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Optimization-based trajectory generation and receding horizon control for systems with convex dynamics

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Abstract— **In this paper we propose an optimization-based control scheme, which can be used for trajectory generation or receding horizon control for system with nonlinear, but convex dynamics, and both explicit and implicit discrete time models. The scheme uses both the nonlinear model and its linearization to construct a tube containing all possible future system trajectories, and uses this tube to predict performance and ensure constraint satisfaction. The controls sequence and tube cross-sections are optimized online in a sequence of convex programs without the need of pre-computed error bounds. We prove feasibility, stability and non-conservativeness of the approach, with the series of convex programs converging to a point which is a local optimum for the original nonlinear optimal control problem. We further present how a structure-preserving model can be implemented within the approach and used to reduce the number of constraints and guarantee a structure-preserving discrete trajectory solution.**

I. INTRODUCTION

Optimization-based control techniques allow for a task to be completed optimally with respect to a given performance index while respecting both system and environment constraints [1], [2]. As such it has often been implemented for online trajectory design or receding horizon control of systems subject to stringent and/or safety critical constraints such as for example autonomous vehicles [3], [4], spacecraft systems [5], [6], and robotics [7]. In its most simple form optimization-based control entails the single solution of a optimal control problem (OCP) resulting in an optimal trajectory and an open-loop control sequence for the entire time period of interest. To allow for feedback a receding horizon strategy can be implemented (also known as model predictive control or MPC) [1]. At each discrete time step, an OCP is solved, the first control of the resulting control sequence is applied and the new state of the system is measured. Based on this measurement a new OCP is solved and the procedure is repeated. Thus for open-loop autonomous trajectory generation of real-time systems or receding horizon implementation, it is critical that the OCP can be solved successfully and reliably within the available time. The nonlinear nature of most system however implies that the resulting OCP is a nonlinear program (NLP) and thus has high computation complexity with no guarantees for a successful solution.

To address this problem many approaches have focused on approximating the problem as a convex program, which can be solved efficiently and reliably [8]. One approach of this kind in the space domain, known as lossless convexification, is capable of obtaining a convex relaxation of the problem, whose solution is guaranteed to coincide with global optima of the original NLP problem but is applicable to a very limited set of problems [2]. Other approaches rely on the solution of a sequence of convex programs at each discrete time step aiming to converge to a solution close to a local optimum of the original NLP. For example successive sequential programming approaches use a sequence of quadratic programs [8], [2], while the tube-based approaches rely on the linearization of the system dynamics and the construction of feasible tubes, which contain all possible trajectories based on the error of the linear approximation [9], [10]. Such approaches and other convexification techniques have been demonstrated to be advantageous for real-time applications ([3], [4], [5], [6] among many others). However, without appropriate step size selection or line search algorithms, there can be a discrepancy between their computed solution and the optimum of the original problem leading to suboptimality or even infeasibility.

Very recently, a tube-based approach was developed by [11] for systems with convex dynamics which is able to guarantees that the sequence of convex programs proposed to approximate the NLP converges to one of the NLP local optima and thus removes the degree of suboptimality other such approaches introduce. It also guarantees that the iteration can be terminated early without loss of feasibility or stability. However, the paper assumes the existence of an explicit discrete time system model with certain convexity properties. When provided with explicit continuous model, this prevents the use of many higher-order standard and structure-preserving model discretization techniques limiting the choice to lower accuracy discrete time model approximation. Such restriction might not be problematic for a receding horizon implementation which relies on feedback to deal with model inaccuracies, but can be detrimental when used for trajectory generation. The assumption also prevents the approach to being applied to a wide range of systems with implicit continuous models such as many multibody systems and other constrained systems.

For this purpose in this paper we provide a reformulation of the nominal receding horizon approach by [11] for implicit convex system models. We present the approach for a receding horizon implementation which can trivially be simplified for single trajectory design use. We prove recursive feasibility and stability of the proposed approach and provide a proof that it also converges to a point which is a local optimum of the original NLP. As such the approach of

[11] can be seen as a specific case of the one proposed here, which uses the existence of an explicit model to simplify the formulation.

In the second part of the paper (Section IV) we focus on a specific use of the proposed approach for structure-preserving discrete models [12] and more specifically variational models [13]. By preserving certain mathematical structures and conservation laws from the continuous into the discrete domain, such discretization schemes allow for better qualitative representation of the system and accurate energy and momentum simulation for exponentially long times even for larger discrete time steps [12]. For this reason, they have often been used to reduce the computational cost of simulations and optimal control strategies and could be of great benefit for fuel and energy optimization problems ([14], [15], [16], [17]). In Section IV we present how such models can readily be used in the proposed optimization-based scheme and show that their characteristics can be employed to simultaneously reduce the number of constraints and guarantee structure-preserving properties of the resulting solution. The advantages of the proposed approach are then demonstrated for two example systems, showing convergence of the sequence of convex programs and the effect of the structure-preservation model on the resulting solution.

II. Problem formulation

Consider a nonlinear system with state $x(t) \in \mathbb{R}^{n_x}$, control input $u(t) \in \mathbb{R}^{n_u}$ and continuous model of the form

$$f(\dot{x}, x, u) = 0 \tag{1}$$

where f is differentiable for all $(\dot{x}, x, u)$ in the operating region $\mathbb{R}^{n_x} \times \mathcal{X} \times \mathcal{U}$ and componentwise convex with respect to $\dot{x}$, x and u.

Assumption 1: We assume a discrete time model can be created of the form

$$g(x_{i+1}, x_i, u_i) = 0 \tag{2}$$

such that g is differentiable for all (x_{i+1}, x_i, u_i) in the operating region $\mathcal{X} \times \mathcal{X} \times \mathcal{U}$ and convexity is preserved, i.e. the elements of $g(\cdot)$ are convex with respect to all $x_{i+1} \in \mathcal{X}$, $x_i \in \mathcal{X}$ and $u_i \in \mathcal{U}$. We further assume the discrete time step is chosen to guarantee $g(\cdot)$ is locally Lipshitz.

We pose the problem of optimally controlling this system's trajectory $\mathbf{x} : \{x_0, x_1, ...\}$, $\mathbf{u} : \{u_0, u_1, ...\}$ to track a given state and input reference trajectory $\mathbf{x^r} : \{x_0^r, x_1^r, ...\}$, $\mathbf{u^r} : \{u_0^r, u_1^r, ...\}$ where $g(x_{i+1}^r, x_i^r, u_i^r) = 0$ for all $i \geq 0$ subject to a quadratic objective

$$\sum_{i=0}^{\infty} \Big(\|x_i - x_i^r\|_Q^2 + \|u_i - u_i^r\|_R^2 \Big) \tag{4}$$

(where $\|x\|_Q^2 = x^T Q x$) and additional initial and polytopic constraints of the form $x_0 = x_{init}$, $x_i \in \mathcal{X} \subset \mathbb{R}^{n_x}$, $u_i \in \mathcal{U} \subset \mathbb{R}^{n_u}$. To solve this problem a MPC optimization strategy can be employed with a receding horizon optimal control problem (RHOCP) of the form

nMPC RHOCP:

$$\mathbf{u}_i^* = \arg \min_{\mathbf{x}_i \in X^{N+1}, \mathbf{u}_i \in U^N} J_{\text{nMPC}}(\mathbf{x}_i, \mathbf{u}_i) \tag{5a}$$

$$\text{subject to:} \qquad x_{0|i} = x_i \tag{5b}$$

$$x_{k|i} \in \mathcal{X}, \qquad u_{k|i} \in \mathcal{U} \qquad \text{for } k = 0, ..., N-1 \tag{5c}$$

$$g(x_{k+1|i}, x_{k|i}, u_{k|i}) = 0 \qquad \text{for } k = 0, ..., N-1 \tag{5d}$$

$$x_{N|i} \in \mathcal{X}_N \subseteq \mathcal{X} \tag{5e}$$

where $x_{k|i}$, $u_{k|i}$ denote respectively the state and input at time $k+i$ predicted at time i and

$$J_{\text{nMPC}}(\mathbf{x}_i, \mathbf{u}_i) = \sum_{k=0}^{N-1} \Big(\|x_{k|i} - x_{k|i}^r\|_Q^2 + \|u_{k|i} - u_{k|i}^r\|_R^2 \Big) + \|x_{N|i} - x_{N|i}^r\|_{Q_N}^2 \tag{6}$$

Assumption 2: We assume the state x_i can be measured at each discrete time step i, the reference trajectory (x_i^r, u_i^r) satisfies (5*c*-5*e*) for all i and Q and R are positive definite. We further assume the prediction horizon N, the terminal weight matrix Q_N and the terminal set $\mathcal{X}_N$ are chosen such that there exists a feedback gain K for which the set $\mathcal{X}_N$ is positively invariant for the system (2) under the control law $u_i = K(x_i - x_i^r) + u_i^r \in \mathcal{U}$, and for all $x_N \in \mathcal{X}_N$

$$\sum_{i=N}^{\infty} (\|x_i - x_i^r\|_Q^2 + \|u_i - u_i^r\|_R^2) \leq \|x_N - x_N^r\|_{Q_N}^2 \tag{7}$$

At each discrete time step i the nonlinear MPC (nMPC) RHOCP could be solved online to obtain an optimal control sequence $\mathbf{u}_i^* : \{u_{0|i}, u_{1|i}, \ldots, u_{N-1|i}\}$ and the first element of the sequence could be used to define a receding control law $u_i = u_{0|i}$. At the next step a new RHOCP can be formulated based on the measurement (or estimate) of x_{i+1} and the process can be repeated. To guarantee stability of the control strategy it is essential that a feasible solution can be obtained within each discrete time step. In its form in (5*a*-5*e*), however, the nMPC RHOCP is a nonlinear program and thus no guarantee can be given on whether it can provide a feasible solution and how long that would require. To address these problems we propose a successive linearization approach to reformulate the problem as a convex program. This approach is a reformulation of [11] allowing its application to a wider variety of systems. The approach [11] can in fact be seen as a special case of the approach proposed here.

III. Proposed approach

Given a prior pair of state and control trajectories $\mathbf{x}_i^0 : \{x_{0|i}^0, x_{1|i}^0, \ldots, x_{N|i}^0\}$ and $\mathbf{u}_i^0 : \{u_{0|i}^0, u_{1|i}^0, \ldots, u_{N-1|i}^0\}$ using Taylor series the discrete time dynamics of the system can be linearized as follows

$$g(x_{k+1|i}, x_{k|i}, u_{k|i}) = g(x_{k+1|i}^0, x_{k|i}^0, u_{k|i}^0) + C_{k|i} s_{k+1|i} + D_{k|i} s_{k|i} + E_{k|i} \nu_{k|i} + e(x_{k+1|i}, x_{k|i}, u_{k|i}) \tag{8}$$

$$\text{where} \qquad s_{k|i} = x_{k|i} - x_{k|i}^0, \qquad \nu_{k|i} = u_{k|i} - u_{k|i}^0 \tag{9}$$

$$C_{k|i} = \frac{\partial g}{\partial x_{k+1|i}} \Big|_{(x_{k+1|i}^0, x_{k|i}^0, u_{k|i}^0)}, \; D_{k|i} = \frac{\partial g}{\partial x_{k|i}} \Big|_{(x_{k+1|i}^0, x_{k|i}^0, u_{k|i}^0)}$$

$$E_{k|i} = \frac{\partial g}{\partial u_{k|i}}|_{(x^0_{k+1|i}, x^0_{k|i}, u^0_{k|i})} \tag{10}$$

Assumption 3: The initial seed trajectory $(\mathbf{x}^0, \mathbf{u}^0)$ is a feasible (but potentially suboptimal) for constraints ($5b$-$5e$).

Remark 1: Due to the convexity of g it is known that $e(x_{k+1|i}, x_{k|i}, u_{k|i})$ is also componentwise convex in $x_{k+1|i}$, $x_{k|i}$, $u_{k|i}$ and $e(x_{k+1|i}, x_k, u_k) \geq 0$. Therefore Assumption 3 ($g(x^0_{k+1}, x^0_k, u^0_k) = 0$) and Equation (8) imply

$$C_k s_{k+1|i} + D_{k|i} s_{k|i} + E_{k|i} \nu_{k|i} \geq 0 \Rightarrow g(x_{k+1|i}, x_{k|i}, u_{k|i}) \geq 0$$

Based on this linearization we can define perturbation sets based on their vertices $S_{k|i} = Co\{s^m_{k|i}, m = 1, ..., n_v\}$, $S^-_{k|i} = Co\{s^{-,m}_{k|i}, m = 1, ..., n_v\}$, $S^+_{k|i} = Co\{s^{+,m}_{k|i}, m = 1, ..., n_v\}$ and formulate a convex program:

cMPC RHOCP :

$$(\mathbf{c}^*_i, \mathbf{S}^*_i, \mathbf{S}^{+,*}_i, \mathbf{S}^{-,*}_i) = \arg \min_{\mathbf{c}_i,\, \mathbf{S}^+_i,\, \mathbf{S}^-_i, \mathbf{S}_i} J_{\text{cMPC}}(\mathbf{c}_i, \mathbf{S}_i, \mathbf{x}^0_i, \mathbf{u}^0_i)$$

subject to:

$$S_{0|i} = \{0\} \tag{11a}$$

$$S_{k|i} \oplus \{x^0_{k|i}\} \subseteq \mathcal{X}, \text{ for } k = 1, ...N-1 \tag{11b}$$

$$S^+_{k|i} \subseteq S_{k|i}, \quad S^-_{k|i} \subseteq S_{k|i}, \text{ for } k = 1, ..., N \tag{11d}$$

$$S_{N|i} \oplus \{x^0_{N|i}\} \subseteq \mathcal{X}_N \tag{11e}$$

$$K s_{k|i} + c_{k|i} + u^0_{k|i} \in \mathcal{U} \; \forall s_{k|i} \in S_{k|i}, k = 0, ..., N-1 \tag{11f}$$

$$\begin{gathered} g(x^0_{k+1|i} + s^-_{k+1|i},\ x^0_{k|i} + s_{k|i},\ K s_{k|i} + c_{k|i} + u^0_{k|i}) \leq 0 \\ C_{k|i} s^+_{k+1|i} + (D_{k|i} + E_{k|i} K) s_{k|i} + E_{k|i} c_{k|i} \geq 0 \\ \text{for } \forall s_{k|i} \in S_{k|i}, \forall s^+_{k+1|i} \in S^+_{k+1|i}, \forall s^-_{k+1|i} \in S^-_{k+1|i} \\ \text{and } k = 0, ...N-1 \end{gathered} \tag{11g}$$

where $\oplus$ is the Minkowski sum, $\mathbf{S}^+_i = \{S^+_{1|i}, \ldots, S^+_{N|i}\}$, $\mathbf{S}_i = \{S^-_{1|i}, \ldots, S^-_{N|i}\}$, $\mathbf{S}_i = \{S_{0|i}, \ldots, S_{N|i}\} \in S^{N+1}$,

$$J_{\text{cMPC}}(\mathbf{c}_i, \mathbf{S}_i, \mathbf{x}^0_i, \mathbf{u}^0_i) = \sum_{k=0}^{N-1} \Big(\max_{s_{k|i} \in S_{k|i}} \|x^0_{k|i} + s_{k|i} - x^r_{k|i}\|^2_Q + \max_{s_{k|i} \in S_{k|i}} \|u_{k|i} - u^r_{k|i}\|^2_R \Big) + \max_{s_{N|i} \in S_{N|i}} \|x^0_{N|i} + s_{N|i} - x^r_{N|i}\|^2_{Q_N}$$

and a dual mode prediction strategy of the form

$$u_{k|i} = u^0_{k|i} + \nu_{k|i} = u^0_{k|i} + K s_{k|i} + c_{k|i} \quad \text{for } k = 0, ...N-1$$

has been used shifting the optimization from the sequence $\mathbf{u}_i$ to the sequence $\mathbf{c}_i : \{c_{0|i}, ..., c_{N-1|i}\}$. In Equation (10) s_k and ν_k can be viewed as perturbations from the original trajectory $\mathbf{x}^0$ and control sequence $\mathbf{u}^0$ respectively. Thus the sets sequence $\mathbf{S}_i \oplus \mathbf{x}_i = \{S_{0|i} \oplus \{x^0_{0|i}\}, \ldots, S_{N|i} \oplus \{x^0_{N|i}\}\}$ can be viewed as a tube containing all possible system trajectories corresponding to the sequence $\mathbf{c}_i$.

Remark 2: In comparison to the explicit model formulation in [11], the use of implicit models dictates that additional sets $S^+_{k|i}, S^-_{k|i}$ have to be defined because implementing both constraints in ($11g$) for a single set $S_{k|i}$ would only be satisfied for a single-trajectory tube ($S_{k|i} = \{0\}$ for all k). In Section III-A we guarantee that the tube obtained with the proposed formulation ($S_{k|i} \supseteq Co\{s^{-,m}_{k|i}, s^{+,m}_{k|i}, m = 1, ..., n_v\}$) contains the feasible solutions of the original NLP ($5a$-$5e$).

Using the convex RHOCP formulation (cMPC RHOCP) a receding horizon strategty can be formulated as described in Algorithm 1, which optimizes both the tube cross sections and control sequences online. As we will prove in the next section, the iterations performed at each discrete time step i converge to a local minimum of the original nMPC RHOCP, eliminating the conservativeness and suboptimality presented by other tube-based approaches. We also show that the iteration can be terminated early without loss of recursive feasibility and stability.

Algorithm 1: cMPC

```
Offline: Determine K, Q_N, X_N, and an initial seed
         trajectory (x_0^0, u_0^0) satisfying (5b)-(5e) with
         x_0 = x_init
Online: At each discrete time step i = 0, 1, ...:
Obtain x_i and set the iteration counter n <- 1;
while n <= maxiters and ||c_i^*|| >= tol do
    Compute C_{k|i}, D_{k|i}, E_{k|i} for k = 0, ..., N-1
      using (10) and (x_i^0, u_i^0);
    Solve nominal RHOCP to obtain
      (c_i^*, S_i^*, S_i^{+,*}, S_i^{-,*});
    for k = 0, ..., N-1 do
        Compute u_{k|i}^* = u_{k|i}^0 + c_{k|i}^* + K(x_{k|i}^* - x_{k|i}^0)
          and g(x_{k+1|i}^*, x_{k|i}^*, u_{k|i}^*) = 0 with x_{0|i}^* = x_i;
    Update the seed: (x_i^0, u_i^0) <- (x_i^*, u_i^*);
    n <- n + 1;
Implement u_i <- u_{0|i}^*;
Set u_{N|i}^* = K(x_{N|i}^* - x_{N+i}^r) + u_{N+i}^r and solve
  g(x_{N+1|i}^*, x_{N|i}^*, u_{N|i}^*) = 0 for x_{N+1|i}^* ;
Set u_{i+1}^0 <- {u_{1|i}^*, ..., u_{N-1|i}^*, u_{N|i}^*}
    x_{i+1}^0 <- {x_{1|i}^*, ..., x_{N|i}^*, x_{N+1|i}^*};
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Remark 3: The initial seed trajectory can be obtained by reformulating the cMPC RHOCP in a strategy analogous to the one outlined in [11]. The constraints and cost can be expressed as a maximization over the vertices of $\mathbf{S}_i, \mathbf{S}^+_i, \mathbf{S}^-_i$, allowing the cost to be reformulated to a linear objective with an additional second-order cone constraint on the vertices of $\mathbf{S}_i$. The resulting problem is convex and can thus be solved efficiently using a variety of solvers (see for example [8]).

Remark 4: The representation of $\mathbf{S}_i, \mathbf{S}^+_i, \mathbf{S}^-_i$ causes exponential growth in the number of variables and constraints in cMPC RHOCP with the state dimension n_x. Alternatively, using homothetic sets as described in [11] allows the numbers of variables and constraints depend linearly on n_x and the chosen number of vertices of $\mathbf{S_i}$, respectively. In either case a reduction in n_x allows for computational savings and a method for achieving this is proposed in Section IV.

A. Feasibility and stability guarantees

In this section we demonstrate the recursive feasibility and stability of the proposed approach. Additionally we prove that the iteration at each horizon time step i converges to a locally optimal solution of the original nMPC RHOCP.

First we prove that for non-zero perturbations, the tube contains the original nonlinear system trajectory and converges to it when the iterations are not terminated early. Let

$$X_{k|i} = Co\{x^m_{k|i}, m = 1, ..., n_v\} = \{x^0_{k|i}\} \oplus S_{k|i}$$
$$X^-_{k|i} = Co\{x^{-,m}_{k|i}, m = 1, ..., n_v\} = \{x^0_{k|i}\} \oplus S^-_{k|i}$$
$$X^+_{k|i} = Co\{x^{+,m}_{k|i}, m = 1, ..., n_v\} = \{x^0_{k|i}\} \oplus S^+_{k|i}$$

Lemma 1: If $S_{k+1|i}$ satisfies ($11d$) and $s^{-,m}_{k+1|i}$, $s^{+,m}_{k+1|i}$ satisfy ($11g$) for $m = 1, ..., n_v$ then there exists a $x_{k+1|i} \in X_{k+1|i}$ satisfying $g(x_{k+1|i}, x_{k|i}, u_{k|i}) = 0$ for $\forall x_{k|i} \in X_{k|i}$ with $u_{k|i} = K(x_{k|i} - x^0_{k|i}) + c_{k|i} + c^0_{k|i}$.

Proof Constraints ($11d$) imply $S_{k|i} \supseteq Co\{s^{-,m}_{k|i}, s^{+,m}_{k|i}, m = 1, ..., n_v\}$. Furthermore let $z^\lambda = \lambda z^m + (1-\lambda) z^v$ for $\lambda \in [0,1]$, $m, v \in \{1, ..., n_v\}$. Then from Remark 1

$$\left.\begin{aligned} g(x^{-,m}_{k+1|i}, x^m_{k|i}, u^m_{k|i}) \le 0 \\ g(x^{-,v}_{k+1|i}, x^v_{k|i}, u^v_{k|i}) \le 0 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} g(x^{-,\lambda}_{k+1|i}, x^\lambda_{k|i}, u^\lambda_{k|i}) \le 0 \\ \text{for any } \lambda \in [0,1] \end{aligned}$$

$$\left.\begin{aligned} C_{k|i} s^{+,m}_{k+1|i} + D_{k|i} s^m_{k|i} + E_{k|i}(u^m_{k|i} - u^0_{k|i}) \ge 0 \\ C_{k|i} s^{+,v}_{k+1|i} + D_{k|i} s^v_{k|i} + E_{k|i}(u^v_{k|i} - u^0_{k|i}) \ge 0 \end{aligned}\right\} \Rightarrow$$
$$g(x^{+,\lambda}_{k+1|i}, x^\lambda_{k|i}, u^\lambda_{k|i}) \ge 0 \text{ for any } \lambda \in [0,1]$$

where $u^m_{k|i} = K(x^m_{k|i} - x^0_{k|i}) + c_{k|i} + c^0_{k|i}$, $u^v_{k|i}$ are defined similarly and $x^\lambda_{k|i} = s^\lambda_{k|i} + x^0_{k|i}$. Based on this and from setting $S_{k+1|i}$ as the convex hull of its vertices it follows that there exists $s^\lambda_{k+1|i} \in Co\{s^{-,\lambda}_{k+1|i}, s^{+,\lambda}_{k+1|i}\} \subseteq S_{k+1|i}$ satisfying $g(x^\lambda_{k+1|i}, x^\lambda_{k|i}, u^\lambda_{k|i}) = 0$ for all $\lambda \in [0,1]$. □

To make notation more precise, at the ith time step and nth iteration of lines 3-8 of Algorithm 1 we denote the seed trajectory as $\mathbf{x}^{0,n}_i$, the seed control sequence as $\mathbf{u}^{0,n}_i$ and the tube as $\mathbf{S}^n_i$. Similarly, we denote the solution of the cMPC RHOCP in line 4 as $(\mathbf{c}^{*,n}_i, \mathbf{S}^{*,n}_i, \mathbf{S}^{+,*,n}_i, \mathbf{S}^{-,*,n}_i)$, and the optimal objective as $J^{*,n}_{\text{cMPC},i} = J_{\text{cMPC}}(\mathbf{c}^{*,n}_i, \mathbf{S}^{*,n}_i, \mathbf{x}^{0,n}_i, \mathbf{u}^{0,n}_i)$.

Theorem 1: (Recursive feasibility): If the initial seed trajectory $(\mathbf{x}^0_i, \mathbf{u}^0_i)$ is feasible w.r.t constraints ($5b$-$5e$) and the assumptions 1, 2, 3 hold, then the nominal RHOCP in step 4) of Algorithm 1 is feasible at each iteration and for $i \ge 0$.

Proof Let $c_{k|i} = 0$, $k = 0, \ldots, N-1$ and $S_{k|i} = \{0\}$ for $k = 0, \ldots, N$ reducing the constraints in nominal RHOCP to $\{x^0_{k|i}\} \subseteq \mathcal{X}$, $u^0_{k|i} \in \mathcal{U}$, $\{x^0_{N|i}\} \subseteq \mathcal{X}_N$. In this case the trajectory takes the form $(\mathbf{x}_i, \mathbf{u}_i) = (\mathbf{x}^0_i, \mathbf{u}^0_i)$ and based on Assumption 3 is a feasible (but possibly suboptimal) solution of nominal RHOCP at $i = 0$. Feasibility of the trajectory $(\mathbf{x}^0_{i+1}, \mathbf{u}^0_{i+1})$ at any subsequent horizon steps is then guaranteed by constraints ($11b$)-($11e$) and the conditions placed on the choice of set $\mathcal{X}_N$ in Assumption 2. □

Theorem 2: For all $i \ge 0$ and for all $n \ge 1$

$$J_{\text{nMPC}}(\mathbf{x}^{0,n+1}_i, \mathbf{u}^{0,n+1}_i) \le J_{\text{cMPC}}(\mathbf{c}_i, \mathbf{S}^n_i, \mathbf{x}^{0,n}_i, \mathbf{u}^{0,n}_i) \quad (12)$$

for $(\mathbf{x}^{0,n+1}_i, \mathbf{u}^{0,n+1}_i)$ obtained by steps 4-7 of Algorithm 1.

Proof It follows from the nominal RHOCP formulation that $\mathbf{x}^{0,n+1}_i$ belongs to the tube $\mathbf{S}^n_i \oplus \{x^{0,n}_k\}$. The statement then follows from Eq (9), steps 5-7 in Alg. 1 and the maximization formulation of the cMPC cost returning the maximum cost over the tube $\mathbf{S}^n_i$. In more detail

$$\|x^{0,n+1}_{k|i} - x^r_{k|i}\|^2_Q \le \max_{s_{k|i} \in S_{k|i}} \|x^{0,n}_{k|i} + s_{k|i} - x^r_{k|i}\|^2_Q$$
$$\|u^{0,n+1}_{k|i} - u^r_{k|i}\|^2_Q \le \max_{s_{k|i} \in S_{k|i}} \|u^{0,n}_{k|i} + K s_{k|i} + c_{k|i} - u^r_{k|i}\|^2_R$$
$$\|x^{0,n+1}_{N|i} - x^r_{N|i}\|^2_Q \le \max_{s_{N|i} \in S_{N|i}} \|x^{0,n}_{N|i} + s_{N|i} - x^r_{N|i}\|^2_{Q_N}$$

and Equation (12) follows from summing the terms before and after the inequality sign for all $k = 0, \ldots, N-1$ to obtain the expressions for $J_{\text{nMPC}}(\mathbf{x}^{0,n+1}_i, \mathbf{u}^{0,n+1}_i)$ and $J_{\text{cMPC}}(\mathbf{c}_i, \mathbf{S}^n_i, \mathbf{x}^{0,n}_i, \mathbf{u}^{0,n}_i)$. □

Theorem 3: For all $i \ge 0$ and for all $n \ge 1$, we have

$$J^{*,n+1}_{\text{cMPC}} \le J^{*,n}_{\text{cMPC}} \quad (13)$$

Proof From Theorem 1 and Theorem 2 it follows that a suboptimal solution exists $\mathbf{c}_i = 0$ and $\mathbf{S}_i = \{\{0\}, \ldots, \{0\}\}$,

$$J_{\text{cMPC}}(\{0\}, \{0\}, \mathbf{x}^{0,n+1}_i, \mathbf{u}^{0,n+1}_i) = J_{\text{nMPC}}(\mathbf{x}^{0,n+1}_i, \mathbf{u}^{0,n+1}_i)$$

and (14)

$$J_{\text{nMPC}}(\mathbf{x}^{0,n+1}_i, \mathbf{u}^{0,n+1}_i) \le J_{\text{cMPC}}(\mathbf{c}^{*,n}_i, \mathbf{S}^{*,n}_i, \mathbf{x}^{0,n}_i, \mathbf{u}^{0,n}_i) \quad (15)$$

From Theorem 1, optimality and (14) it follows that

$$J_{\text{cMPC}}(\mathbf{c}^{*,n}_i, \mathbf{S}^{*,n}_i, \mathbf{x}^{0,n}_i, \mathbf{u}^{0,n}_i) \le J_{\text{nMPC}}(\mathbf{x}^{0,n}_i, \mathbf{u}^{0,n}_i) \quad (16)$$

(13) follows from the combination of (14), (15), (16). □

Theorem 4: For all $i \ge 0$, the iteration on lines 2-8 of Algorithm 1 converges to a seed trajectory $(\mathbf{x}^{0,n}_i, \mathbf{u}^{0,n}_i)$ such that $\mathbf{c}^{*,n}_i = 0$ and $\mathbf{S}^{*,n}_i = \{\{0\}, \ldots, \{0\}\}$ is an optimal solution of the cMPC RHOCP in the limit as $n \to \infty$.

Proof Assumption 2 implies $J^{*,n}_{\text{cMPC},i} \ge 0$ and from Theorem 3 it follows that $J^{*,n}_{\text{cMPC},i} - J^{*,n+1}_{\text{cMPC},i} \to 0$.Thus from (13) and (16) it follows that $J^{*,n}_{\text{cMPC},i} - J_{\text{nMPC}}(\mathbf{x}^{0,n}_i, \mathbf{u}^{0,n}_i) \to 0$ as $n \to \infty$ implying that $(\mathbf{c}^{*,n}_i, \mathbf{S}^{*,n}_i) = (0, \{\{0\}, \ldots, \{0\}\})$ is the optimal solution of the cMPC RHOCP in the limit as $n \to \infty$.

Theorem 5: The iteration defined in steps 2-9 of Algorithm 1 converges to a local minimum of nMPC RHOCP ($5a$-$5e$).

Proof The proof can be found in the Appendix.

Theorem 6: Given Assumptions 1 and 2, the control law of Algorithm 1 ensures that $x^\delta_{i+1} = x^\delta_i = u^\delta_i = 0$ with $x^\delta_i = x_i - x^r_i$ and $u^\delta_i = u_i - u^r_i$ is an asymptotically stable equilibrium of the system (2) (i.e. $g(x^\delta_{i+1}, x^\delta_i, u^\delta_i) = 0$), with region of attraction consisting of the initial conditions x_0 of (2) for which there exists a control sequence such that $u_i \in \mathcal{U}$ and $x_i \in \mathcal{X}$ for $k = 0, \ldots, N-1$, and $x_N \in \mathcal{X}_N$.

Proof The proof is analogous to that of Theorem 8 in [11].

IV. Proposed variational reformulation

In this section we present how a structure-preserving model can be accommodated by the proposed approach. We further discuss how such a model can be used to reduce the number of variables and constraints in the respective RHOCP. These developments are demonstrated for a system with explicit continuous model, but the extension to implicit models (such as for systems with constraints) is trivial.

A. Model formulation and terminal set calculation

We consider a general mechanical system with configuration $q(t) \in \mathbb{R}^{n_q}$ and Lagrangian of the form

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2}\dot{q}^T M \dot{q} - V(q) \tag{17}$$

Using the Continuous Forced Euler-Lagrange equations the continuous model of the system motion under the influence of control forces $u(t) \in \mathbb{R}^{n_u}$ can be derived as

$$\ddot{q} = -M^{-1}\nabla_q V(q) + u \tag{18}$$

Assumption 4: We assume a regular $\mathcal{L}$, symmetric invertible M, polytopic configuration constraints $q \in \mathcal{Q}$ and assume the expression $M^{-1}\nabla_q V(q)$ is concave.

To discretize the equations of motion we formulate a discrete approximation of the Lagrangian and derive a discrete time system model using a variational integrator [13]:

$$g(q_{i+1}, q_i, q_{i-1}, u_i, u_{i-1}) = -q_{i+1} + 2q_i - q_{i-1} - \frac{h^2}{2}(M^{-1}\nabla_{\eta_i} V(\eta_i) + M^{-1}\nabla_{\eta_{i-1}} V(\eta_{i-1}) + \frac{h^2 M^{-1}}{2}(u_{i-1} + u_i)) \tag{19}$$

for $\eta_i = \frac{q_{i+1}+q_i}{2}$. To implement initial and terminal conditions on the configuration and velocity (or momentum) an equivalent variational model formulation can be used :

$$y(x_{i+1}, x_i, u_i) = \begin{bmatrix} r(q_{i+1}, x_i, u_i) \\ f(x_{i+1}, q_i, u_i) \end{bmatrix} = \begin{bmatrix} -q_{i+1} + q_i - \frac{h^2}{2}M^{-1}\nabla_{\eta_i} V(\eta_i) + hM^{-1}p_i - \frac{h^2}{2}M^{-1}u_i \\ q_{i+1} - q_i - \frac{h^2}{2}M^{-1}\nabla_{\eta_i} V(\eta_i) - hM^{-1}p_{i+1} + \frac{h^2}{2}M^{-1}u_i \end{bmatrix} = 0 \tag{20}$$

Here h denotes the discrete time step and $x_i = [q_i^T, p_i^T]$ where $p_i \in \mathbb{R}^{n_q}$ is the discrete conjugate momentum [13].

Remark 5: Notably, the model (19) has reduced dimensionality ($g \in \mathbb{R}^{n_q}$) compared to its equivalent reformulation (20) ($y \in \mathbb{R}^{2n_q}$) and to other standard Euler or Runge-Kutta models used to discretize (18) in which both the configuration and the velocity are simulated in time.

Remark 6: Based on Assumption 4 the continuous model (18) can be rewritten as a convex implicit function. Using both (19) and (20) the discrete model remains a convex function, something that is not guaranteed with higher stage Runge-Kutta methods or other higher order standard methods. Thus the variational model provides a convex, second order approximation of the system dynamics which can simulate energy and momenta accurately for exponentially long times (in the presence of forcing, any momentum conservation laws from the continuous case are accurately preserved in the discrete simulation as guaranteed by the Discrete Noether Theorem with Forcing [18]). As we will discuss in Section IV-B, it also allow for a reduction in the number of optimization variables and constraints of an OCP.

B. Variational RHOCP formulation

A cMPC RHOCP formulation can be created using only the implicit model (20) for $k = 0, ..., N-1$ as presented in (11*a*-11*g*) with a state vector comprising of both the configuration and the momentum instances $x_{k|i} = [q_{k|i}^T, p_{k|i}^T]$. Alternatively in the cases where no cost weights are placed on the velocity/momentum a second formulation can be made which reduces the number of constraints and variables based on the configuration-only model in (19) as follows:

VcMPC RHOCP :

$$(\mathbf{c}_i^*, \mathbf{S}_i^*, \mathbf{S}_i^{+,*}, \mathbf{S}_i^{-,*}, Z_{0|i}^*, Z_{N|i}^*, Z_{N|i}^{+,*}, Z_{N|i}^{-,*}) = \arg \min_{\substack{\mathbf{c}_i, \mathbf{S}_i^+, \mathbf{S}_i^-, \mathbf{S}_i \\ Z_{0|i}, Z_{N|i}, Z_{N|i}^+, Z_{N|i}^-}} J_{\text{cMPC}}(\nu_i, \mathbf{S}_i, Z_{N|i}, \mathbf{q}_i^0, x_N^0, \mathbf{u}_i^0)$$

subject to:

$$Z_{0|i} = \{0\} \tag{21a}$$

$$S_{k|i} \oplus \{q_{k|i}^0\} \subseteq \mathcal{Q}, \text{ for } k = 1, ...N-1 \tag{21b}$$

$$S_{k|i}^+ \subseteq S_{k|i}, \ \ S_{k|i}^- \subseteq S_{k|i}, \text{ for } k = 1, ..., N-1 \tag{21c}$$

$$Z_{N|i}^+ \oplus \{x_{N|i}^0\} \subseteq \mathcal{X}_N, \ \ Z_{N|i}^- \oplus \{x_{N|i}^0\} \subseteq \mathcal{X}_N \tag{21d}$$

$$Z_{N|i}^+ \subseteq Z_{N|i}, \ \ Z_{N|i}^- \subseteq Z_{N|i} \tag{21e}$$

$$\nu_{k|i} + u_{k|i}^0 \in \mathcal{U} \quad \text{for } k = 0, ..., N-1 \tag{21f}$$

$$\begin{gathered} r(q_{1|i}^0 + s_{1|i}^-, \ x_{0|i}^0 + z_{0|i}, \ \nu_{0|i} + u_{0|i}^0) \leq 0 \\ F_{0|i}s_{1|i}^+ + G_{0|i}z_{0|i} + Y_{0|i}\nu_{0|i} \geq 0 \\ \text{for } \forall z_{0|i} \in Z_{0|i}, \forall s_{1|i}^+ \in S_{1|i}^+, \forall s_{1|i}^- \in S_{1|i}^- \end{gathered} \tag{21g}$$

$$\begin{gathered} g(q_{k+1|i}^0 + s_{k+1|i}^-, \ q_{k|i}^0 + q_{k|i}, \\ q_{k-1|i}^0 + s_{k-1|i}^-, \ \nu_{k|i} + u_{k|i}^0, \ \nu_{k-1|i} + u_{k-1|i}^0) \leq 0 \\ C_{k|i}s_{k+1|i}^+ + D_{k|i}s_{k|i} + H_{k|i}s_{k-1|i} + E_{k|i}\nu_{k|i} + P_{k|i}\nu_{k-1|i} \geq 0 \\ \text{for } \forall s_{k-1|i} \in S_{k-1|i}, \forall s_{k|i} \in S_{k|i}, \forall s_{k+1|i}^+ \in S_{k+1|i}^+, \\ \forall s_{k+1|i}^- \in S_{k+1|i}^- \text{ and } k = 1, ...N-1 \end{gathered} \tag{21h}$$

$$\begin{gathered} f(x_{N|i}^0 + z_{N|i}^-, \ q_{N-1|i}^0 + s_{N-1|i}, \ \nu_{N-1|i} + u_{N-1|i}^0) \leq 0 \\ L_{N-1|i}z_{N|i}^+ + P_{N-1|i}s_{N-1|i} + T_{N-1|i}\nu_{N-1|i} \geq 0 \\ \text{for } \forall s_{N-1|i} \in S_{N-1|i}, \forall z_{N|i}^+ \in Z_{N|i}^+, \forall z_{N|i}^- \in Z_{N|i}^- \end{gathered} \tag{21i}$$

where the variational convex RHOCP is denoted VcMPC RHOCP, $s_{k|i} = q_{k|i} - q_{k|i}^0$, $x_{k|i} = x_{k|i} - x_{k|i}^0$, $s_{0,i} = z_{0,i}[0 : n_q]$, $Z_{k,i} = Co\{z_{k,i}^m \in \mathbb{R}^{2n_q}, m = 0, ..., 2n_q\}$, $S_{k,i} = Co\{s_{k,i}^m \in \mathbb{R}^{n_q}, m = 0, ..., n_q\}$ and the plus and minus sets are defined similarly. Additionally the matrices $(F_{0|i}, G_{0|i}, Y_{0|i})$, $(C_{k|i}, D_{k|i}, H_{k|i}, E_{k|i}, P_{k|i})$, $(L_{N-1|i}, P_{N-1|i}, T_{N-1|i})$ are defined similarly to (8) with respect to the elements of $r(\cdot)$, $g(\cdot)$, and $f(\cdot)$ respectively with the linearization performed around the seed configuration trajectory $\mathbf{q}^0$ and the given initial state $x_i^0 =$

$[(q_i^0)^T, (p_i^0)^T]$. Here $\mathbf{q}_i^0 : \{q_{0|i}^0, q_{1|i}^0, \dots, q_{N-1|i}^0\}$ and

$$J_{\text{VcMPC}}(\nu_i, \mathbf{S}_i, Z_{N|i}, \mathbf{q}_i^0, x_N^0, \mathbf{u}_i^0) = \textstyle\sum_{k=0}^{N-1}(\|u_{k|i}^0 + \nu_{0|i} - u_{k|i}^r\|_R^2 + \max_{s_{k|i}\in S_{k|i}} \|q_{k|i}^0 + s_{k|i} - q_{k|i}^r\|_Q^2) + \max_{z_{N|i}\in Z_{N|i}} \|x_{N|i}^0 + z_{N|i} - x_{N|i}^r\|_{Q_N}^2$$

but constraints on the velocity can also be placed using forward, backward or central difference approximation. In this formulation we have used (19) to define the model constraints for $k = 1, ..., N{-}1$ and the equations in (20) to define the initial and terminal constraints. The terminal set and cost can in this case are computed based on (20) guaranteeing convergence of both the configuration and momentum to an equilibrium point. Such formulation maintains feasibility guarantees due to the equivalence of the model (19) and (20). Steps 5-6 in Algorithm 1 must also be replaced with

$$\begin{aligned}
&\text{At } k = 0 && u_{0|i}^* = u_{0|i}^0 + \nu_{0|i}^* \\
& && r(q_{1|i}^*, x_{0|i}^*, u_{0|i}^*) = 0 \text{ with } x_{0|i}^* = x_i \\
&\text{For } k = 1, ..., N-1 && u_{k|i}^* = u_{k|i}^0 + \nu_{k|i}^* \\
& && g(q_{k+1|i}^*, q_{k|i}^*, q_{k-1|i}^*, u_{k|i}^*, u_{k-1|i}^*) = 0
\end{aligned}$$

Despite the fact that constraints ($21g$) have to be encoded for three sets rather than two sets in the standard formulation in ($11g$), as we will demonstrate in Section V, due to the smaller dimensionality of the elements of the sets in the variational formulation, such formulation allows for the reduction in both the number of optimization variables and constraints. Furthermore due to Theorem 5 it is guaranteed that the sequence of convex problem will converge to a local optimum of the original nonlinear OCP with ($5d$) replaced by the combination of (19) for steps $1, .., N-1$ and (20) for steps $0, N$. Such variational NLP formulations have been proved to demonstrate structure-preserving properties [18] in their solutions and thus the sequence of convex programs is guaranteed to converge to such solution.

V. Numerical Examples

In this section the numerical results are obtained by applying the proposed approach with a variational RHOCP formulation using python and the cvxpy library [19]. The RHOCP constraints are implemented by solving each constraint for each of the vertices the perturbation sets. For all experiments $maxiter = 3$ and $tol = 10^{-6}$.

Example 1 First we consider an example system with

$$\mathcal{L}(q, \dot{q}) = \tfrac{1}{2}(\dot{q})^T \dot{q} - 0.2e^{-q[1]}, \; f(\dot{q}, u) = \begin{bmatrix} u[1] - \dot{q}[1] - 0.2 \\ u[2] \end{bmatrix}$$

where $n_q = 2$, $n_u = 2$ and f represents the generalized forces acting on the system. The discrete model is obtained using the forced variational integrator from [18] and discrete time step $h = 8 \times 10^{-3}$. A VcMPC RHOCP with $N = 14$, $Q = 1$, $R = \text{diag}\{1, 1\}$, initial condition $q[0](0) = q[1](0) = 2$, $\dot{q}[1](0) = \dot{q}[2](0) = 2$, constraints $|q[1](t)| \leq 10$, $|q[2](t)| \leq 10$, $|u[1](t)| \leq 150$, $|u[2](t)| \leq 150$ and reference solution $q_i^r = 0, u_i^r = 0$ for all i.

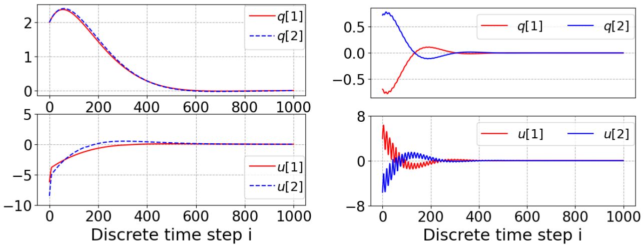

Fig. 1. Closed-loop state trajectories. Left: Example 1. Right: Example 2

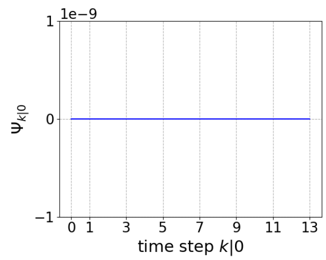

Fig. 2. Demonstration of the conservation properties of the variational cMPC approach for Example 1

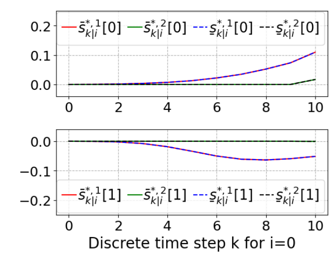

Fig. 3. Convergence of the tube shown through the size of the perturbations between iterations 1 and 2 at $i = 0$ for Example 2

Example 2 We also consider the famous Fermi-Pasta-Ulam (FPU) problem with $n_q = 2$, $n_u = 2$ and a change of coordinates as follows:

$$\mathcal{L}(q, \dot{q}) = \tfrac{1}{2}(\dot{q})^T \dot{q} - \tfrac{\eta^2(q[1]+q[2])^2}{4} - q[1]^3 - q[2]^3, \quad f(u) = u$$

with $\eta = 50$. The discrete model is again obtained using the forced variational integrator and a discrete time step of $h = 10^{-2}$. The VcMPC RHOCP is solved with $N = 10$, $Q = 0.1$, $R = \text{diag}\{0.1, 0.1\}$, reference solution $q_i^r = 0, u_i^r = 0$ for all i, constraints $|q[1](t)| \leq 10$, $|q[2](t)| \leq 10$, $|u[1](t)| \leq 33$, $|u[2](t)| \leq 33$ and initial conditions

$$\begin{bmatrix} q[1](0) \\ q[2](0) \end{bmatrix} = E \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; \begin{bmatrix} \dot{q}[1](0) \\ \dot{q}[2](0) \end{bmatrix} = E \begin{bmatrix} 1/\eta \\ 1 \end{bmatrix}, \; E = \tfrac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Figure 1 demonstrates the closed-loop trajectories for both examples obtained using Algorithm 1 with the variational changes discussed in Section IV-B. In general the conjugate momentum and its change are defined as $p = \frac{\partial \mathcal{L}(q,\dot{q})}{\partial q}$, $\dot{p} = \frac{\partial \mathcal{L}(q,\dot{q})}{\partial \dot{q}}$ and thus for the system in Example 1 $p[1]$ is conserved. Based the Forced Noether Theorem this is equivalent to the following discrete conservation law

$$\Psi_{k|i} = p_{k|i} - p_{0|i} - h \sum_{k=0}^{k-1} f(\tfrac{q_{k+1|i} - q_{k|i}}{h}, u_{k|i}) = 0$$

Using the trajectory and control sequence result from the last iteration at $i = 0$, in Figure 2 it is demonstrated that the discrete conservation law is indeed preserved by the successive approach. In Table I it is also shown that the proposed variational formulation ($21g$-$21i$) reduces the number of model constraints and optimization variables compared to an Euler formulation of ($11g$). The number of constraints will be higher with a multi-stage Runge-Kutta scheme.

Next the performance of the receding horizon scheme itself is demonstrated. In Figure 4 it is shown that for both examples

Example	Number of model constraints		Number of optimization variables	
	cMPC RHOCP	VcMPC RHOCP	cMPC RHOCP	VcMPC RHOCP
	$Nn_q 2^{4n_q+2}$	$\frac{N+1}{2} n_q 2^{3n_q+2}$	$Nn_u+n_q(12N+4)$	$Nn_u+n_q(6N+10)$
1	28672	3840	372	216
2	20480	2816	268	160

TABLE I

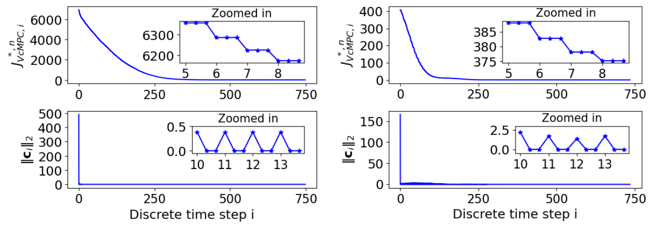

Fig. 4. Convergence of the cMPC optimal cost and control perturbation sequence norm $\|\mathbf{c}_i^*\|$ for Left: Example 1, Right: Example 2.

at successive iterations and discrete time steps the optimal predicted cost is non-increasing and $\|\mathbf{c}_i^*\| \to 0$ as expected from Theorems 3 and 4. In Figure 3 one can also see the rapid convergence of the tube to a single trajectory within a single iteration.

VI. Conclusion

In this paper we present a formulation of a tube-based optimization-based approach for trajectory design and receding horizon control of system with nonlinear, but convex dynamics. It is a reformulation of the approach presented in [11] applicable to a wider range of systems with both explicit and implicit discrete models. The approach relies on both the nonlinear dynamics directly and their linearization to formulate a sequence of convex programs solved at each discrete time step, which define a tube of possible trajectories and optimize their cross-sections online without the need of pre-defined linearization bounds. In the paper we prove that the approach guarantees stability and feasibility even when the iteration of convex programs is terminated early. If the iteration is not terminated early, we prove that it converges to a point that is a local optimum for the original nonlinear optimal control problem. We further describe how the approach can accommodate for a structure-preserving discrete system model while reducing the number of constraints and providing a structure-preserving trajectory generation and control approach. The advantages of the proposed scheme are then demonstrated for two example systems, showing convergence of the successive convex problem and the effect of the structure-preservation model on the resulting trajectory and control solution.

Appendix

Here we present the proof for Theorem 5 for a system with $n_x = 1$ and omit the $\{\cdot|i\}$ notation to conserve space. The proof for higher-dimensional systems is analogous. We also focus on simplified problems which incorporate only the constraints which do not clearly lead to the same Karush-Kuhn-Tucker (KKT) first-order necessary conditions [8]:

nMPC RHOCP simplified :

$$\mathbf{c}^* = \arg\ \min J^c_{nMPC}(\mathbf{x}, \mathbf{c})$$

subject to: $\quad \phi(x_{k+1}, x_k, c_k) = 0 \ \text{ for } k = 0, \ldots, N-1$

cMPC RHOCP simplified :

$$(\mathbf{c}^*, \mathbf{S}^*, \mathbf{S}^{+,*}, \mathbf{S}^{-,*}) = \arg \min_{\substack{\mathbf{c},\ \mathbf{S} \in S^{N+1}, \\ \mathbf{S}^+ \in (S^+)^N, \mathbf{S}^- \in (S^-)^N}} J_{\text{cMPC}}(\mathbf{c}, \mathbf{S}, \mathbf{x}^0, \mathbf{c}^0)$$

subject to:

$$C_k \bar{s}^+_{k+1} + \Phi_k \bar{s}_k + E_k c_k \geq 0, \ \ C_k \underline{s}^+_{k+1} + \Phi_k \bar{s}_k + E_k c_k \geq 0$$
$$C_k \bar{s}^+_{k+1} + \Phi_k \underline{s}_k + E_k c_k \geq 0, \ \ C_k \underline{s}^+_{k+1} + \Phi_k \underline{s}_k + E_k c_k \geq 0$$
$$\phi(x^0_{k+1} + \bar{s}^-_{k+1},\ x^0_k + \bar{s}_{k|i},\ c_{k|i} + u^0_k - K x^0_k) \leq 0$$
$$\phi(x^0_{k+1} + \underline{s}^-_{k+1},\ x^0_k + \bar{s}_{k|i},\ c_{k|i} + u^0_k - K x^0_k) \leq 0$$
$$\phi(x^0_{k+1} + \bar{s}^-_{k+1},\ x^0_k + \underline{s}_{k|i},\ c_{k|i} + u^0_k - K x^0_k) \leq 0$$
$$\phi(x^0_{k+1} + \underline{s}^-_{k+1},\ x^0_k + \underline{s}_{k|i},\ c_{k|i} + u^0_k - K x^0_k) \leq 0$$

for $\forall s^-_{k+1} \in S^-_{k+1}, \forall s^+_{k+1} \in S^+_{k+1}, \forall s_k \in S_k, k = 0, ..., N-1$

Here the nMPC problem has been reformulated using $u_k = K x_k + c_k$, $\Phi_k = D_k + E_k K$, $J_{nMPC}(\mathbf{x}, \mathbf{u}) = J^c_{nMPC}(\mathbf{x}, \mathbf{c}) = \sum_{k=0}^{\infty} \Big(\|x_k - x^r_k\|^2_Q + \|K x_k + c_k - u^r_k\|^2_R \Big)$, $g(x_{k+1}, x_k, u_k) = \phi(x_{k+1}, x_k, c_k)$. To simplify the cMPC RHOCP constraints ($11d$) have been removed as the at the cMPC convergence point $S^+_k = S_k = S^-_k = \{0\}$ and the sets have been implemented using their bounds $\underline{s}^+_{k|i}$, $\bar{s}^+_{k|i}$,$\underline{s}^-_{k|i}$, $\bar{s}^-_{k|i} \in \mathbb{R}^{n_x}$ as optimization variables where $S^+_{k|i} = \{s : \underline{s}^+_{k|i} \leq s^+ \leq \bar{s}^+_{k|i}\}$, $S^-_{k|i} = \{s : \underline{s}^-_{k|i} \leq s^- \leq \bar{s}^-_{k|i}\}$. The augumented Lagrangian and the KKT conditions for the simplified NLP problem are

$$L_{\text{nMPC}}(\mathbf{x}, \mathbf{c}, \lambda) = J^c_{\text{nMPC}}(\mathbf{x}, \mathbf{c}) - \textstyle\sum_{k=0}^{N-1} \lambda_k \phi(x_{k+1}, x_k, c_k)$$
$$0 = \tfrac{\partial J^c_{\text{nMPC}}(\mathbf{x}_i, \mathbf{c}_i)}{\partial x_0}\big|_{(\mathbf{x}^{*,n}, \mathbf{c}^{*,n})} - \lambda_0^{*,n} \tfrac{\partial \phi(r,y,c)}{\partial y}\big|_{(x_1^{*,n}, x_0^{*,n}, c_0^{*,n})}$$

for $k = 1, \ldots, N-1$

$$0 = \tfrac{\partial J^c_{\text{nMPC}}(\mathbf{x}_i, \mathbf{c}_i)}{\partial x_k}\big|_{(\mathbf{x}^{*}, \mathbf{c}^{*,n})} - \lambda_k^{*,n} \tfrac{\partial \phi(r,y,c)}{\partial y}\big|_{(x_{k+1}^{*,n}, x_k^{*,n}, c_k^{*,n})} - \lambda_{k-1}^{*,n} \tfrac{\partial \phi(r,y,c)}{\partial r}\big|_{(x_k^{*,n}, x_{k-1}^{*,n}, c_{k-1}^{*,n})}$$

and for $k = 0, \ldots, N-1$

$$0 = \tfrac{\partial J^c_{\text{nMPC}}(\mathbf{x}_i, \mathbf{c})}{\partial c_k}\big|_{(\mathbf{x}^{*,n}, \mathbf{c}^{*,n})} - \lambda_k^{*,n} \tfrac{\partial \phi(r,y,c)}{\partial c}\big|_{(x_{k+1}^{*,n}, x_k^{*,n}, u_k^{*,n})}$$
$$0 = \phi(x_{k+1}^{*,n}, x_k^{*,n}, c_k^{*,n})$$

where $(\cdot)^{*,n}$ denotes the value at the local optimum point for nMPC simplified and $\mathbf{x}_i^{*,n}$ is obtained from $\mathbf{c}_i^{*,n}$ and $\phi(x_{k+1}, x_k, c_k)$. The augumented Lagrangian and the KKT conditions for the simplified cMPC RHOCP problem become

$$L_{\text{cMPC}}(\mathbf{c}, \mathbf{S}, \mathbf{x}^0, \mathbf{u}^0, \alpha, \beta, \gamma, \mu, \psi, \eta, \xi, \tau) = J_{\text{cMPC}}(\mathbf{c}, \mathbf{S}, \mathbf{x}^0, \mathbf{u}^0) + $$
$$+ \textstyle\sum_{k=0}^{N-1} \Big(\alpha_k \phi(z^1_k) + \beta_k \phi(z^2_k) + \gamma_k \phi(z^3_k) + \mu_k \phi(z^4_k)$$
$$-(\psi_k (C_k \bar{s}^-_{k+1} + \Phi_k \bar{s}_k + E_k c_k) + \eta_k (C_k \underline{s}^-_{k+1} + \Phi_k \bar{s}_k + E_k c_k)$$
$$+ \xi_k (C_k \bar{s}^-_{k+1} + \Phi_k \underline{s}_k + E_k c_k) + \tau_k (C_k \underline{s}^-_{k+1} + \Phi_k \underline{s}_k + E_k c_k)) \Big)$$

for $k = 1, \ldots, N-1$

$$0 = \alpha_{k-1}^{*,c} \tfrac{\partial \phi(r,y,c)}{\partial r}\big|_{z_{k-1}^{1,*,c}} + \gamma_{k-1}^{*,c} \tfrac{\partial \phi(r,y,c)}{\partial r}\big|_{z_{k-1}^{3,*,c}} \tag{26a}$$
$$0 = \beta_{k-1}^{*,c} \tfrac{\partial \phi(r,y,c)}{\partial r}\big|_{z_{k-1}^{2,*,c}} + \mu_{k-1}^{*,c} \tfrac{\partial \phi(r,y,c)}{\partial r}\big|_{z_{k-1}^{4,*,c}} \tag{26b}$$

$$0 = -\psi_{k-1}^{*,c} \frac{\partial \phi(r,y,c)}{\partial r}\big|_{z_{k-1}^{5,*,c}} - \xi_{k-1}^{*,c} \frac{\partial \phi(r,y,c)}{\partial r}\big|_{z_{k-1}^{7,*,c}} \tag{26c}$$

$$0 = -\eta_{k-1}^{*,c} \frac{\partial \phi(r,y,c)}{\partial r}\big|_{z_{k-1}^{6,*,c}} - \tau_{k-1}^{*,c} \frac{\partial \phi(r,y,c)}{\partial r}\big|_{z_{k-1}^{8,*,c}} \tag{26d}$$

and for $k = 0, \ldots, N-1$

$$\begin{aligned} 0 = \frac{\partial J_{\text{cMPC}}}{\partial \bar{s}_k}\big|_{\mathbf{z}_i^{*,\mathbf{c}}} + \alpha_k^{*,c} \frac{\partial \phi(r,y,c)}{\partial y}\big|_{z_k^{1,*,c}} + \beta_k^{*,c} \frac{\partial \phi(r,y,c)}{\partial y}\big|_{z_k^{2,*,c}} \\ - \psi_k^{*,c} \frac{\partial \phi(r,y,c)}{\partial y}\big|_{z_k^{5,*,c}} - \eta_k^{*,c} \frac{\partial \phi(r,y,c)}{\partial y}\big|_{z_k^{6,*,c}} \end{aligned} \tag{26e}$$

$$\begin{aligned} 0 = \frac{\partial J_{\text{cMPC}}}{\partial \underline{s}_k}\big|_{\mathbf{z}_i^{*,\mathbf{c}}} + \gamma_k^{*,c} \frac{\partial \phi(r,y,c)}{\partial y}\big|_{z_k^{3,*,c}} + \mu_k^{*,c} \frac{\partial \phi(r,y,c)}{\partial y}\big|_{z_k^{4,*,c}} \\ - \xi_k^{*,c} \frac{\partial \phi(r,y,c)}{\partial y}\big|_{z_k^{7,*,c}} - \tau_k^{*,c} \frac{\partial \phi(r,y,c)}{\partial y}\big|_{z_k^{8,*,c}} \end{aligned} \tag{26f}$$

$$\begin{aligned} 0 = \frac{\partial J_{\text{cMPC}}}{\partial c_k}\big|_{\mathbf{z}_i^{*,\mathbf{c}}} + \alpha_k^{*,c} \frac{\partial \phi(r,y,c)}{\partial c}\big|_{z_k^{1,*,c}} + \beta_k^{*,c} \frac{\partial \phi(r,y,c)}{\partial c}\big|_{z_k^{2,*,c}} \\ +\gamma_k^{*,c} \frac{\partial \phi(r,y,c)}{\partial c}\big|_{z_k^{3,*,c}} + \mu_k^{*,c} \frac{\partial \phi(r,y,c)}{\partial c}\big|_{z_k^{4,*,c}} - \psi_k^{*,c} \frac{\partial \phi(r,y,c)}{\partial c}\big|_{z_k^{5,*,c}} \\ -\eta_k^{*,c} \frac{\partial \phi(r,y,c)}{\partial c}\big|_{z_k^{6,*,c}} - \xi_k^{*,c} \frac{\partial \phi(r,y,c)}{\partial c}\big|_{z_k^{7,*,c}} - \tau_k^{*,c} \frac{\partial \phi(r,y,c)}{\partial c}\big|_{z_k^{8,*,c}} \end{aligned} \tag{26g}$$

$$0=\alpha_k^{c,*}\phi(z_k^{1,c,*})=\beta_k^{c,*}\phi(z_k^{2,c,*})=\gamma_k^{c,*}\phi(z_k^{3,c,*})=\mu_k^{c,*}\phi(z_k^{4,c,*}) \tag{26h}$$

$$0=\psi_k^{c,*}(C_k \bar{s}_{k+1}^{-,c,*}+\Phi_k \bar{s}_k^{c,*}+E_k c_k^{c,*})=\eta_k^{c,*}(C_k \underline{s}_{k+1}^{-,c,*}+\Phi_k \bar{s}_k^{c,*}+E_k c_k^{c,*})$$

$$0=\xi_k^{c,*}(C_k \bar{s}_{k+1}^{-,c,*}+\Phi_k \underline{s}_k^{c,*}+E_k c_k^{c,*})=\tau_k^{c,*}(C_k \underline{s}_{k+1}^{-,c,*}+\Phi_k \underline{s}_k^{c,*}+E_k c_k^{c,*})$$

where: $\mathbf{z}_i^{*}=(\mathbf{c}^{c,*},\mathbf{S}^{c,*},\mathbf{S}^{+,c,*},\mathbf{S}^{-,c,*},\mathbf{x}_i^0,\mathbf{u}_i^*)$

$$z_k^1=(x_{k+1}^0+\bar{s}_{k+1}^-,\ x_k^0+\bar{s}_k,\ \sigma_k),\quad z_k^2=(x_{k+1}^0+\underline{s}_{k+1}^-,\ x_k^0+\bar{s}_k,\ \sigma_k)$$

$$z_k^3=(x_{k+1}^0+\bar{s}_{k+1}^-,\ x_k^0+\underline{s}_k,\ \sigma_k),\quad z_k^4=(x_{k+1}^0+\underline{s}_{k+1}^-,\ x_k^0+\underline{s}_k,\ \sigma_k)$$

$$z_k^5=(x_{k+1}^0+\bar{s}_{k+1}^+,\ x_k^0+\bar{s}_k,\ \sigma_k),\quad z_k^6=(x_{k+1}^0+\underline{s}_{k+1}^+,\ x_k^0+\bar{s}_k,\ \sigma_k)$$

$$z_k^7=(x_{k+1}^0+\bar{s}_{k+1}^+,\ x_k^0+\underline{s}_k,\ \sigma_k),\quad z_k^8=(x_{k+1}^0+\underline{s}_{k+1}^+,\ x_k^0+\underline{s}_k,\ \sigma_k)$$

and $z_k^{1,*,c} - z_k^{8,*,c}$ are analogously defined with $\sigma_k = c_k + u_k^0 - Kx_k^0$. Now at the convergence point $(\mathbf{c}_i^{*,n}, \mathbf{S}_i^{*,n}) = (0, \{\{0\}, \ldots, \{0\}\})$ summing (26*a* - 26*d*) for $k = 0$ and (26*a* - 26*f*) for $k > 1$ we obtain

$$\begin{aligned} \frac{\partial J_{\text{cMPC}}}{\partial x_0}\big|_{\mathbf{z}_i^{*,\mathbf{c}}} + (\alpha_0^{*,c}+\beta_0^{*,c}+\gamma_0^{*,c}+\mu_0^{*,c}-\psi_0^{*,c} \\ -\eta_0^{*,c}-\xi_0^{*,c}-\tau_0^{*,c}) \frac{\partial \phi(r,y,c)}{\partial y}\big|_{(x_1^0,\ x_0^0,\ c_0^0)}=0 \end{aligned} \tag{27}$$

$$\begin{aligned} \frac{\partial J_{\text{cMPC}}}{\partial x_k}\big|_{\mathbf{z}_i^{*,\mathbf{c}}} + (\alpha_k^{*,c}+\beta_k^{*,c}+\gamma_k^{*,c}+\mu_k^{*,c}-\psi_k^{*,c}-\eta_k^{*,c}-\xi_k^{*,c} \\ -\tau_k^{*,c}) \frac{\partial \phi(r,y,c)}{\partial y}\big|_{(x_{k+1}^0,\ x_k^0,\ c_k^0)} + (\alpha_{k-1}^{*,c}+\gamma_{k-1}^{*,c}+\beta_{k-1}^{*,c}+\mu_{k-1}^{*,c}-\psi_{k-1}^{*,c} \\ -\eta_{k-1}^{*,c}-\xi_{k-1}^{*,c}-\tau_{k-1}^{*,c}) \frac{\partial \phi(r,y,c)}{\partial r}\big|_{(x_k^0,\ x_{k-1}^0,\ c_{k-1}^0)}=0 \text{ for } k \geq 1 \end{aligned} \tag{28}$$

with $c_k^0 = u_k^0 - Kx_k^0$ because at the point of interest $z_k^{1,*,c} = z_k^{2,*,c} = \ldots = z_k^{8,*,c}$ and the cost derivative for a given k is non-zero for either the bar or the underbar variable, but never both at the same time due to the min-max cost formulation. Similarly for $k \geq 0$ Equation (26*g*) becomes

$$\begin{aligned} \frac{\partial J_{\text{cMPC}}}{\partial c_k}\big|_{\mathbf{z}_i^{*,\mathbf{c}}} + (\alpha_k^{*,c}+\beta_k^{*,c}+\gamma_k^{*,c}+\mu_k^{*,c}-\psi_k^{*,c}-\eta_k^{*,c} \\ -\xi_k^{*,c}-\tau_k^{*,c}) \frac{\partial \phi(r,y,c)}{\partial c}\big|_{(x_{k+1}^0,\ x_k^0,\ c_k^0)}=0 \end{aligned} \tag{29}$$

Now using (27), (28), and (29) together and comparing to the nMPC KKT for the simplified problem with $\mathbf{x} = \mathbf{x}^0, \mathbf{c} = \mathbf{c}^0$ it is clear that the point $(\mathbf{x}_i^0, \mathbf{c}_i^0, \{-\alpha_k^{*,c} - \beta_k^{*,c} - \gamma_k^{*,c} - \mu_k^{*,c} + \psi_k^{*,c} + \eta_k^{*,c} + \xi_k^{*,c} + \tau_k^{*,c}$ for $k = 0, \ldots, N\})$ satisfies the simplified KKT conditions for nMPC RHOCP simplified. This point, to which the successive convex programs converge, is also a local optimum for the original NLP problem.

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