2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

Data-driven Gaussian process output regulation for a class nonlinear systems

Telema Harry and Martin Guay

Abstract— **In this paper, we solve the robust output regulation problem (RORP) for a class of nonlinear systems using a data-driven approach to reconstruct the internal model unknown nonlinear continuous map online from some input and output data. The data-driven model is then used to estimate the ideal feed-forward steady-state control inputs obtained by solving the regulator equation instead of implementing it with an extended observer as in previous studies. Secondly, we implement an output feedback stabilizer that does not rely on the complete knowledge of the system but on output measurement of the regulated output, making the proposed approach suitable for systems with modelling errors. Finally, we showed through detailed Lyapunov analysis that under certain conditions the closed-loop system achieves practical asymptotic stability.**

I. INTRODUCTION

Output regulation is the standard theory for solving disturbance rejection and reference signal tracking problems. *internal model principle* is the main technique for solving such problems, it can be intuitively be stated as a controller must incorporate a suitably reduplicated model of the exosystem dynamic structure with feedback from the regulated variable. The output regulation problem for linear systems subject to external disturbance was elegantly solved in the works of Davison, Francis, Wonham [1], [2].

Extending the internal model principle to nonlinear systems is a very challenging problem. In fact, unlike linear systems, the knowledge of the exosystem alone is neither a sufficient condition nor necessary condition for solving output regulation problems in nonlinear systems [3], [4], [5]. Several significant contributions have been made to enhance the nonlinear output regulation theory using the concepts of immersion and steady state generators (see [3], [4], [6], [7] and reference therein for more details).

All these works solves the RORP by describing the regulator equation with some equations whose analytical solutions are difficult to solve even for simple problems [8]. The alternative is to approximate the internal model using data-driven approach. Recently, [9], [8] proposed an elegant adaptive data-driven regulator in which the unknown internal model steady-state continuous mapping is formulated as a continuous-time [9] and a discrete-time [8] system identification problem based on least-squares. Though the least squares algorithm is shown to be effective, it constrains the unknown nonlinear map to finite-dimensional sets, which may be difficult to determine for complex problems. In [10], a Gaussian Process algorithm, which did not constrain the unknown nonlinear map to a finite-dimensional set was used. But like in [8], it was implemented with an extended observer, which requires the Gaussian Process algorithm to compute the time derivative of the unknown continuous steady-state mappings, which may be challenging for complex systems.

In this work, we first reconstruct the ideal internal model unknown steady-state continuous map using some input and output data and use the model to directly estimate the ideal feed-forward steady-state control inputs obtained by solving the regulator equation [6] instead of implementing it with an extended observer as in previous studies. Secondly, we implement an output feedback stabilizer that does not rely on the complete knowledge of the system but on output measurement of the regulated output, making our proposed approach robust and suitable for systems with modeling errors. We also relax the assumption that the model set of the steady-state continuous nonlinear map is known a priori by developing a non-parametric model that does not constrain the approximate model to a pre-determined model set.

This article is organized as follows. Problem formulation, linear internal model, and the Gaussian Process-based identifiers are presented in Section II. The proposed regulator and the main results of this article are presented in Section IV. Finally, a numerical example to illustrate the effectiveness of a proposed algorithm in Section V and conclusions in Section VI.

Notations: If $\mathcal{A} \subset \mathbb{R}^n$, then $|x|_{\mathcal{A}} \coloneqq \inf_{a\mathcal{A}} |x-a|$ denotes the distance of $x \in \mathbb{R}^n$ to $\mathcal{A}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}_\infty$ if $\lim_{s\to\infty} \gamma(s) = \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each $t \in \mathbb{R}_{\geq 0}$ and, for each $s \in \mathbb{R}_{\geq 0}$, $\beta(s, \cdot)$ is continuous and strictly decreasing to zero as $t \to \infty$

II. PROBLEM FORMULATION

In this paper, we consider the practical output regulation problem for a class of nonlinear systems of the form:

$$\dot{z} = f(z, y, \varphi), \quad \dot{y} = q(z, y, \varphi) + b(z, y, \varphi)u \tag{1}$$

with system states $\text{col}(z, y) \in \mathbb{R}^{n_z} \times \mathbb{R}$, control input $u \in \mathbb{R}$, measured output to be regulated y, and $\varphi = \text{col}(w, m, \wp)$, containing the state of the exosystem $w \in \mathbb{W} \subset \mathbb{R}^{n_w}$ representing the disturbance signal to be rejected and the reference signal to be tracked. The variable $m \subset \mathbb{M} \in \mathbb{R}^{n_m}$ denote the uncertainties belonging to a known compact set $\mathbb{M}$ and $\wp \in \mathbb{P} \subset \mathbb{R}^{n_\wp}$ is some unknown parameter arising from the exosystem. We assume, the exosystem w is generated by the autonomous system:

$$\dot{w} = s(w, \wp), \quad y_0 = h_0(w, m) \tag{2}$$

where $y_0 \in \mathbb{R}$ is the output of the exosystem. We assume that the functions $f(\cdot)$, $q(\cdot)$ and $h_0(\cdot)$ are globally defined sufficiently smooth functions satisfying $f(0,0,\varphi) = 0$, $q(0,0,\varphi) = 0$, $h_0(0,m) = 0$. The function $b(z,y,\varphi)$ is lower bounded by some positive integer $\bar{b}^*$. We define the *regulated output* $e \in \mathbb{R}$ composed of systems (1) and (2) as

$$e = y - y_0 = h(z, y, \varphi) \tag{3}$$

We formally define the robust approximate output regulation problem composed of systems (1), (2) and (3) as follows:

Problem 1. *Consider systems* (1), (2) *and* (3) *and given any compact set* $\mathbb{W}$, $\mathbb{M}$ *and* $\mathbb{P} \subset \mathbb{R}^{n_\wp}$ *containing the origin, design an output feedback control law such that for any initial conditions* $w(0) \in \mathbb{W}$, $m(0) \in \mathbb{M}$, *and* $\wp(0) \in \mathbb{P}$ *and any initial states* $col(z(0), y(0)) \in \mathbb{R}^{n_z} \times \mathbb{R}$, *the closed-loop system is bounded for all* $t \geq 0$, *and the regulated output* $e(t)$ *achieves practical asymptotic stability in the following sense* $\lim_{t\to\infty} |e(t)| \leq \varepsilon$.

Next, we will outline the standard assumptions necessary to address the output regulation problem.

Assumption 1. *The dynamics of the exosystem* (2) *are such that* $w(t)$ *evolves on a compact invariant set* $\mathbb{W} \subset \mathbb{R}^p$ *as* $t \to \infty$.

Assumption 2. *(Solvability of regulator equation). There exists a sufficiently smooth functions* $\mathbf{z}^\star(w, m, \wp)$ *defined on an open set* $col(w, m, \wp) \in \mathbb{W} \times \mathbb{M} \times \mathbb{P}$ *satisfying*

$$\frac{\partial \mathbf{z}(w,m)}{\partial w} s(w,\wp) = f(\mathbf{z}^\star(w,m,\wp), 0, \varphi) \tag{4}$$

Assumption 3. *Given a compact set* $\Upsilon \subset \mathbb{R}^{n_w} \times \mathbb{M} \times \mathbb{P}$, *there exist a smooth function* $V_{\bar{z}} : \Upsilon \to \mathbb{R}_{\geq 0}$, *satisfying* $\alpha_{1z}(\|\bar{z}\|) \leq V_{\bar{z}}(\bar{z}) \leq \alpha_{2z}(\|\bar{z}\|)$ *for some class functions* $\alpha_{1z}, \alpha_{2z} \in \mathcal{K}_\infty$, *some known positive definite function* $\rho(\cdot)$ *function such that for any* φ *the* $\bar{z}$ *subsystem in system* (11) *satisfies*

$$\dot{V}_{\bar{z}} \leq -\alpha(\|\bar{z}\|) + \delta\rho(e), \tag{5}$$

where $\alpha(\cdot)$ *satisfies* $\lim_{s\to 0^+} \sup(\alpha_{iz}^{-1}(s^2)/s)$

Under Assumptions 1 and 2, let $\mathbf{y}(w, m, \wp) = h_0(w, m, \wp)$, then the ideal control input required to regulate e to the origin is given by:

$$\begin{aligned}\mathbf{u}^\star(w, m,\wp) = b^{-1}&(\mathbf{z}(w, m, \wp), h(w, m), \varphi) \\ &\left(\frac{\partial h(w,m)}{\partial w} s(w,\wp) - q(\mathbf{z}(w,m,\wp), 0, \varphi)\right)\end{aligned} \tag{6}$$

$\mathbf{y}^*(w, m)$, $\mathbf{z}^\star(w, m, \wp)$ and $\mathbf{u}^\star(w, m, \wp)$ constitutes the solution to the so called regulator equation [6]. The function $\mathbf{u}^\star(w, m, \wp)$ provides the necessary feed-forward information to solve the output regulation problem 1. Although the expression of $\mathbf{u}^\star(w, m, \wp)$ can be easily derived, it cannot be used for feedback control design, as it contains the exogenous signal w and unknown parameter m. In practice, an internal model is designed to reproduce $\mathbf{u}(w, m, \wp)$ asymptotically. Under Assumption 3, the state $\bar{z}$ is ISS with respect to the error e.

III. Preliminaries

A. Regulator Model Analysis

In this paper, we propose a nonlinear regulator that utilizes a data-driven internal model, generated at discrete-time intervals to generate control signal for a continuous-time system. In this context, we formulate our regulator as a hybrid system using the formalism described in [11]. Consider a hybrid system $\mathcal{H}$ with state $x \in \mathbb{X}$ and input $u \in \mathcal{U}$ of the form:

$$\mathcal{H} : \quad \begin{cases} (x,u) \in \mathcal{C} & \dot{x} = f(x,u) \\ (x,u) \in \mathcal{D} & x^+ = g(x,u) \end{cases} \tag{7}$$

where $\mathcal{C}$ and $\mathcal{D}$ denotes the *flow set* and *jump set* respectively. The vector fields $f(\cdot)$ and $g(\cdot)$ are assumed to be continuous on $\mathcal{C}$ and $\mathcal{D}$, respectively. The solution of system (7) are defined on hybrid time domain. We recall some definitions related to the hybrid formulation from [11]

Definition 1. *(Hybrid time domains) A subset* $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ *is a compact hybrid time domain if for all* $(T, J) \in E$, $E \cap ([0,T] \times \{0, 1, \cdots, J\}) = \bigcup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ *for some finite sequence of times* $0 = t_0 \leq t_1 \leq \cdots \leq t_J$.

B. Linear Internal Model and Augmented System

Motivated by the works of references [4], [7], we define a linear internal model of the form:

$$\dot{\eta} = M\eta + Nu, \quad \eta \in \mathbb{R}^{n_\eta} \tag{8}$$

where $(M, N) \in \mathbb{R}^{n_\eta \times n_\eta} \times \mathbb{R}^{n_\eta \times 1}$ is a controllable pair, with M a Hurwitz matrix and n_η is a sufficiently large positive number. We design $n_\eta = 2(n_w + n_z + 1)$ per result in [4]. As shown in reference [4] system (8) can be generated using the observation mappings [12], [13].

$$\begin{aligned}\boldsymbol{\eta}^\star(w(t), m, \wp) &= \int_{-\infty}^{t} \exp(M(t-\tau)N\mathbf{u}(w(\tau), m, \wp)d\tau \\ \mathbf{u}(w(t), m, \wp) &= \varpi(\boldsymbol{\eta}^\star(w(t), m, \wp)), \quad \boldsymbol{\eta} \in \mathbb{R}^{n_\eta}\end{aligned}$$

Since $\varpi(w(t), m, \wp)$ is an integral of past outputs by definition, the current value can be generated asymptotically following the observer structure of the form [12]:

$$\begin{aligned}\dot{\boldsymbol{\eta}}^\star(w(t), m, \wp) &= M\boldsymbol{\eta}^\star(w(t), m, \wp) + N\mathbf{u}(w(t), m, \wp) \\ \mathbf{u}(w(t), m, \wp) &= \varpi(\boldsymbol{\eta}^\star(w(t), m, \wp))\end{aligned} \tag{9}$$

where $\varpi(\boldsymbol{\eta}^\star(w(t), m, \wp))$ is a continuous nonlinear mapping. Next, we implement a coordinate transformation on the augmented system composed of systems (1), (3) and (8) by defining new states:

$$\begin{aligned}\bar{z} &= z - \mathbf{z}(w, m, \wp) \\ \bar{\eta} &= \eta - \boldsymbol{\eta} - b^{-1}(\bar{z} + \mathbf{z}, e + h(w, m, \wp), \varphi)Ne\end{aligned} \tag{10}$$

We can obtain the so-called augmented system by operating system (10) on system (1), (3) and (8):

$$\dot{\bar{z}} = \bar{f}(\bar{z}, e, \varphi), \quad \dot{\bar{\eta}} = M\bar{\eta} + \bar{r}(\bar{z}, e, \varphi) \tag{11a}$$

$$\dot{e} = \bar{q}(\bar{z}, \bar{\eta}, e, \varphi) + \bar{b}(\bar{z}, e, \varphi)(u - \mathbf{u}^\star(w, m, \wp)) \tag{11b}$$

where

$$\bar{f}(\bar{z},e,\varphi) = f(\bar{z}+\mathbf{z}, e+h(w,\sigma),\varphi) - f(\mathbf{z},0,\varphi)$$
$$\bar{b}(\bar{z},e,\varphi) = b(\bar{z}+\mathbf{z}, e+h(w,\sigma),\varphi)$$
$$\bar{r}(\bar{z},e,\varphi) = \bar{b}^{-1}(\bar{z},e,\varphi)(MNe - N\bar{q}(\bar{z},e,\varphi)) - \frac{\mathrm{d}\bar{b}(\bar{z},e,\varphi)}{\mathrm{d}t}Ne$$
$$\bar{q}(\bar{z},\bar{\eta},e,\varphi) = q(\bar{z}+\mathbf{z}, e+h(w,m,\wp),\varphi) - q(\boldsymbol{z},0,\varphi)$$

It is easy to show that for any $\varphi \in \mathbb{W}\times\mathbb{M}\times\mathbb{P}$, the origin of the augmented system (11) is an equilibrium point and the regulated variable e is equal to zero at the origin with $\bar{f}(0,0,\varphi)=0, \quad \bar{r}(0,0,\varphi)=0, \quad \bar{q}(0,0,0,\varphi)=0$

Although the nonlinear map $\varpi(\cdot)$ in (9) is shown to exist, its construction is challenging and still an open research area as highlighted above. In this paper, we propose a data-driven model $\hat{\varpi}(\cdot)$ to estimate the continuous mapping $\varpi(\cdot)$. The model $\hat{\varpi}(\cdot)$ would then be used to generate the ideal feedforward control action (6).

C. Gaussian Process Regression Identifier

Motivated by the works of [8], [9], we develop a model using a supervised learning technique to approximate the unknown nonlinear map $\varpi(\cdot)$ shown in (9) online using some input–output data set. Consider the steady-state map in system (9), written as

$$\alpha^\star_{out} = \varpi(\alpha^\star_{in}) \tag{12}$$

with $\alpha^\star_{out} := \mathbf{u}^\star(w(t),m,\wp)$ and $\alpha^\star_{in} := \boldsymbol{\eta}(w(t),m,\wp)$. System (12) can be interpreted as a regression model between α_{in} and α_{out} [8]. The problem is to find an approximate model $\hat{\varpi}(\cdot)$ such that the input–output data pair $\{(\alpha_{in}(i),\alpha_{out}(i)\}_{i\in\mathbb{N}}$ fits. Since the input–output signals are unknown, we replace these signals with some "proxies" for $\alpha^\star_{in}$ and $\alpha^\star_{out}$ by using η and u respectively as proposed in [8], [9]. The proxies are then used to train the data-driven internal model.

Our goal is to approximate the unknown continuous nonlinear map $\varpi(\cdot)$ with a Gaussian process $\mu(\boldsymbol{\eta}) \sim \mathcal{GP}(m(\boldsymbol{\eta}),K(\boldsymbol{\eta},\boldsymbol{\eta'}))$ model. i.e., $\mu(\eta)\approx\varpi(\cdot)$. where $m(\cdot)$ and $K(\cdot,\cdot)$ are the mean function and kernel covariance function defined respectively as

$$\begin{aligned} m(\boldsymbol{\eta}) &= \mathbb{E}[\mu(\boldsymbol{\eta})] \\ K(\boldsymbol{\eta},\boldsymbol{\eta'}) &= \mathbb{E}[(\mu(\boldsymbol{\eta})-m(\boldsymbol{\eta'}))(\mu(\boldsymbol{\eta})-m(\boldsymbol{\eta'}))]. \end{aligned} \tag{13}$$

A Gaussian process ($\mathcal{GP}$) is a probabilistic non-parametric method of modelling complex functions from observed data. It is formally defined as a collection of random variables, any finite number of which have a joint Gaussian distribution [14]. Instead of providing a single "best fit" to the data, the $\mathcal{GP}$ model provides a distribution over functions.

The $\mathcal{GP}$ approximation of the unknown nonlinear map $\varpi(\cdot)$ is a two-step process, comprising of *learning* and *prediction*. In the learning phase, input–output datasets generated by a continuous-time system are used to train the $\mathcal{GP}$ model. The training data $\mathcal{DS} := \{\boldsymbol{\eta},\boldsymbol{u}\}_{i\in\mathbb{N}}$ contains input data set $\boldsymbol{\eta} = [\eta(t_1),\eta(t_2),\cdots,\eta(t_N)]^T \in \mathbb{R}^{n_\eta}$ with corresponding output data set $\boldsymbol{u} = [u(t_1),u(t_2),\cdots,u(t_N)]^T \in \mathbb{R}$. where $u(t_i) = \mu(\eta(t_i)) + \varepsilon(t_i), \quad \forall\ i = 1,\cdots,N$. The output data might be corrupted by a Gaussian noise $\varepsilon(t_i) \sim \mathcal{N}(0,\sigma_n^2)$. As typical in the literature, we assume a zero mean $\mathcal{GP}$, without loss of generality. A squared exponential kernel given below is used in this work.

$$K(\boldsymbol{\eta},\boldsymbol{\eta'}) = \sigma_f^2 \exp\left(-\frac{\|\boldsymbol{\eta}-\boldsymbol{\eta'}\|^2}{2\lambda^2}\right) \tag{14}$$

where $\sigma_f^2 \in \mathbb{R}_{\geq 0}$ is the signal variance and $\lambda_i \in \mathbb{R}_{\geq 0}$, $i = 1,\cdots,n$ is the length scales. σ_f^2 and λ are commonly referred to as hyperparameters. The hyperparameters are obtained from maximum likelihood according to Bayesian principles. In the prediction phase, the model is used to predict the output over some test data $\boldsymbol{\eta_*} = [\eta_*(t_i),\cdots,\eta_*(t_n)]^T$. The predictive mean and variance over some test data $\boldsymbol{\eta_\star}$ is given by:

$$\mu(\boldsymbol{\eta}_*) = \boldsymbol{K_*}^T[\boldsymbol{K}+\lambda_n^2 I]^{-1}\boldsymbol{u} \tag{15}$$
$$\sigma^2(\boldsymbol{\eta}_*) = \boldsymbol{K}(\boldsymbol{\eta_*},\boldsymbol{\eta_*}) - \boldsymbol{K_*}^T[\boldsymbol{K}+\sigma_n^2 I]^{-1}\boldsymbol{K_*} \tag{16}$$

where we have used a compact notation setting $\boldsymbol{K_*} = \boldsymbol{K_*}(\boldsymbol{\eta},\boldsymbol{\eta}_*)$, $\boldsymbol{K} = \boldsymbol{K}(\boldsymbol{\eta},\boldsymbol{\eta})$. $\boldsymbol{K}(\boldsymbol{\eta_*},\boldsymbol{\eta_*})$ is the prior covariance of the test data with itself. The $\mathcal{GP}$ model continuously learns the unknown nonlinear map from the training data $\mathcal{DS}$ and the more the training data the better the model's predictive performance. The learning curve of the $\mathcal{GP}$ model relates the Bayesian generalization error ϵ_D to the number of training data and it is independent of the test points [15]. The generalization error ϵ_D for given test point $\mathbf{u}^\star$, is given as:

$$\epsilon_D = (\mu(\boldsymbol{\eta}^\star) - \mathbf{u}^\star))^2 \tag{17}$$

It can be easily shown that the posterior variance, (16) gives the expected Bayesian generalization error ϵ_D given a training data $\mathcal{DS}$, (17) at test point at $\varpi(\boldsymbol{\eta}^\star)$. The reader should see [15] for additional information.

Assumption 4. *The unknown function $\varpi(\cdot)$ has a bounded norm under the **RKHS** generated by the kernel $K(\boldsymbol{\eta},\boldsymbol{\eta'})$.*

Assumption 5. *The covariance kernel $K(\boldsymbol{\eta},\boldsymbol{\eta'})$ is smooth and Lipschitz continuous with a constant $\mathcal{L}_k$.*

Assumptions 4 and 5 are standard assumptions in the literature, see [16], [17]. Assumption 4, ensures that the unknown function is not discontinuous in the compact set. Most commonly used covariance functions such as the squared exponential kernel (14) already fulfill the requirement imposed by Assumption 5. Next, we state some important results from the literature, which will be used to show our results.

Lemma 1. *[18, Lemma 2] For any compact set $\mathbb{H}$ and assuming the probability $\delta \in (0,1)$ holds, then:*

$$\mathcal{P}\{\|\mu(\boldsymbol{\eta}) - \varpi(\boldsymbol{\eta})\| \leq \|\boldsymbol{\beta}\|\|\sigma(\boldsymbol{\eta})\|, \forall \boldsymbol{\eta}\in\mathbb{H}\} \geq 1-\delta \tag{18}$$

where $\mu(\boldsymbol{\eta})$ and $\boldsymbol{\sigma}(\boldsymbol{\eta})$ are the mean and standard deviation of the posterior function, with $\boldsymbol{\beta} = [\beta_1,\cdots,\beta_n]^T$: $\beta =$

$\sqrt{2\|f\|_k^2 + 3000\varrho \log^3\left(\frac{N+1}{\delta}\right)}$, *where the information gain* ϱ_i *is given as* $\varrho = \max_{\boldsymbol{\eta}\in\mathbb{H}} \frac{1}{2}\log|I_N + \sigma_{on}^{-2}\boldsymbol{K}(\boldsymbol{\eta},\boldsymbol{\eta}')|$ $\boldsymbol{K}$ *is the kernel defined in equation* (14)

Remark 1. *Although* $\boldsymbol{\beta}$ *increases with the number of observations* N*, the bounds given in system* (18) *implies that the norm of the model error* $\|\boldsymbol{\mu}(\boldsymbol{\eta}) - \boldsymbol{\varpi}(\boldsymbol{\eta})\|$ *of a* $\mathcal{GP}$ *estimates is upper bounded by some constants* $\bar{\mathcal{M}} := \|\boldsymbol{\beta}\|\|\sigma(\boldsymbol{\eta})\|$ *that decreases as* $N \to \infty$

IV. MAIN RESULT

A. Gaussian Process-Based Nonlinear Regulator

We propose the regulator described by the hybrid dynamical system given by:

$$\left.\begin{cases} \dot{\tau} &= 1 \\ \dot{\eta} &= M\eta + Nu \\ \dot{\tilde{u}} &= 0 \end{cases}\right\} \tau \in [0, T] \tag{19a}$$

$$C := [0, T] \times \mathbb{R}^{n_\eta} \times \mathbb{U}$$

$$\left.\begin{cases} \tau^+ &= 0, \quad \eta^+ = \eta \\ \tilde{\boldsymbol{u}}^+ &= \mu(\boldsymbol{\eta}) + \varepsilon \end{cases}\right\} \tau = T \tag{19b}$$

$$D := \{T\} \times \mathbb{R}^{n_\eta} \times \mathbb{U}$$

where the feedback controller is given by:

$$u = \text{sat}\,(-k\rho(e)e + \tilde{u}), \quad \tilde{u} = \mu(\eta) \tag{20}$$

where $k \in \mathbb{R}_{\geq 0}$ is the controller gain to be designed and $\tilde{u}$ is an approximate of the ideal feedforward control input defined in (6). $\tilde{u}$ is estimated from a $\mathcal{GP}$ model $\mu(\eta)$ presented in Section III-C. The regulator is composed of an hybrid clock whose states is defined by τ, internal model unit η, and a $\mathcal{GP}$ model $\mu(\eta)$.

The states trajectories evolve according to the dynamical system (19a) during flows (i.e. whenever the timer τ is less than 1) and when $\tau = 1$, the timer τ resets to zero triggering the discrete-time system (19b) which updates the $\mathcal{GP}$ model $\mu(\boldsymbol{\eta})$. Note: it takes T seconds for τ to increase from zero to one and resets to zero at $\tau = T$.

The closed-loop system composed of the augmented system (11), regulator (19) and control law (20) is given below:

$$\begin{cases} \dot{\tau} &= 1 \\ \dot{\bar{z}} &= \bar{f}(\bar{z}, e, \varphi) \\ \dot{\bar{\eta}} &= M\bar{\eta} + \bar{r}(\bar{z}, e, \varphi) \\ \dot{e} &= \bar{q}(\bar{z}, \bar{\eta}, e, \varphi) + \bar{b}(\bar{z}, e, \varphi)(\text{sat}(-k\rho(e)e + \tilde{u} \\ & \quad -\varpi(\eta^*)) \\ \dot{\tilde{\boldsymbol{u}}} &= 0 \end{cases} \tag{21a}$$

$$\mathcal{C} := [0, T] \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_\eta} \times \mathbb{R} \times \mathbb{U}$$

$$\begin{cases} \tau^+ &= 0, \\ \bar{z}^+ &= \bar{z}, \quad \bar{\eta}^+ = \bar{\eta}, \quad e^+ = e \\ \tilde{\boldsymbol{u}}^+ &= \mu(\boldsymbol{\eta}) \end{cases} \tag{21b}$$

$$\mathcal{D} := \{T\} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_\eta} \times \mathbb{R} \times \mathbb{U}$$

with $\hat{\eta} = \bar{\eta} + \boldsymbol{\eta}^\star + b^{-1}(\bar{z} + \mathbf{z}, e + h(w, m, \wp), m)Ne$. We present our first result for the $(\bar{z}, \bar{\eta})$–subsystem.

Lemma 2. *Consider the* $\bar{Z} := col(\bar{z}, \bar{\eta})$ *subsystem in* (21)*, there exist a smooth function* $V_2(\bar{Z}) : \mathbb{R}^{n_z} \times \mathbb{R}^{n_\eta} \to \mathbb{R}_{\geq 0}$*, some class* $\mathcal{K}_\infty$ *functions* $\underline{\alpha}_1(\cdot)$*,* $\bar{\alpha}_1(\cdot)$*, such that*

$$\underline{\alpha}_1(\bar{Z}) \leq V_2(\bar{Z}) \leq \bar{\alpha}_1(\bar{Z}) \tag{22a}$$

$$\langle \nabla V_2(\bar{Z}), f(\bar{Z}) \rangle \leq -\Psi(\bar{Z})\bar{Z}^2 + \gamma_r(e)e^2 \tag{22b}$$

where $\Psi(\bar{Z}) > 0$ *and* $\bar{\Psi}(\bar{Z}) > 0$ *are some smooth functions.* $\gamma_r(e) > 0$ *is some smooth positive definite functions.*

This can easily be shown with a Lyapunov candidate $V_2(\bar{Z}) = V_{1z} + \bar{\eta}^T P \bar{\eta}$ where V_{1z} is taken from Assumption 2. The proof has been omitted here due to space constraints. We now present the main result of this article with the following theorem.

Theorem 1. *Suppose Assumptions 1 – 5 hold, then the closed-loop system obtained by the interconnection of systems* (1)*,* (2) *and* (3) *with the Gaussian process-based regulator* (19) *and control law* (20) *is such that there exist a sufficiently large positive constant* k*, such that any solution* ϕ *for the hybrid system* $\mathcal{H}_u$ *in equation* (21) *originating from the compact set* $\mathcal{A} := \{0, T\} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_\eta} \times \mathbb{E} \times \mathbb{U}$*,* (τ, z, y, η, e) *is bounded for all* $(t, j) \in$ *dom* ϕ *and the solution* ϕ *satisfies* $|\phi(t, j)|_{\mathcal{A}} \leq \varepsilon_\mu$ *as (t, j)* $\to \infty \times \infty$.

Proof: Let $\chi := (\tau, \bar{z}, \bar{\eta}, e, \alpha)$. We start our stability analysis by posing the Lyapunov function:

$$V(\chi) = \exp(T - \tau)(V_2(\bar{Z}) + e^2 + (\tilde{u} - \mathbf{u}^\star)^2) \tag{23}$$

Where $\mathbf{u}^\star$ is the ideal feedforward control input defined in (6) and $\tilde{u}$ is its approximation. We start our analysis by showing that $V(\chi)$ satisfies the condition $\alpha_5(\|\chi\|) \leq V(\|\chi\|) \leq \alpha_6(\|\chi\|)$ with $T > 0$. Where $\alpha_5(\cdot)$ and $\alpha_6(\cdot)$ are some class $\mathcal{K}_\infty$ functions.

$$V(\chi) \geq \underline{c}_0(\underline{\alpha}_{1z}(\|\bar{z}\|) + \lambda_{\min}(P)\|\bar{\eta}\|^2 + \|e\|^2)$$
$$V(\chi) \leq \bar{c}_0(\bar{\alpha}_{2z}(\|\bar{z}|) + \lambda_{\max}(P)\|\bar{\eta}\|^2 + \|e\|^2)$$

We see that α_5 and α_6 can be taken as

$$\alpha_5(s) = \underline{c}_0(\underline{\alpha}_{1z}(s) + (\lambda_{\min}(P) + 1)s^2) \tag{24a}$$

$$\alpha_6(s) = \bar{c}_0(\bar{\alpha}_{1z}(s) + (\lambda_{\max}(P) + 1)s^2) \tag{24b}$$

Next, we consider the flow equations (21a) on set $\mathcal{C}$. Recall: $\tilde{u} = \mu(\hat{\eta})$ and $\mathbf{u}^\star = \varpi(\boldsymbol{\eta}^\star)$. The Lyapunov function $V(\chi)$ evaluated on the flow set is given by:

$$\begin{aligned} \langle \nabla V(\chi), f(\chi) \rangle \leq & -V(\chi) + \exp(T - \tau)\Big(\langle \nabla V_2(\bar{Z}), f(\bar{Z}) \rangle \\ & + e^2 + \|\bar{q}(\bar{z}, e, \varphi)\|^2 - 2\text{sat}(k\rho(e)e)e\bar{b}^* \\ & + 2\bar{b}^* e\|\text{sat}(\mu(\hat{\eta})) - \varpi(\boldsymbol{\eta}^\star)\|\Big) \end{aligned}$$

Since $\bar{q}(0, 0, \varphi) = 0$, by [19, Lemma 11.1], there exist some smooth positive functions $\pi_2(\cdot) \geq 1$ and $\gamma_z(\cdot) \geq 1$ such that for all $\bar{z} \in \mathbb{R}^{n_z}$, $\bar{\eta} \in \mathbb{R}^{n_\eta}$, $e \in \mathbb{R}$, $\varphi \in \mathbb{W} \times \mathbb{M} \times \mathbb{P}$.

$$\|\bar{q}(\bar{Z}, e, \varphi)\|^2 \leq \pi_2(\bar{Z})\|\bar{Z}\|^2 + \gamma_z(e)e^2 \tag{25}$$

We re-write $\hat{\eta} = \boldsymbol{\eta}^\star + \zeta$ with $\zeta = \bar{\eta} + b^{-1}(\bar{z} + \mathbf{z}, e + h(w, m, \wp), m)Ne$. Under Assumption 5 the $\mathcal{GP}$ posterior mean prediction is Lipschitz continuous. Hence:

$$\|\text{sat}(\mu(\hat{\eta}) - \varpi(\boldsymbol{\eta}^\star))\|^2 \leq \|\mu(\boldsymbol{\eta}^\star) - \varpi(\boldsymbol{\eta}^\star)\|^2 + L_\mu\|\bar{\zeta}\|^2$$

Applying results from Lemma 1 and setting $\bar{\mathcal{M}} = \|\boldsymbol{\beta}\|\|\boldsymbol{\sigma(\eta)}\|$. We can design $k\rho(e)$ such that $k\rho(e) \geq \max\{\gamma(e)\} + 2$ and let $C_\mu = L_\mu\|\bar{\zeta}\|^2 + \bar{\mathcal{M}}^2 \in \mathbb{R}_{\geq 0}$. It can easily be shown that

$$\begin{aligned}\langle \nabla V(\chi), f(\chi)\rangle \leq -V(\chi) - \Big(&\Psi(\bar{Z})\|\bar{Z}\|^2 + (2k\rho(e) \\ &\bar{b}^*)e^2\Big)\exp(T-\tau) + \bar{b}^* C_\mu \exp(T-\tau)\end{aligned} \tag{26}$$

It follows that there exists a class $\mathcal{K}^\infty$ $\bar{\alpha}_1(\|\chi\|)$ such that:

$$\langle \nabla V(\chi), f(\chi)\rangle \leq -\bar{\alpha}_1\|\chi\| + \bar{b}^* C_\mu \exp(T) \tag{27}$$

As a result, the states of the system will enter a ball of radius $\varepsilon_\mu = \bar{\alpha}_1^{-1}(\bar{b}^* C_\mu \exp(T))$ centered at the origin.

During jumps, the $\mathcal{GP}$ model is updated with $N_s \times 1$ training datasets. Let $\mu_{n+1}(\eta)$ and $\mu_n(\eta)$ denote the former and later predictive mean respectively, with a little abuse of notation. For any $(\tau, \bar{z}, \bar{\eta}, e, \alpha) \in \mathcal{D}$. We have:

$$\begin{aligned}V(g(\chi)) - V(\chi) = ((&\mu_{n+1}(\eta) - \mathbf{u}^\star)^2 \\ &- (\mu_n(\eta) - \mathbf{u}^\star)^2)\exp(T)\end{aligned}$$

If we apply equation (17) and substitute ϵ_D with the posterior variance $\sigma^2(\eta)$, we will have:

$$V(g(\chi)) - V(\chi) = (\sigma_{n+1}^2(\eta) - \sigma_n^2(\eta))\exp(T) \tag{28}$$

We show that $\sigma_{n+1}^2(\eta) < \sigma_n^2(\eta)$. We recall the predictive variance given in (16) and let the Gram matrix $\bar{K} = K + \sigma_n^2 I$. We can partition the $(n+1)\times 1$ vector $\boldsymbol{K}_*$ into a vector $\mathbf{k}_n$ and a scalar $\mathbf{c}$. i.e. $\boldsymbol{K}_*(\eta) = \begin{bmatrix}\mathbf{k}_n(\eta) & \mathbf{c}(\eta)\end{bmatrix}^T$. with $\mathbf{k}_n(\eta) = (\boldsymbol{K}(\boldsymbol{\eta}, \boldsymbol{\eta}_1), \cdots, \boldsymbol{K}(\boldsymbol{\eta}, \boldsymbol{\eta}_n))$ and $\mathbf{c}(\eta) = \boldsymbol{K}(\boldsymbol{\eta}, \boldsymbol{\eta}_{n+1})$. Let the $(n+1)^{\text{th}}$ element be denoted by η'. The $(n+1)\times(n+1)$ Gram matrix can be partitioned into four sub-matrices:

$$\bar{\boldsymbol{K}}_{n+1} = \begin{bmatrix} Q_n & \mathbf{k}_n(\eta') \\ \mathbf{k}_n^T(\eta') & \mathbf{c}(\eta')\end{bmatrix}, \bar{\boldsymbol{K}}_{n+1}^{-1} = \begin{bmatrix}\tilde{Q}_n & \tilde{\mathbf{k}}_n(\eta') \\ \tilde{\mathbf{k}}_n^T(\eta') & \tilde{\mathbf{c}}(\eta')\end{bmatrix}$$

The matrix Q_n is the $n\times n$ covariance matrix of a $\mathcal{GP}$ trained with n dataset. The inverse of the Gram matrix is obtained applying matrix inversion lemma [14, Section **A.3**]. where

$$\begin{aligned}\tilde{Q}_n &= Q_n^{-1} + Q_n^{-1}\mathbf{k}_n(\eta')M^{-1}\mathbf{k}_n^T(\eta')Q_n^{-1} \\ M &= \tilde{\mathbf{c}}(\eta') - \mathbf{k}_n^T(\eta')Q_n^{-1}\mathbf{k}_n(\eta') \\ \tilde{\mathbf{k}}_n(\eta') &= -Q_n^{-1}\mathbf{k}_n(\eta')M^{-1}, \quad \tilde{\mathbf{c}}_n(\eta') = M^{-1}\end{aligned}$$

We now compute the posterior variance $\sigma_{n+1}^2(\eta)$ using (16))

$$\begin{aligned}\sigma_{n+1}^2(\eta) &= \boldsymbol{K}(\eta,\eta) - \begin{bmatrix}\mathbf{k}_n(\eta) \\ \mathbf{c}(\eta)\end{bmatrix}^T \bar{\boldsymbol{K}}_{n+1}^{-1}\begin{bmatrix}\mathbf{k}_n(\eta) \\ \mathbf{c}(\eta)\end{bmatrix} \\ &= \boldsymbol{K}(\eta,\eta) - \mathbf{k}_n^T(\eta)Q^{-1}\mathbf{k}_n(\eta) \\ &\quad - \frac{(\mathbf{k}_n^T(\eta)Q^{-1}\mathbf{k}_n(\eta') - \mathbf{c}(\eta'))^2}{\sigma_n^2(\eta')}\end{aligned}$$

The first two terms is the posterior variance $\sigma_n^2(\eta)$, hence:

$$\sigma_{n+1}^2(\eta) = \sigma_n^2(\eta) - \frac{(\mathbf{k}_n^T(\eta)Q^{-1}\mathbf{k}_n(\eta') - \mathbf{c}(\eta'))^2}{\sigma_n^2(\eta')} \tag{29}$$

$\frac{(\mathbf{k}_n^T(\eta)Q^{-1}\mathbf{k}_n(\eta') - \mathbf{c}(\eta'))^2}{\sigma_n^2(\eta')} \geq 0$. If $\mathbf{k}_n^T(\eta)Q^{-1}\mathbf{k}_n(\eta') > \mathbf{c}(\eta')$, then $\sigma_{n+1}^2(\eta) < \sigma_n^2(\eta)$. Therefore

$$V(g(\chi)) - V(\chi) < 0 \tag{30}$$

If $\mathbf{k}_n^T(\eta)Q^{-1}\mathbf{k}_n(\eta') < \mathbf{c}(\eta')$, then $\sigma_{n+1}^2(\eta) < \sigma_n^2(\eta)$ which implies that more training data results in a better GP model performance and no further improvement in the $\mathcal{GP}$ model is achievable when this condition is satisfied. As a result, the constant C_μ achieves its minimal value. Thus the solution ϕ of the hybrid system $\mathcal{H}_\mu$, is bounded and converges to ε_μ as $t\to\infty$ and $j\to\infty$. We prove that $|\phi(t,j)|_{\mathcal{A}} \leq \varepsilon_\mu$ as (t, j) $\to \infty\times\infty$, which implies that the error signal enters the set where $|e| \leq \varepsilon_\mu$. This completes the proof. □

Remark 2. *Complete knowledge of the system dynamics is not required, unlike the pioneering work in [8]. Here, we require direct measurement of the regulated variable and the closed-loop system is stabilized using a high-gain stabilization technique, which we attenuate with a saturation function. Finally, the algorithm performance depends on the $\mathcal{GP}$ modelling error which is shown to be bounded and reduces as the number of training data increases.*

V. Simulation Example

We present a numerical example to illustrate the effectiveness of our proposed regulator. In what follows, we solve the generalized Lorenz system problem presented in [20].

$$\begin{aligned}&\dot{z}_1 = a_{11}z_1 + a_{12}y, \quad \dot{z}_2 = a_3 z_2 + z_1 y \\ &\dot{y} = z_1(a_{21} - z_2) + a_{22}y + u, \quad e = y - w_1\end{aligned} \tag{31}$$

and the exogenous signal is given by:

$$\dot{w}_1 = \wp w_2, \quad \dot{w}_2 = -\wp w_1 \tag{32}$$

where (z_1, z_2) and y are the state, e is the regulated output, $a = \text{col}(a_{11}, a_{12}, a_{21}, a_{22}, a_3)$ are some constant parameters satisfying $a_{11} < 0$, $a_3 < 0$. a. a is allowed to undergo some perturbation: $a = (\bar{a}_{11}, \bar{a}_{12}, \bar{a}_{21}, \bar{a}_{22}, \bar{a}_3) + (m_1, m_2, \cdots, m_5)$. where $m = (m_1, m_2, \cdots, m_5)$ is an uncertain parameter and $(\bar{a}_{11}, \bar{a}_{12}, \bar{a}_{21}, \bar{a}_{22}, \bar{a}_3)$ is the nominal value of a.

It can be verified the composite system composed of system (31) and (32) satisfies Assumptions 1 - 3. Under Assumption 2, it is straightforward to show that the steady-state states $\mathbf{y}(w,m,\wp)$, $\mathbf{z}_1(w,m,\wp)$, and $\mathbf{z}_2(w,m,\wp)$ are given as: $\mathbf{y}(w,m,\wp) = w_1$, $\mathbf{z}_1(w,m,\wp) = r_{11}w_1 + r_{12}w_2$, $\mathbf{z}_2(w,m,\wp) = r_{21}w_1^2 + r_{22}w^2 + r_{23}w_1w_2$. where

$$\begin{aligned}r_{11}(m,\wp) &= -\frac{a_{11}a_{12}}{\wp^2 + a_{11}}, \quad r_{12}(m,\wp) = -\frac{a_{12}\wp}{\wp^2 + a_{11}^2} \\ r_{21}(m,\wp) &= -\frac{a_3^2 r_{11} - a_3\wp r_{12} + 2\wp^2 r_{11}}{a_3(a_3^2 + 4\sigma_2^2)} \\ r_{22}(m,\wp) &= -\frac{\wp}{a_3}r_{23}, \quad r_{23}(m,\wp) = -\frac{r_{12}a_3 + 2\wp r_{11}}{4\wp^2 + a_3^2}\end{aligned}$$

Finally, the ideal feedforward control input $\mathbf{u}(w,\wp)$ presented in system (6) is given as:

$$\begin{aligned}\mathbf{u}(w,m,\wp) = r_{31}w_1 + r_{32}w_2 + r_{33}w_1^3 + r_{34}w_2^3 \\ + r_{35}w_1^2w_2 + r_{36}w_1w_2^2\end{aligned} \tag{33}$$

$$\begin{aligned}&r_{31}(\cdot) = -b^{-1}(a_{22} + a_{21}r_{11}), \; r_{32}(\cdot) = b^{-1}(\wp - a_{21}r_{12}) \\ &r_{33}(\cdot) = b^{-1}r_{11}r_{21}, \quad r_{34}(\cdot) = b^{-1}r_{12}r_{22}\end{aligned}$$

$$r_{35}(\cdot) = b^{-1}(r_{12}r_{21} + r_{11}r_{23}),\ r_{36}(\cdot) = b^{-1}(r_{11}r_{22} + r_{12}r_{23})$$

There is no analytical approach to obtaining the ideal control input (33). We now solve the robust output regulation problem using our proposed regulator.

1) We performed the simulation with $k = 500, 700$ and implemented a normal saturation function with saturation limit set as sat $= 100$
2) $n_w = 2$, $n_z = 2$. Thus, $n_\eta = 2(n_w + n_z + 1) = 10$, let

$$M = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & \cdots & \cdots & -1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

3) Per Remark 1, the model error decreases with the number of training points. In theory, we can use all the past historical data to train or model. Thus, the number of training data N_n increases as (t, j) increases, thereby improving the model performance. For convenience, we have used input-output data sets generated in the previous $\mathcal{P}$ flow events. NOTE: no adaptation with the GP model at the first flow event. $\mathcal{P} = 10$.
4) The hybrid clock periodic interval is set as $T = 0.2$

We performed the simulation with $\wp = 0.8$, $a = [-10, 10, 28, -1, 2.6667]^T$. Initial conditions are chosen to be $z(0) = [0.6589, -1.3279]^T$, $y(0) = 0.2439$, $\eta(0) = 0$ and $w(0) = [2.1579, -0.8240]^T$. $\rho(e) = e^2 + 1$

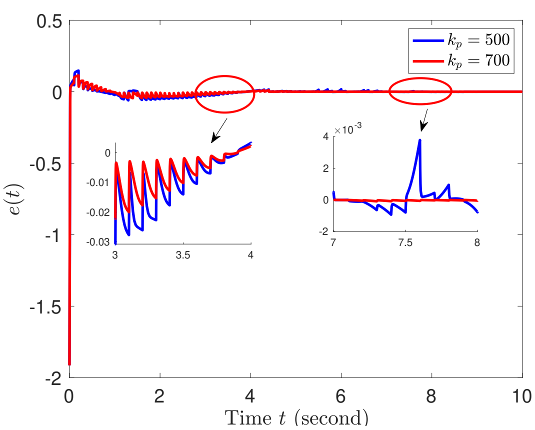

Fig. 1. Tracking error trajectories when the $\varpi(\cdot)$ is estimated with the $\mathcal{GP}$ at different k_p

Figure 1 shows the trajectory of the tracking error of the ideal feedforward control input $\mathbf{u}^\star(w, m, \wp)$ and its estimation $\mu(\eta)$ obtained from our proposed regulator at different values of gain k. The regulator performance $\mu(\eta)$ improves with higher gain k. Secondly, regulator performance improves over time as more training datasets are added to the $\mathcal{GP}$.

VI. Conclusion

The RORP is solved using a data-driven algorithm to approximate the unknown continuous nonlinear map online, which is then used to predict or estimate the ideal steady-state feedforward control action. The proposed method does not require a complete knowledge of the system, making it robust and suitable for systems with modelling errors. Future works will look at systems with unknown optimal points.

VII. Acknowledge

The authors would like to express their sincere gratitude to Michelangelo Bin for his help while writing this paper.

References

[1] B. Francis and W. Wonham, "The internal model principle of control theory," *Automatica*, vol. 12, no. 5, pp. 457–465, 1976.

[2] B. A. Francis, "The linear multivariable regulator problem," *SIAM Journal on Control and Optimization*, vol. 15, no. 3, pp. 486–505, 1977.

[3] C. Byrnes and A. Isidori, "Limit sets, zero dynamics, and internal models in the problem of nonlinear output regulation," *IEEE Transactions on Automatic Control*, vol. 48, no. 10, pp. 1712–1723, 2003.

[4] L. Marconi, L. Praly, and A. Isidori, "Output stabilization via nonlinear luenberger observers," *SIAM Journal on Control and Optimization*, vol. 45, no. 6, pp. 2277–2298, 2007.

[5] M. Bin and L. Marconi, "Output regulation by postprocessing internal models for a class of multivariable nonlinear systems," *International Journal of Robust and Nonlinear Control*, vol. 30, no. 3, pp. 1115–1140, 2020.

[6] A. Isidori and C. Byrnes, "Output regulation of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 35, no. 2, pp. 131–140, 1990.

[7] J. Huang and Z. Chen, "A general framework for tackling the output regulation problem," *IEEE Transactions on Automatic Control*, vol. 49, no. 12, pp. 2203–2218, 2004.

[8] M. Bin, P. Bernard, and L. Marconi, "Approximate nonlinear regulation via identification-based adaptive internal models," *IEEE Transactions on Automatic Control*, vol. 66, no. 8, pp. 3534–3549, 2021.

[9] P. Bernard, M. Bin, and L. Marconi, "Adaptive output regulation via nonlinear luenberger observer-based internal models and continuous-time identifiers," *Automatica*, vol. 122, p. 109261, 2020.

[10] L. Gentilini, M. Bin, and L. Marconi, "Data-driven output regulation via gaussian processes and luenberger internal models," *IFAC-PapersOnLine*, vol. 56, no. 1, pp. 367–372, 2023. 12th IFAC Symposium on Nonlinear Control Systems NOLCOS 2022.

[11] R. Goedel, R. G. Sanfelice, and A. R. Teel, *Hybrid dynamical systems: modeling stability, and robustness*. Princeton University Press, NJ, USA, 2012.

[12] G. Kreisselmeier and R. Engel, "Nonlinear observers for autonomous lipschitz continuous systems," *IEEE Transactions on Automatic Control*, vol. 48, pp. 451–464, March 2003.

[13] S. Wang, M. Guay, Z. Chen, and R. D. Braatz, "A nonparametric learning framework for nonlinear robust output regulation," *archivePrefix=arXiv*, 2024.

[14] C. E. Rasmussen and C. K. Williams, *Gaussian processes for machine learning*. Adaptive computation and machine learning, Cambridge, Mass: MIT Press, 2006.

[15] F. Vivarelli, *Studies on the generalisation of Gaussian processes and Bayesian neural networks*. PhD thesis, Aston University, Sept 1998.

[16] A. Lederer, J. Umlauft, and S. Hirche, "Uniform error bounds for gaussian process regression with application to safe control," *Advances in Neural Information Processing Systems*, vol. 32, pp. 659–669, 2019.

[17] L. Gentilini, M. Bin, and L. Marconi, "Adaptive nonlinear regulation via gaussian processes," in *2022 IEEE 61st Conference on Decision and Control (CDC)*, pp. 2532–2537, 2022.

[18] J. Umlauft, L. Pöhler, and S. Hirche, "An uncertainty-based control lyapunov approach for control-affine systems modeled by gaussian process," *IEEE Control Systems Letters*, vol. 2, no. 3, pp. 483–488, 2018.

[19] Z. Chen and J. Huang, *Stabilization and Regulation of Nonlinear Systems*. Springer International Publishing, 2015.

[20] D. Xu and J. Huang, "Robust adaptive control of a class of nonlinear systems and its applications," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 57, no. 3, pp. 691–702, 2010.