UWD shown below:



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Note that we consider compositions of two subsystems for simplicity; our results generalize to any number of subsystems in an obvious way.

A benefit of this graphical syntax is that it uniquely defines the dynamics of the composite system based on the dynamics of the subsystems. For linear systems, the composition is achieved by the construction of a *composition matrix* K. To define it, consider two open-loop linear control subsystems S_1 and S_2, where S_1 has state space $\mathbb{R}^{n_1}$, control surface $\mathbb{R}^{m_1}$, and dynamics

$$\dot{x}^1(t) = A^1 x^1(t) + B^1 u^1(t), \tag{1}$$

and S_2 has state space $\mathbb{R}^{n_2}$, control surface $\mathbb{R}^{m_2}$, and dynamics

$$\dot{x}^2(t) = A^2 x^2(t) + B^2 u^2(t). \tag{2}$$

Note that $x^i \in \mathbb{R}^{n_i}$, $A^i \in \mathbb{R}^{n_i \times n_i}$, $B^i \in \mathbb{R}^{n_i \times m_i}$, $u^i \in \mathbb{R}^{m_i}$, $i \in \underline{2}$. Suppose further that S_1 and S_2 share $k \leq \min\{n_1, n_2\}$ states defined by the composition pattern of some UWD. This sharing allows us to define a lower-dimensional state vector for the composite system, denoted $\bar{x}$. That is, the composition pattern between S_1 and S_2 defines an identification between the new state variable $\bar{x} \in \mathbb{R}^{n_1+n_2-k}$ and the state variables of S_1 and S_2 themselves. The composition pattern can be used to construct the composition matrix $K \in \mathbb{R}^{(n_1+n_2)\times(n_1+n_2-k)}$ as follows:

- if $\bar{x}_i$ is a non-shared variable and $\bar{x}_i = x^1_j$ for some $j \in \underline{n_1}$ then we have $K_{j,:} = \delta_i^\top$,
- if $\bar{x}_i$ is a non-shared variable and $\bar{x}_i = x^2_k$ for some $k \in \underline{n_2}$ then we have $K_{k+n_1,:} = \delta_i^\top$,
- if $\bar{x}_i$ is a shared variable with $\bar{x}_i = x^1_j = x^2_k$ for some $j \in \underline{n_1}$ and $k \in \underline{n_2}$, then we have $K_{j,:} = K_{k+n_1,:} = \delta_i^\top$.

We can now define the composition of open-loop linear subsystems based on the UWD syntax.

Definition 2 (Composition of linear open-loop control subsystems)**.** *Let S_1 and S_2 be the linear open-loop control subsystems in* (1) *and* (2)*, respectively. Suppose we have computed the composition matrix K based on a UWD defining S_1 and S_2's composition pattern. The composite system $S_1 \Box S_2$ has state space $\mathbb{R}^{n_1+n_2-k}$, control surface $\mathbb{R}^{m_1+m_2}$, and dynamics*

$$\dot{\bar{x}}(t) = \underbrace{K^\top \underbrace{\begin{bmatrix} A^1 & \mathbf{0} \\ \mathbf{0} & A^2 \end{bmatrix}}_{\bar{A}} K}_{\mathcal{A}} \, \bar{x}(t) + \underbrace{K^\top \underbrace{\begin{bmatrix} B^1 & \mathbf{0} \\ \mathbf{0} & B^2 \end{bmatrix}}_{\bar{B}}}_{\mathcal{B}} \underbrace{\begin{bmatrix} u^1(t) \\ u^2(t) \end{bmatrix}}_{u(t)} . \tag{3}$$

Remark 1. *We point out two aspects of* (3)*. First, the dynamics are not separable because the system matrix of* (3) *is $K^\top \bar{A} K$, which is non-diagonal. Second, control inputs are not shared. Instead, the control input of a shared state variable is obtained by summing up the control inputs from the subsystems that share that state.*

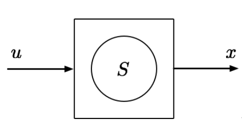
Remark 2. *The composition matrix K encodes the resource sharing machine between two linear subsystems. In* (3)*, the term $K\bar{x}(t)$ replicates the shared variables while preserving the non-shared variables, and it maps the replicated shared variables to the dynamics of each of the two subsystems. Using this formulation, the dynamics of a shared variable in the composite system can be obtained by summing the dynamics of the corresponding duplicated variables over each subsystem in which they appear. This is done in the term $K^\top \bar{A} K \bar{x}(t)$. On the other hand, the dynamics of a non-shared variable in the composite system are identical to those of the corresponding variable in the subsystem.*

To reason about LQR on composite systems, we need a formal setting for reasoning about feedback systems. For this, we use the notion of (I, O)-Systems developed in [26].

Definition 3. *Let I and O be Euclidean spaces. An (I, O)-**System** is a 3-tuple (X, f^{upd}, f^{rdt}), where*

- *X is the system's state space and is a Euclidean space,*
- *f^{upd} is an I-parameterized vector field $I \times X \to TX$ called the update function defining the system dynamics,*
- *f^{rdt} is a function $X \to O$ called the readout function.*

Observe that given an open-loop control system S with state space X, control surface U, and dynamics f, we can construct a (U, X)-System as the 3-tuple (X, f, id_X), where the readout map id_X is the identity map on X, which corresponds to a system's output simply equaling its state vector. We call this construction a "boxed open-loop system," and we represent it graphically by wrapping an open-loop control system in a box with input and output wires:



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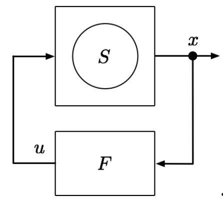
C. Compositionality of LQR

We can now define closed-loop feedback control subsystems for boxed open-loop subsystems.

Definition 4. *Given a (U, X)-System $(X, f \colon X \times U \to TX, id_X)$ and a full state feedback controller $F \colon X \to U$, the **closed-loop feedback control system** defined by this data is a $(\emptyset, X)$-System with state space X, update function given by $x \mapsto f(F(x), x)$, and readout function id_X. Graphically, we represent the system by*



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In the linear case, this gives $\dot{x} = Ax + BFx$, where $F \in \mathbb{R}^{m \times n}$ is a gain matrix. Note that the input space is the empty set to signify the closed-loop nature of the system, i.e., each input is uniquely determined by the system's state.

In light of the above definitions, we now have two ways of designing LQR controllers for composite dynamical systems: we can either design controllers for each subsystem individually, then compose the resultant controllers, or compose the subsystem dynamics, and design a single controller for the overall composite system. This observation sets up the central definition of the paper.

Definition 5 (Compositionality of LQR)**.** *Given open-loop subsystems S_1 and S_2, an LQR controller is **compositional** if the following diagram commutes:*

$$\begin{tikzcd} S_1, S_2 \arrow[r, "LQR{,}LQR"] \arrow[d, "compose"'] & LQR(S_1), LQR(S_2) \arrow[d, "compose"] \\ S_1 \Box S_2 \arrow[r, "LQR"'] & LQR(S_1 \Box S_2) \end{tikzcd}$$

where "LQR" denotes the synthesis of an LQR controller, and "compose" denotes the composition of open-loop systems on the left of the diagram and composition of closed-loop systems on the right of the diagram.

Unpacking this definition, we have that in order for an LQR controller to be compositional for given subsystems S_1 and S_2, the controller of the composite of the subsystems must be the same as composing the controllers designed for S_1 and S_2 individually.

D. Problem Statements

In light of the above, we must formalize the composition of closed-loop systems, then evaluate the compositionality of their properties. In particular, we will solve the following:

Problem 1. *Under the definition of closed-loop control systems in Definition 4, how can we compose closed-loop control inputs using the framework of resource sharing machines?*

Problem 2. *Under the definition of compositionality in Definition 5, does compositionality of stability, which holds* *for systems without control, also hold when control inputs are present? That is, must the composition of stable subsystems give a stable composite system?*

Problem 3. *Under what conditions will the compositionality of the LQR design given by Definition 5 be preserved when using the resource sharing machines formulation?*

We will solve Problems 1-3 in the next two sections.

E. Illustrative Example

We close this section with the following example to illustrate the composition of two linear systems with no control inputs through the composition matrix K; we omit inputs for simplicity, but composition of systems with inputs can be done using the same composition matrix K constructed in the example (as shown in Definition 2).

Example 1. *Consider the following two subsystems*

$$\underbrace{\begin{bmatrix} \dot{x}_1^i(t) \\ \dot{x}_2^i(t) \end{bmatrix}}_{\dot{x}^i(t)} = \underbrace{\begin{bmatrix} a_{11}^i & a_{12}^i \\ a_{21}^i & a_{22}^i \end{bmatrix}}_{A^i} \underbrace{\begin{bmatrix} x_1^i(t) \\ x_2^i(t) \end{bmatrix}}_{x^i(t)}, \tag{4}$$

where $i \in \underline{2}$. In this example, the two subsystems share x_2^1 and x_1^2, i.e., $x_2^1 = x_1^2 = \bar{x}_2$ in the composed system. Based on (3)*, we have the composite system*

$$\underbrace{\begin{bmatrix} \dot{\bar{x}}_1(t) \\ \dot{\bar{x}}_2(t) \\ \dot{\bar{x}}_3(t) \end{bmatrix}}_{\dot{\bar{x}}(t)} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{K^\top} \underbrace{\begin{bmatrix} A^1 & 0 \\ 0 & A^2 \end{bmatrix}}_{\bar{A}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \\ \bar{x}_3(t) \end{bmatrix}}_{\bar{x}(t)}. \tag{5}$$

Note that in the composite system, the composition matrix K duplicates the shared state $\bar{x}_2$ twice, and maps the copies to the dynamics of the two subsystems separately. Meanwhile, K maps the non-shared state $\bar{x}_1(t)$ to x_1^1, and it maps the non-shared state $\bar{x}_3(t)$ to x_2^2. After updating the states through A_1 and A_2, we compose the changes of all states through $K^\top$. The matrix $K^\top$ maps the dynamics of the non-shared variable $\dot{x}_1^1$ to $\dot{\bar{x}}_1$. It also maps the dynamics of the non-shared variable $\dot{x}_2^2$ to $\dot{\bar{x}}_3$. For dynamics of the shared variable $\dot{\bar{x}}_2(t)$, the matrix $K^\top$ sums the dynamics of the variables from the subsystems, i.e., $\dot{\bar{x}}_2(t) = \dot{x}_2^1(t) + \dot{x}_1^2(t)$.

Non-shared states have the same dynamics as the corresponding states in the subsystems. Shared states, on the other hand, are duplicated and summed, leading to identical values for them within the subsystems that share them. Further, the dynamics of a shared state are the summation of the dynamics of its corresponding states in the subsystems.

III. Compositionality of Linear Control Systems

In this section, we propose a way of composing control inputs based on the framework of the resource sharing machines. In order to first solve Problems 1 and 2, we will propose a mechanism to compose the control inputs of closed-loop subsystems based on Definition 2. Then, we will investigate the conditions under which the resulting composite control system preserves the compositionality of the feedback laws of the subsystems. These are generic feedback laws and we do not consider LQR explicitly until the next section.

We consider the case where the control signals can be designed through a full state feedback mechanism, where

$$u^1(t) = F^1 x^1(t) \quad \text{and} \quad u^2(t) = F^2 x^2(t), \tag{6}$$

where the matrices $F^i \in \mathbb{R}^{m_i \times n_i}$, $i \in \underline{2}$, are gain matrices. Building on Definition 2, we introduce the following definition for the composite closed-loop control system.

Definition 6 (Composite Closed-loop Feedback Control Systems)**.** *For two closed-loop feedback control subsystems $\dot{x}^i(t) = A^i x^i(t) + B^i F^i x^i(t)$, $i \in \underline{2}$, the composite closed-loop system is given by*

$$\dot{\bar{x}}(t) = \mathcal{A}\bar{x}(t) + \mathcal{B} \underbrace{\underbrace{\begin{bmatrix} F_1 & \mathbf{0} \\ \mathbf{0} & F_2 \end{bmatrix}}_{\bar{F}} K}_{\mathcal{F}} \bar{x}(t). \tag{7}$$

Our choice to compose closed-loop systems in this way is based on wanting to preserve the dynamics of the individual subsystems. The following result shows that this choice of composition does indeed do so.

Lemma 1. *The dynamics of the composed closed-loop control systems $\dot{\bar{x}}(t) = \mathcal{A}\bar{x}(t) + \mathcal{B}\mathcal{F}\bar{x}(t)$ in* (7)*, are equivalent to the composition of the subsystems $\dot{x}^i(t) = A^i x^i(t) + B^i F^i x^i(t)$, $i \in \underline{2}$.*

Proof. Please check [25]. □

Lemma 1 shows that we can leverage the composite closed-loop control system given in (7) to study subsystems, and vice versa. Hence, Lemma 1 solves Problem 1.

Using this formulation, we next solve Problem 2.

Theorem 1. *Consider two asymptotically stable closed-loop feedback control systems $\dot{x}^i(t) = A^i x^i(t) + B^i F^i x^i(t)$ for $i \in \underline{2}$. If the matrices $A^i + B^i F^i$ for $i \in \underline{2}$ are symmetric, then the composite closed-loop system $\dot{\bar{x}}(t) = \mathcal{A}\bar{x}(t) + \mathcal{B}\mathcal{F}\bar{x}(t)$ with $\mathcal{A} = K^\top \bar{A} K$ and $\mathcal{B}\mathcal{F} = K^\top \bar{B}\bar{F}K$ is asymptotically stable, where $\bar{F}$ and $\mathcal{F}$ are from Definition 6 and $\bar{A}$ and $\bar{B}$ are from Definition 2.*

Proof. Please check [25]. □

Theorem 1 shows that the stability of the closed-loop control systems under the composition of resource sharing machines will be preserved if the system matrices of the closed-loop control systems are symmetric. However, Theorem 1 does not determine if feedback control laws designed directly for the composite system are the same as the compositions of control laws designed for each subsystem. We will answer this question in the following theorem.

Theorem 2. *Consider the composite feedback control system $\dot{\bar{x}}(t) = \mathcal{A}\bar{x}(t) + \mathcal{B}\bar{u}(t)$, where $\bar{u}(t)$ is designed for the composite system directly so that $\bar{u}(t) = \hat{\mathcal{F}}\bar{x}(t)$. Consider another composite feedback control system $\dot{\bar{x}}(t) = \mathcal{A}\bar{x}(t) + \mathcal{B}\bar{F}K\bar{x}(t)$, where the control design of the composite system* *is obtained through the composition of the two closed-loop control subsystems via Definition 6. The two systems are equivalent if $\hat{\mathcal{F}} = \bar{F}K$.*

Proof. Please check [25]. ☐

Theorem 2 provides a way of testing the compositionality of a feedback control design, namely by comparing the two feedback control gain matrices $\hat{\mathcal{F}}$ and $\mathcal{F} = \bar{F}K$. The theorem lays a foundation for designing controllers for composed systems via resource sharing machines through designing controllers for their subsystems. Based on the results in this section, we will use resource sharing machines to study the compositionality of LQR designs specifically.

IV. Compositionality of LQR

In this section, we study the compositionality of LQR design problems through resource sharing machines. We will analyze differences between (i) the LQR controller designed for a composite system and the (ii) controller that results from composing LQR controllers designed for the subsystems of the composite. Together, these analyses will solve Problem 3.

A. LQR Design for Subsystems

First we introduce the LQR problem for each subsystem. Again, we consider two linear subsystems $\dot{x}^i(t) = A^i x^i(t) + B^i u^i(t)$ for $i \in \{1, 2\}$, and we define cost functions

$$J^i(x^i, u^i) = \int_0^\infty x^i(\tau)^\top Q^i x^i(\tau)\, d\tau + \int_0^\infty u^i(\tau)^\top R^i u^i(\tau)\, d\tau, \quad (8)$$

where $Q^i \in \mathbb{R}^{n_i \times n_i}$ and $R^i \in \mathbb{R}^{m_i \times m_i}$ for $i \in \underline{2}$. Additionally, the matrix Q^i is positive-semidefinite while R^i is positive definite for $i \in \underline{2}$. The first term on the right-hand side is the penalty on excessive state size, and the second term on the right-hand side is the penalty on control effort. The optimal solutions are given by the following state-feedback representations [22]:

$$u^i(t) = -(R^i)^{-1}(B^i)^\top P^i x^i(t) = -F^i x^i(t), \quad (9)$$

for $i \in \underline{2}$, where P^i for $i \in \underline{2}$ satisfies an algebraic Riccati equation given by

$$\mathbf{0} = -P^i A^i - (A^i)^\top P^i - Q^i + P^i B^i (R^i)^{-1}(B^i)^\top P^i, \quad (10)$$

where the solution P^i is positive semidefinite.

B. LQR Design for the Composite System

We now define the LQR problem for the composite system in (7). The cost function is defined as

$$\mathcal{J}(\bar{x}, \bar{u}) = \int_0^\infty \bar{x}(\tau)^\top \underbrace{K^\top \begin{bmatrix} Q^1 & \mathbf{0} \\ \mathbf{0} & Q^2 \end{bmatrix} K}_{\mathcal{Q}} \bar{x}(\tau)\, d\tau + \int_0^\infty \bar{u}(\tau)^\top \underbrace{\begin{bmatrix} R^1 & \mathbf{0} \\ \mathbf{0} & R^2 \end{bmatrix}}_{\bar{R}} \bar{u}(\tau)\, d\tau, \quad (11)$$

where $\mathcal{Q}$ and $\bar{R}$ are composite weight matrices. Note that following the proof of Theorem 1, the matrix $\mathcal{Q}$ is positive semidefinite. And $\bar{R}$ is positive definite. Further, the optimal solution of the control problem in (7) with the objective function $\mathcal{J}$ is

$$\bar{u}(t) = -\bar{R}^{-1}\mathcal{B}^\top \mathcal{P}\bar{x}(t) = -\mathcal{F}\bar{x}(t), \quad (12)$$

where $\mathcal{P}$ is a symmetric positive semidefinite matrix that solves the algebraic Riccati equation

$$\mathbf{0} = -\mathcal{P}\mathcal{A} - \mathcal{A}^\top \mathcal{P} - \mathcal{Q} + \mathcal{P}\mathcal{B}\bar{R}^{-1}\mathcal{B}^\top \mathcal{P}. \quad (13)$$

C. Comparison of LQR Approaches

This subsection compares the LQR designs from Sections IV-A and IV-B. The following theorem gives a necessary and sufficient condition for the compositionality of the LQR controller, i.e., conditions under which the composition of LQR controllers designed for subsystems is equivalent to an LQR controller designed for the composite system.

Theorem 3. *The LQR solution of the composite system in* (7) *and the composition of the LQR solutions for the subsystems, given in* (9)*, are equivalent if and only if*

$$\underbrace{\begin{bmatrix} P_1 & \mathbf{0} \\ \mathbf{0} & P_2 \end{bmatrix}}_{\bar{P}} K = K\mathcal{P}. \quad (14)$$

Proof. Please check [25]. ☐

Theorem 3 can be regarded as a special case of Theorem 2 in which the control law is based on LQR design. However, Theorem 3 further illustrates that, for a composite system, it is possible to design an LQR controller for the system using the LQR designs of the corresponding control subsystems, if we can construct a composition matrix K to satisfy (14). Note that the matrices P_1 and P_2 that comprise $\bar{P}$ and $\mathcal{P}$ are from the steady-state solutions of the algebraic Riccati equations given in (10) and (13), respectively. Therefore, we further investigate the compositionality of LQR through exploring the properties of these algebraic Riccati equations. Specifically, the next theorem unifies them into a single Generalized Algebraic Riccati equation.

Theorem 4. *The matrix $K\mathcal{P}$ in the LQR design for the composite system and the matrix $\bar{P}K$ in the composition of LQR designs for the corresponding subsystems are both solutions to the General Algebraic Riccati equation*

$$\mathbf{0} = -X^\top \bar{A}K - K^\top \bar{A}^\top X - \mathcal{Q} + X^\top \bar{B}\bar{R}^{-1}\bar{B}^\top X, \quad (15)$$

where the matrix X is to be solved for.

Proof. Please check [25]. ☐

Theorem 4 connects the LQR design for the composite system and the composed LQR designs for the corresponding subsystems. Note that it is common for an algebraic Riccati equation to have more than one solution [27]. Hence, we are able to consider the case in which $\bar{P}K \neq K\mathcal{P}$ in the Generalized Algebraic Riccati equation in (15). We next further leverage (15) to determine conditions under which $\bar{P}K = K\mathcal{P}$ holds.

Corollary 1. *The equation $\bar{P}K = K\mathcal{P}$ holds only if $\bar{P}KK^\top$ is symmetric and positive semidefinite.*

Proof. Please check [25]. $\square$

Corollary 1 provides a necessary condition to determine whether a composed LQR design from control subsystems is the same as the LQR design of the composite system. In particular, based on Corollary 1, it is unnecessary for us to compute the LQR controller of the composite system to check compositionality. Instead, we can explore the compositionality of the LQR design through studying the algebraic Riccati equations of the LQR designs of the subsystems in (10) and the composition matrix K.

Following this idea, we present the following theorem, where we can leverage information from the control subsystems to determine the compositionality of LQR, without needing to actually generate any LQR designs or solve any Riccati equations.

Theorem 5. *Let $\bar{P}KK^\top$ be a positive semidefinite, symmetric matrix. If the pair $(\bar{A}KK^\top, \bar{B}\Sigma\Lambda^{-\frac{1}{2}})$ is a controllable pair, and the pair $(\bar{A}KK^\top, \Delta)$ is an observable pair, where Σ and Λ are obtained through the eigendecomposition of the symmetric positive definite matrix $\bar{R} = \Sigma\Lambda\Sigma^\top$ in* (11)*, and $KQK^\top = \Delta^\top\Delta$, then the diagram in Definition 5 commutes. That is, the following two objects are equivalent: (i) the composition of the LQR solutions* (9) *of the control subsystems via* (7) *and (ii) the LQR solution of the composite system in* (7)*.*

Proof. Please check [25]. $\square$

Remark 3. *Theorem 5 transforms the study of compositionality of LQR design to the investigation of the controllability of the constructed pair $(\bar{A}KK^\top, \bar{B}\Sigma\Lambda^{-\frac{1}{2}})$ and the investigation of the observability of the pair $(\bar{A}KK^\top, \Delta)$. Note that the controllability and observability analyses of the constructed pairs are not directly performed on the composed linear systems in* (7)*. The advantage of leveraging Theorem 5 is that, instead of comparing control designs that involve solving potentially many algebraic Riccati equations, we can directly leverage information from the system and input matrices of the subsystems, the optimal control problem formulation of the subsystems, and the composition structure, to verify the compositionality of LQR design.*

V. Conclusion

In this work, we introduce the composition of linear systems using the formulation of resource sharing machines. We explore the compositionality of open-loop and closed-loop control systems, specifically focusing on the LQR design problem on linear systems. In future work, we aim to investigate how other system properties, such as stability and controllability, are affected by the composition of resource sharing machines. We are motivated to study the optimality gap of the LQR designs in the cases in which compositionality fails. Furthermore, we are interested in implementing resource-sharing machines in engineering applications.

References

[1] X. Li, R. Lu, X. Liang, X. Shen, J. Chen, and X. Lin, "Smart community: An Internet of things application," *IEEE Comm. Mag.*, vol. 49, no. 11, pp. 68–75, 2011.

[2] W. M. Haddad and V. Chellaboina, *Nonlinear Dynamical Systems and control: A Lyapunov-based Approach*. Princeton University Press, 2011.

[3] C. Ebenbauer, T. Raff, and F. Allgöwer, "Dissipation inequalities in systems theory: An introduction and recent results," in *Invited Lectures of the Int. Con. on Indus. and App. Math.*, vol. 2007, 2009, pp. 23–42.

[4] A. Chapman, M. Nabi-Abdolyousefi, and M. Mesbahi, "Controllability and observability of network-of-networks via Cartesian products," *IEEE Trans. on Autom. Cont.*, vol. 59, no. 10, pp. 2668–2679, 2014.

[5] T. N. Tran and A. Chapman, "Generalized graph product: Spectrum, trajectories and controllability," in *Proc. 58th IEEE Conf. on Deci. and Cont. (CDC)*. IEEE, 2018, pp. 5358–5363.

[6] B. She, S. S. Mehta, E. Doucette, C. Ton, and Z. Kan, "Characterizing energy-related controllability of composite complex networks via graph product," *IEEE Trans. on Autom. Cont.*, vol. 66, no. 7, pp. 3205–3212, 2020.

[7] B. Fong and D. I. Spivak, *An invitation to applied category theory: Seven sketches in compositionality*. Cambridge University Press, 2019.

[8] J. Culbertson, P. Gustafson, D. E. Koditschek, and P. F. Stiller, "Formal composition of hybrid systems," *arXiv preprint arXiv:1911.01267*, 2019.

[9] G. Bakirtzis, C. H. Fleming, and C. Vasilakopoulou, "Categorical semantics of cyber-physical systems theory," *ACM Trans. on Cyber-Phy. Syst.*, vol. 5, no. 3, pp. 1–32, 2021.

[10] M. D. Kvalheim, P. Gustafson, and D. E. Koditschek, "Conley's fundamental theorem for a class of hybrid systems," *SIAM Journal on Applied Dynamical Systems*, vol. 20, no. 2, pp. 784–825, 2021.

[11] E. Lerman, "A category of hybrid systems," *arXiv preprint arXiv:1612.01950*, 2016.

[12] S. Libkind, A. Baas, M. Halter, E. Patterson, and J. Fairbanks, "An algebraic framework for structured epidemic modeling," *arXiv preprint arXiv:2203.16345*, 2022.

[13] M. Halter, E. Patterson, A. Baas, and J. Fairbanks, "Compositional scientific computing with Catlab and semanticmodels," *arXiv preprint arXiv:2005.04831*, 2020.

[14] E. Patterson, O. Lynch, and J. Fairbanks, "Categorical data structures for technical computing," *arXiv preprint arXiv:2106.04703*, 2021.

[15] S. Awodey, *Category Theory*. Oxford University press, 2010.

[16] D. Vagner, D. I. Spivak, and E. Lerman, "Algebras of open dynamical systems on the operad of wiring diagrams," *arXiv preprint arXiv:1408.1598*, 2014.

[17] P. Tabuada and G. J. Pappas, "Quotients of fully nonlinear control systems," *SIAM J. on Contr. and Opt.*, vol. 43, no. 5, pp. 1844–1866, 2005.

[18] E. Lerman, "Networks of open systems," *Journal of Geometry and Physics*, vol. 130, pp. 81–112, 2018.

[19] E. Lerman and J. Schmidt, "Networks of hybrid open systems," *Journal of Geometry and Physics*, vol. 149, p. 103582, 2020.

[20] S. Libkind, A. Baas, E. Patterson, and J. Fairbanks, "Operadic modeling of dynamical systems: Mathematics and computation," *arXiv preprint arXiv:2105.12282*, 2021.

[21] S. Libkind, "An algebra of resource sharing machines," *arXiv preprint arXiv:2007.14442*, 2020.

[22] B. D. Anderson and J. B. Moore, *Optimal Control: Linear Quadratic Methods*. Courier Corporation, 2007.

[23] F. Dörfler, P. Tesi, and C. De Persis, "On the role of regularization in direct data-driven LQR control," in *Proc. 61st IEEE Conf. on Dec. and Cont. (CDC)*. IEEE, 2022, pp. 1091–1098.

[24] X. Zhang and T. Başar, "Revisiting LQR control from the perspective of receding-horizon policy gradient," *arXiv preprint arXiv:2302.13144*, 2023.

[25] B. She, T. Hanks, J. Fairbanks, and M. Hale, "Characterizing compositionality of LQR from the categorical perspective," *arXiv preprint arXiv:2305.01811*, 2023.

[26] D. I. Spivak, "The steady states of coupled dynamical systems compose according to matrix arithmetic," 2015. [Online]. Available: https://arxiv.org/abs/1512.00802

[27] V. Kučera, "A review of the matrix Riccati equation," *Kybernetika*, vol. 9, no. 1, pp. 42–61, 1973.