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Controllability properties of a continuously monitored qubit

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Abstract— **We consider a stochastic dynamics describing the evolution of a qubit controlled by an external field and subject to continuous-time measurements. Motivated by stabilization techniques recently developed in, e.g., [10], [11], we investigate the support of the corresponding solution, which is a random variable taking values on the space of two-by-two density matrices. By making use of the Strook-Varadhan support theorem and by classical geometric control arguments we compute the support for two possible choices of the measurement and Hamiltonian operators. In one case we show that, in Bloch coordinates, the support is always contained inside an ellipsoid depending on the physical parameters of the system. More precisely, a solution starting from the ellipsoid never exit it, with probability one, and every open subset of the ellipsoid is visited with nonzero probability for some choice of the control function. In the second case the support coincides with (the interior of) the Bloch ball: every open subset of the Bloch ball is visited with nonzero probability up to suitably choosing the control function.**

I. Introduction

One of the fundamental question in control theory concerns the controllability properties of the systems of interest, i.e., the characterization of the set of target states that can be reached from a given initial state by applying appropriate controls. This question is of great interest also in the framework of quantum dynamical systems, and it has given rise, in the last decades, to an active research field. We note that, due to the great variety of quantum dynamical systems, the problem and the techniques involved branch out in very different frameworks. A first difference concerns the nature of the Hilbert space of the system, that is, if it is finite or infinite-dimensional. In this paper, we focus on the finite-dimensional case only. Then, we must distinguish between *closed* and *open* quantum systems, i.e. if they can be considered as "isolated" or if they interact with other systems. The controllability properties of closed quantum system have already been investigated since the last three decades and are completely understood (see, e.g. [4], [9]). On the other hand, this is far from being true for open quantum systems, i.e., for systems interacting with an unmonitored environment and/or with measurement devices which provide continuous-time information about the state of the qubit. Firstly, it was observed in [2] that an open quantum system is never small-time locally controllable (STLC) by means of coherent control, and that some configurations are not reachable in finite time (this is an expected consequence of the relaxation and decoherence phenomena in open quantum systems); moreover, in [9], it has been pointed out that the Lie Algebraic Rank Condition (LARC) does not guarantee the controllability of open quantum systems.

Concerning a characterization of the reachable set, few results are available, and they often focus on qubits (i.e., two-dimensional quantum systems) and/or on specific interactions (see, for instance, [5], [6], [12], [14]).

When we consider a quantum system subject to (weak or strong) measurements, the laws of quantum mechanics impose to take into account the backaction induced by the measurement operations, and this leads to a very different mathematical framework. In particular, when we consider *continuous-time* quantum measurement, the evolution of the system is defined by a stochastic differential equation (SDE) (see, for instance, [17]).

In the context of SDEs several meaningful controllability notions may be considered. Suppose, for instance, that a stochastic process X_t is the solution of a controlled SDE in $\mathbb{R}^n$ and that the initial value X_0 admits, as density function, a Dirac distribution at a given point of $\mathbb{R}^n$. Then one may say that the system is controllable if the set of density functions attainable in an arbitrary time by suitably tuning the control function coincides with the space $\{f \in L^1(\mathbb{R}^n)\} \mid f \geq 0, \|f\|_{L^1} = 1\}$ or with a dense subset of it. This property can be stated equivalently as a controllability (or approximate controllability) property of a partial differential equation, i.e., the Fokker-Planck equation associated with the initial SDE. In particular, this problem appears to be far out of reach in view of the current state of the art in the domain of controlled partial differential equations.

This leads one to ask for weaker properties, such as the following: is it possible to find a control function such that every target point in $\mathbb{R}^n$ is contained in the support of the probability density of X_t, for t large enough? Rather surprisingly, this apparently weak property may be extremely helpful in order to prove global stability features of an equilibrium of the system. In particular, for specific open quantum systems, such a property has been used e.g. in [10], [11] in combination with suitable local Lyapunov arguments in order to prove global exponential stabilization (by measurement-based feedback) to the target equilibrium. In the special case of continuously monitored qubits, a partial obstruction to this controllability property has been observed in [15], which is motivated by the experimental results obtained in [3]; the authors prove that, for two-level systems and for some specific choices of the measurement channels,

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the stochastic dynamics on the Bloch ball at time t starting from a given state remains confined on a submanifold (a curve or a surface) evolving with t. In that article, the authors also give an algebraic characterization of the decoherence channel that give rise to such a phenomenon. In the systems considered in [15] there is no Hamiltonian evolution. In this paper, we study what happens in presence of a controlled Hamiltonian evolution, affine in the control, together with the continuous-time measurement.

In Proposition 4, we prove that, if both the free and the controlled Hamiltonian commute with the dissipation channel, then a result similar to the one proved in [15] holds true. We further investigate what may occur when the hypotheses of Proposition 4 are not met, through the analysis of two specific two-level systems, corresponding to continuous-time measurement along the z-axis and fluorescent measurement, respectively. Relying on the Strook-Varadhan support theorem and by classical geometric control arguments, we are able to compute the support of the solutions. In particular, we show that, in the first case, the support is always contained inside an ellipsoid depending on the physical parameters of the system (Theorem 7). In the second case, the support coincides with (the interior of) the Bloch ball (Theorem 12).

II. Setting and Problem Formulation

The state of a finite-dimensional quantum system is represented by a positive semi-definite Hermitian operator with trace one, acting on $\mathbb{C}^N$, for some positive integer $N > 1$; such operator is called *density matrix* (or density operator) and is usually denoted by ρ. When $N = 2$, the quantum system is known as *qubit*.

For an open quantum system undergoing continuous-time measurements the evolution of the density matrix is described by the quantum master stochastic differential equation

$$d\rho_t = -i[H, \rho_t]dt + \mathcal{L}_M(\rho_t)dt + \sqrt{\eta}\mathcal{D}_M(\rho_t)dW_t, \qquad (1)$$

where

$$\mathcal{L}_M(\rho) = M\rho M^\dagger - \frac{1}{2}M^\dagger M\rho - \frac{1}{2}\rho M^\dagger M$$

is the Lindbladian operator associated with the channel $M \in \mathbb{C}^{N\times N}$ and

$$\mathcal{D}_M(\rho) = M\rho + \rho M^\dagger - \mathrm{tr}(M\rho + \rho M^\dagger)\rho$$

corresponds to the measurement backaction, W_t is a Wiener process, and H is the (possibly controlled) Hamiltonian operator. The parameter η represents the measurement efficiency of the model. This equation has to be understood in Itô sense. For simplicity, in equation (1) we take into account a single monitored channel, associated with the operator M, although the equation can be generalized to the case of multiple monitored channels, and we assume no further interaction with the environment.

In this paper we will mainly focus on the two-level case, that is, the case $N = 2$.

The purpose of this paper is to study the support of the probability distribution corresponding to the solution ρ_t of (1) starting from a given density operator ρ_0. In particular, assuming that the Hamiltonian is of the form $H = H(u) = H_0 + uH_1$, we want to characterize the set of target states $\hat\rho$ belonging to the support of the random variable ρ_T at some time T and for some control function $t \mapsto u(t)$, i.e., the density matrices $\hat\rho$ such that any neighborhood can be attained with a nonzero probability in finite time for a suitable choice of $u(\cdot)$. We further investigate the choices of the measurement and Hamiltonian operators such that all density matrices satisfy the above property.

In order to study this problem, we take advantage of the Strook-Varadhan's support theorem, that we recall here below in an adapted form.

Theorem 1 (Support theorem [7], [16]): Consider the following SDE in Stratonovich form[1]

$$\begin{cases} dx_t = \mathcal{F}_0(x_t)dt + \mathcal{F}_1(x_t) \circ dW_t, \\ x_0 = x, \end{cases}$$

where $\mathcal{F}_0, \mathcal{F}_1$ are smooth vector fields and the initial condition x belongs to K, a compact subset of $\mathbb{R}^N$ (almost-surely) forward invariant for the dynamics. Let $\mathbb{P}_x$ be the probability law of the solution x_t starting at x. Let $\mathcal{C}$ be the space of continuous paths from $\mathbb{R}_+$ to K endowed with the topology of uniform convergence on compact sets, and $\mathcal{S}_x$ be the smallest closed subset of $\mathcal{C}$ such that $\mathbb{P}_x(x_\cdot \in \mathcal{S}_x) = 1$ (i.e., the support of the stochastic process x_t). With the previous stochastic dynamics we associate the following deterministic control system

$$\begin{cases} \dot x_v(t) = \mathcal{F}_0(x_v(t)) + \mathcal{F}_1(x_v(t))v(t), \\ x_v(0) = x \end{cases}$$

with $v : \mathbb{R}_+ \to \mathbb{R}$. Then

$$\mathcal{S}_x = \overline{\{x_v(\cdot) \mid v \text{ piecewise constant}\}}^{\mathcal{C}}.$$

Thus, to study the support of the solutions of (1), we set $\mathcal{L}'_M(\cdot) = \mathcal{L}_M(\cdot) - \frac{1}{2}\eta\frac{d\mathcal{D}_M}{d\rho}(\cdot)\mathcal{D}_M(\cdot)$ and consider the control-affine system

$$\dot\rho = -i[H_0, \rho] + \mathcal{L}'_M(\rho) - i[H_1, \rho]u + \sqrt{\eta}\mathcal{D}_M(\rho)v. \qquad (2)$$

We then study reachable sets and controllability properties of systems of the form (2). In particular, we will make use of the classical notions recalled here below.

Definition 2: Consider the control-affine system

$$\dot q(t) = \mathcal{F}_0(q(t)) + \sum_{k=1}^{m}\mathcal{F}_k(q(t))u_k(t), \quad q \in \mathcal{M}, \qquad (3)$$

where $\mathcal{F}_k$ is a smooth vector field on a connected smooth manifold $\mathcal{M}$ for each $k = 0, \dots, m$ and $u = (u_1, \dots, u_m)$

[1] We recall that any SDE in Itô form

$$dx_t = \hat{\mathcal{F}}_0(x_t)dt + \hat{\mathcal{F}}_1(x_t)dW_t$$

can be equivalently written in Stratonovich form as

$$dx_t = \mathcal{F}_0(x_t)dt + \mathcal{F}_1(x_t) \circ dW_t,$$

where $\mathcal{F}_0(\cdot) = \hat{\mathcal{F}}_0(\cdot) - \frac{1}{2}\frac{d\hat{\mathcal{F}}_1}{dx}(\cdot)\hat{\mathcal{F}}_1(\cdot)$ and $\mathcal{F}_1(\cdot) = \hat{\mathcal{F}}_1(\cdot)$.

takes values in $U \subset \mathbb{R}^m$. Then, the *reachable set* of (3) from an initial condition q_0 is defined as the set of states which can be reached in finite time by a solution of (3) starting from q_0 with u piecewise constant taking values in U. Equation (3) is said to be *controllable in* $\mathcal{M}$ if, for every $q_0 \in \mathcal{M}$, the reachable set from q_0 coincides with $\mathcal{M}$. In the case in which $\mathcal{F}_0 = 0$ and $U = \mathbb{R}^m$, the reachable set from q_0 is said to be the *orbit* of $\mathcal{F}_1, \dots, \mathcal{F}_m$ through q_0.

As a direct consequence of Theorem 1, we can state the following

Proposition 3: Consider some density matrix ρ_0 and let ρ_t be the solution at time t of equation (1), with initial condition ρ_0. Let $\hat{\rho}$ be contained in the closure of the reachable set from ρ_0 of (2). Then, for every neighborhood $\mathcal{U}$ of $\hat{\rho}$, there exists $\hat{\rho}' \in \mathcal{U}$, a piecewise-constant control function $u(\cdot)$ and a time $T > 0$ such that $\hat{\rho}'$ belongs to the support of the random variable ρ_T.

Proof: First of all we note that Theorem 1 may be extended to the case in which the vector fields $\mathcal{F}_0, \mathcal{F}_1$ are time-dependent and piecewise constant with respect to time (i.e., every interval $[0, T]$ may be split into a finite number of subintervals in each of which $\mathcal{F}_0, \mathcal{F}_1$ are smooth autonomous vector fields). This may be shown by classical arguments, e.g., by the Markov property of the trajectories and the continuity of the solution of a SDE with respect to the initial condition. By assumption, for every neighborhood $\mathcal{U}$ of $\hat{\rho}$ there exists a density matrix $\hat{\rho}' \in \mathcal{U}$ which is reachable in a time $T > 0$ by a trajectory of (2), with initial condition equal to ρ_0 and using piecewise constant controls $u(\cdot), v(\cdot)$. Let now $\mathcal{U}'$ be an open neighborhood of $\hat{\rho}'$. As the set of trajectories taking values in $\mathcal{U}'$ at time T form an open set with respect to the topology of uniform convergence on compact sets (i.e., the compact-open topology), it follows from Theorem 1 that ρ_T belongs to $\mathcal{U}'$ with nonzero probability, so that, by arbitrariness of $\mathcal{U}'$ one obtains that $\hat{\rho}'$ belongs to the support of the random variable ρ_T. ∎

III. Analysis of the Reachable Set

The purpose of this section is to investigate how the support of the random variable ρ_t depends on the Hamiltonians H_0, H_1 and on the measurement operator M in some special cases. We start with the following result.

Proposition 4: Assume that

$$[H_0, M] = [H_1, M] = [M, M^\dagger] = 0$$

and let ρ_t be the solution of (1) starting from a density matrix ρ_0. Then, for every $t > 0$ and every control function $u(\cdot)$ there exists a one-dimensional submanifold $\mathcal{R}_t$ of the space of density matrices such that $\mathbb{P}_{\rho_0}(\rho_t \in \mathcal{R}_t) = 1$.

Proof: The proposition could be obtained by adapting the arguments presented in [15], with the difference that we consider an additional Hamiltonian drift term. Here we provide a slightly different argument which allows to prove the result through significantly less demanding computations. Instead of directly studying the nonlinear equation (1), we consider the evolution of an unnormalized state, i.e., a positive semi-definite Hermitian operator $\hat{\rho}_t$ such that $\rho_t = \frac{\hat{\rho}_t}{\mathrm{tr}(\hat{\rho}_t)}$, satisfying the linear SDE

$$d\hat{\rho}_t = -i[H, \hat{\rho}_t]dt + \mathcal{L}_M(\hat{\rho}_t)dt + \sqrt{\eta}\tilde{\mathcal{D}}_M(\hat{\rho}_t)dY_t, \qquad (4)$$

on the cone of positive semidefinite Hermitian operators, where

$$\begin{aligned} dY_t &= dW_t + \sqrt{\eta}\,\mathrm{tr}(\rho_t(M + M^\dagger))dt, \\ \tilde{\mathcal{D}}_M &= M\hat{\rho}_t + \hat{\rho}_t M^\dagger, \end{aligned}$$

see, e.g., [13], [10].[2] By Girsanov theorem the process Y_t is a Brownian motion with respect to a probability measure which differs from the one associated with W_t. Note that the previous equation is analogous to the Zakai equation in classical filtering. In Stratonovich form (4) writes as

$$d\hat{\rho}_t = -i[H, \hat{\rho}_t]dt + \tilde{\mathcal{L}}_M(\hat{\rho}_t)dt + \sqrt{\eta}\tilde{\mathcal{D}}_M(\hat{\rho}_t) \circ dY_t$$

where $\tilde{\mathcal{L}}_M(\rho) = \mathcal{L}_M(\rho) - \frac{1}{2}\tilde{\mathcal{D}}_M(\tilde{\mathcal{D}}_M(\rho))$. Next, we compute the Lie bracket between $\tilde{\mathcal{L}}_M$ and $\tilde{\mathcal{D}}_M$:

$$\begin{aligned} [\tilde{\mathcal{L}}_M, \tilde{\mathcal{D}}_M](\rho) &= \tilde{\mathcal{L}}_M(\tilde{\mathcal{D}}_M(\rho)) - \tilde{\mathcal{D}}_M(\tilde{\mathcal{L}}_M(\rho)) \\ &= \mathcal{L}_M(\tilde{\mathcal{D}}_M(\rho)) - \tilde{\mathcal{D}}_M(\mathcal{L}_M(\rho)) \\ &= \frac{1}{2}[M, M^\dagger]M\rho + \frac{1}{2}\rho M^\dagger[M, M^\dagger] = 0. \end{aligned}$$

We deduce that the vector fields $\tilde{\mathcal{L}}_M, \tilde{\mathcal{D}}_M$ commute. Assume now that the Hamiltonian term is absent. From Theorem 1 and the commutativity of the flows of $\tilde{\mathcal{L}}_M$ and $\tilde{\mathcal{D}}_M$ we deduce that the support of the random variable $\hat{\rho}_t$ with initial condition ρ_0 coincides with the one-dimensional manifold

$$\hat{\mathcal{R}}_t = \{\Phi^s_{\tilde{\mathcal{D}}_M}(\Phi^t_{\tilde{\mathcal{L}}_M}(\rho_0)) \mid s \in \mathbb{R}\},$$

where Φ^τ_X denotes the flow of a vector field X at time τ.

Let us now consider the general case and let $U(\cdot)$ be the evolution operator associated with the Schrödinger equation

$$i\dot{\psi}(t) = (H_0 + u(t)H_1)\psi(t), \quad \psi(t) \in \mathbb{C}^N,$$

i.e., the unitary operator $U : \mathbb{C}^N \to \mathbb{C}^N$ such that $\psi(t) = U(t)\psi(0)$ for every initial condition $\psi(0)$. In particular, $\dot{U}(t) = -i(H_0 + u(t)H_1)U(t)$ with $U(0) = \mathbf{1}$. Moreover, under the assumptions of the proposition it follows that $U(\cdot)$ commutes with both M and $M^\dagger$. It is then easy to see that, if $\hat{\rho}_t$ is a solution of (4) then $U(t)^\dagger\hat{\rho}_t U(t)$ solves the equation

$$d\rho_t = \mathcal{L}_M(\rho_t) + \sqrt{\eta}\tilde{\mathcal{D}}_M(\rho_t)dY_t.$$

The result then follows as a consequence of the first part of the proof and by projecting into the space of density matrices via the normalization $\hat{\rho} \mapsto \frac{\hat{\rho}}{\mathrm{tr}(\hat{\rho})}$. ∎

The proof of Proposition 4 shows that every matrix in the support of ρ_t is unitarily equivalent to an element of a two-parameter family of density matrices. This suggests that, at least for large values of N, under the assumptions of the proposition, the support is far from covering the whole space of density matrices.

[2] In the experimental settings leading to the model (1), the process Y_t actually corresponds to the measurement output.

We now focus on the case $N = 2$ and investigate few relevant cases for which the assumptions of Proposition 4 are not verified. We recall that every density matrix acting on $\mathbb{C}^2$ can be written as

$$\rho = \frac{1}{2}\big(\mathbf{1} + x\sigma_x + y\sigma_y + z\sigma_z\big),$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices and $\mathbf{1}$ is the identity matrix. Due to the positivity constraint, we have that the space of density matrices is identified with the set of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x^2 + y^2 + z^2 \leq 1$. This set, called *Bloch ball*, will be denoted in the following as B. The variables x, y, z are called *Bloch coordinates* and can be computed using the fact that $\star = \mathrm{tr}(\rho\sigma_\star)$ for $\star = x, y, z$.

A. First case study: continuous-time measurement along the z-axis

We consider a monitored decoherence channel corresponding to the operator $M = \frac{\sqrt{\mu}}{2}\sigma_z$, μ representing the strength of the interaction between the probe and the quantum system to be controlled, and Hamiltonian operators $H_0 = \frac{1}{2}\omega\sigma_z$, $H_1 = \frac{1}{2}\sigma_y$, where we assume that $\omega > 0$ (the case $\omega < 0$ can be treated similarly). For $u = 0$, this model corresponds to the case of quantum non demolition measurements.

By simple computations, it follows that in Bloch coordinates the system takes the form:

$$\begin{aligned} dx_t &= \big(-\omega y_t - \frac{\mu}{2}x_t + u z_t\big)dt - \sqrt{\eta\mu}\, x_t z_t dW_t \\ dy_t &= \big(\omega x_t - \frac{\mu}{2}y_t\big)dt - \sqrt{\eta\mu}\, y_t z_t dW_t \\ dz_t &= -u x_t dt + \sqrt{\eta\mu}(1 - z_t^2)dW_t. \end{aligned} \tag{5}$$

Setting $q = (x, y, z)$, the system above can be written in Stratonovich form as

$$dq_t = (f_0(q_t) + f_1(q_t)u)dt + g(q_t) \circ dW_t,$$

where

$$\begin{aligned} f_0(q) &= \begin{pmatrix} -\omega y - \frac{\mu}{2}(1-\eta)x - \eta\mu x z^2 \\ \omega x - \frac{\mu}{2}(1-\eta)y - \eta\mu y z^2 \\ \eta\mu z(1 - z^2) \end{pmatrix}, \\ f_1(q) &= \begin{pmatrix} z \\ 0 \\ -x \end{pmatrix}, \quad g(q) = \begin{pmatrix} -\sqrt{\eta\mu}\, xz \\ -\sqrt{\eta\mu}\, yz \\ \sqrt{\eta\mu}(1 - z^2) \end{pmatrix}. \end{aligned} \tag{6}$$

Thanks to Theorem 1 and Proposition 3, the support of the random variable q_t can be investigated by computing the reachable set of the deterministic equation

$$\dot{q}(t) = f_0(q(t)) + f_1(q(t))u(t) + g(q(t))v(t). \tag{7}$$

To do so, we apply classical arguments from geometric control theory. Focusing on the controlled fields f_1 and g, by easy computations and by applying Rashevski-Chow Theorem [1, Theorem 5.9] to each invariant submanifold, we obtain the following result.

Proposition 5: Outside the circle

$$\mathcal{C} = \{q \in B \mid y = 0 \text{ and } x^2 + z^2 = 1\}$$

the Lie algebra generated by f_1, g has dimension constantly equal to two. It is of dimension one when restricted to $\mathcal{C}$.

In particular, the unit ball B can be decomposed into the following manifolds, corresponding to the orbits of the family of vector fields $\{f_1, g\}$:

- the circle $\mathcal{C}$,
- the sets

  $$\mathcal{H}_\alpha = \{q \in B \mid y = \alpha\sqrt{1 - x^2 - z^2},\ x^2 + z^2 < 1\},$$

  for $\alpha \in [-1, 1]$.

For perfect measurements (corresponding to $\eta = 1$), the system is controllable, as the following proposition shows.

Proposition 6: Let $\eta = 1$ and assume that (u, v) take their values in a set U containing an open neighborhood of the segment $\{0\}\times[-\sqrt{\mu}, \sqrt{\mu}]$. Then the control-affine system (7) with (u, v) taking values in U is controllable in the interior of B and leaves the unit sphere invariant.

Sketch of the proof. Setting $w = v + \sqrt{\mu}z$, the system takes the form

$$\dot{q} = \tilde{f}_0(q) + f_1(q)u + g(q)w, \quad (u, w) \in V(q) \tag{8}$$

where, by assumption, $V(q)$ contains an open neighborhood $\mathcal{O}$ of 0 independent of $q \in B$ and $\tilde{f}_0(q) = (-\omega y, \omega x, 0)^\top$.

It is easy to show that the vector fields $\tilde{f}_0, f_1, g$ are linearly independent if and only if $(|q|^2 - 1)xz \neq 0$. In particular, the vector fields $\tilde{f}_0, f_1, g$ are tangent to the unit sphere, which implies that the latter is an invariant set for the dynamics of (8).

In order to show the controllability of (8) in the interior of B, we apply standard controllability arguments. We first note that $\tilde{f}_0$ is a recurrent vector field [8, Definition 3, page 113] since its trajectories are periodic. Let us now consider a point $q = (x, y, z)$ in the interior of B such that $xz = 0$. It is easy to see that there exists an admissible trajectory of (8) starting from q and leaving the set $L = \{(x, y, z) \in B \mid xz = 0\}$. Since the Lie algebra generated by $\tilde{f}_0, f_1, g$ has dimension three outside L and since, by the analiticity of the vector fields, the Lie algebra has constant dimension on each orbit [8, Theorem 6, page 48], it must have dimension three on L, hence on the whole interior of B. By applying [8, Theorem 5, page 114] we deduce that the control system (8) with $(u, v) \in U$ is controllable in the interior of B. ■

We now focus on the case of an imperfect measurement, i.e., we assume that $\eta < 1$. In this case, the system is not controllable in the whole ball, as the following result shows.

Theorem 7: Assume that $\eta < 1$ and that (u, v) can take any value in $\mathbb{R}^2$. Then the system (7) is controllable in the set

$$\mathcal{A} = \{q \in B \mid x^2 + \frac{y^2}{\bar{\alpha}^2} + z^2 < 1\},$$

where

$$\bar{\alpha} = \frac{-\mu(1-\eta) + \sqrt{\mu^2(1-\eta)^2 + 4\omega^2}}{2\omega}.$$

Moreover, the reachable set of (7) starting from a point outside $\mathcal{A}$ strictly contains $\mathcal{A}$.

In order to prove the theorem, we take advantage of the following lemma, which is a consequence of Proposition 5 and the fact that, as we do not impose any bound on the controls, the orbit of the vector fields f_1, g through a point q belongs to the closure of the reachable set of (7) from q.

Lemma 8: Let $\alpha \in [-1, 1]$ and consider $q \in \mathcal{H}_\alpha$. Then $\mathcal{H}_\alpha$ belongs to the closure of the reachable set of (7) from q.

Proof of Theorem 7: It follows from Lemma 8 that, whenever there exists a trajectory of (7) from a point of $\mathcal{H}_{\alpha_1}$ to a point of $\mathcal{H}_{\alpha_2}$ for some $\alpha_1, \alpha_2 \in [-1, 1]$, then, for every $q_0 \in \mathcal{H}_{\alpha_1}$ the closure of the reachable set from q_0 contains the union of all $\mathcal{H}_\alpha$ with α comprised between α_1 and α_2. Since

$$\mathcal{A} = \cup_{\alpha \in (-\bar{\alpha}, \bar{\alpha})} \mathcal{H}_\alpha,$$

in order to prove that the closure of the reachable set from $q_0 \in \mathcal{A}$ coincides with the closure of $\mathcal{A}$, it is then enough to show that

- for every $\alpha \in (-\bar{\alpha}, \bar{\alpha})$ and every α' in a small enough neighborhood of α there exists a trajectory of (7) from a point of $\mathcal{H}_\alpha$ to $\mathcal{H}_{\alpha'}$,
- if $|\alpha| > \bar{\alpha}$ then there is no trajectory from $\mathcal{H}_{\pm\bar{\alpha}}$ to $\mathcal{H}_\alpha$.

Let us define on $B \setminus \mathcal{C}$ the function

$$\beta(q) = \frac{y^2}{1 - x^2 - z^2}.$$

The level sets $\{q \in B \setminus \mathcal{C} \mid \beta(q) = \alpha^2\}$ coincide with the union $\mathcal{H}_\alpha \cup \mathcal{H}_{-\alpha}$ for every $\alpha \in [-1, 1]$. In particular f_1, g are tangent to such level sets.

It is easy to check that $\nabla\beta(q)^\top f_0(q) < 0$ if $q = (x, y, z) \in B \setminus \mathcal{C}$ is such that $xy < 0$. Furthermore, one can show by direct computations that $\nabla\beta(q_\alpha)^\top f_0(q_\alpha) > 0$ whenever $0 < |\alpha| < \bar{\alpha}$, where $q_\alpha = (\frac{\operatorname{sign}(\alpha)}{\sqrt{2}}, \frac{\alpha}{\sqrt{2}}, 0)^\top$. Hence in this case $\mathcal{H}_{\alpha'}$ can be reached from $\mathcal{H}_\alpha$ for every α' in a small enough neighborhood of α. Similarly, it can be easily shown that $\mathcal{H}_\varepsilon$ can be reached from $\mathcal{H}_0$ if $|\varepsilon|$ is small enough. Finally, by direct maximization over $\mathcal{H}_\alpha$, one obtains that $\nabla\beta(q)^\top f_0(q) \leq 0$ for every $q \in \mathcal{H}_\alpha$ whenever $|\alpha| \geq \bar{\alpha}$. This shows that the closure of the reachable set from a point q coincides with the closure of $\mathcal{A}$ if $q \in \mathcal{A}$, or contains it if $q \notin \mathcal{A}$. The theorem then follows from Krener's Theorem (see, e.g., [1, Theorem 8.1]) and the fact that $\mathcal{A}$ is an open subset of B invariant for the dynamics. ∎

Concerning the stochastic system (5), we deduce the following.

Corollary 9: If $\eta < 1$, then for every open subset $\mathcal{O}$ of $\mathcal{A}$ and every $T > 0$ there exists $t > T$ and a piecewise-constant control function $u(\cdot)$ such that the solution of (5) visits $\mathcal{O}$ at time t with nonzero probability. For $\eta = 1$ the conclusion holds true for every open subset of B.

A simulation of 50 trajectories of (5) is depicted in Figure 1. The simulation appears to be consistent with the conclusions of Corollary 9.

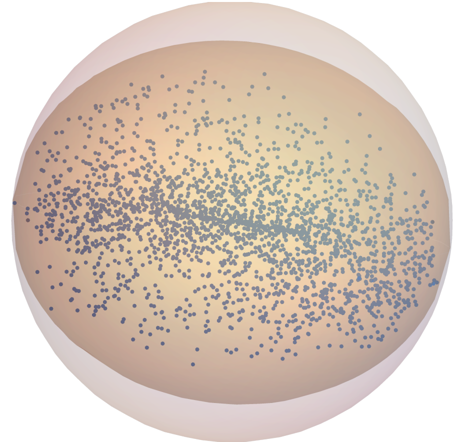

Fig. 1. Simulation of 50 solutions of (5) using randomly chosen piecewise controls $u(\cdot)$ and starting from the origin (completely mixed state). The parameters of the systems have been chosen as $\omega = \mu = 1$ and $\eta = 0.6$, and the trajectories are depicted at times $0.1k$ for $k = 1, \ldots, 50$. The orange ellipsoid corresponds to $\mathcal{A}$.

Remark 10: In the case $\omega = u = 0$ it has been shown in [15] that the dynamics of (5) is almost surely confined on a time-dependent one-dimensional manifold of the form $\{q \in B \mid (x, y) = \zeta_0(1 - z^2)e^{-\gamma_0 t}(\cos\theta_0, \sin\theta_0),\ z \in (-1, 1)\}$ for some angle θ_0 and positive ζ_0, γ_0 (θ_0, ζ_0 are determined by the initial data). In contrast, for the general controlled system (5), Theorem 7 asserts that any open subset of the three-dimensional set $\mathcal{A}$ can be reached with nonzero probability by appropriately choosing the control $u(\cdot)$.

Remark 11: Let us modify the model above by considering an additional controlled field, that is, we replace the Hamiltonian $H_0 + uH_1$ by $H_0 + u_1 H_1 + u_2 H_2$, where $H_2 = \frac{1}{2}\sigma_x$. In Stratonovich form, the system writes as

$$dq_t = (f_0(q_t) + u_1 f_1(q_t) + u_2 f_2(q_t))dt + g(q_t) \circ dW_t,$$

where f_0, f_1 and g are as in (6) and $f_2 = (0, -z, y)^\top$. The vector fields f_1, f_2 and g are linearly independent everywhere, except on the unit sphere and on the plane defined by $z = 0$. It follows that the vector field f_2 is transversal to each $\mathcal{H}_\alpha$ at almost every point and for every $\alpha \in (-1, 1)$. By proceeding similarly as in the proof of Theorem 7 one then deduces that the system is controllable in the interior of the Bloch ball B.

B. Second case study: fluorescence measurement

In this section, we consider system (1) with monitored decoherence channel $M = \sqrt{\mu}\sigma_- = \frac{1}{2}\sqrt{\mu}(\sigma_x - i\sigma_y)$, corresponding to fluorescence measurements, and controlled Hamiltonian $H = \omega\sigma_z/2 + u\sigma_y/2$, with $\omega > 0$ (the case $\omega < 0$ can be treated similarly). In Bloch coordinates, the system is described by the following SDE in Itô form:

$$\begin{aligned} dx_t &= \big(-\omega y_t - \frac{\mu}{2}x_t + u z_t\big)dt + \sqrt{\eta\mu}(1 + z_t - x_t^2)dW_t \\ dy_t &= \big(\omega x_t - \frac{\mu}{2}y_t\big)dt - \sqrt{\eta\mu}\, y_t x_t dW_t \\ dz_t &= -\big(\mu(1 + z_t) + u x_t\big)dt - \sqrt{\eta\mu}\, x_t(1 + z_t)dW_t, \end{aligned} \tag{9}$$

which can be converted into the Stratonovich SDE

$$dq_t = (\hat{f}_0(q_t) + \hat{f}_1(q_t)u)dt + \hat{g}(q_t) \circ dW_t,$$

with

$$\hat{f}_1(q) = \begin{pmatrix} z \\ 0 \\ -x \end{pmatrix}, \quad \hat{g}(q) = \begin{pmatrix} \sqrt{\eta\mu}(1+z-x^2) \\ -\sqrt{\eta\mu}xy \\ -\sqrt{\eta\mu}x(1+z) \end{pmatrix},$$
$$\hat{f}_0(q) = \begin{pmatrix} -\omega y + \frac{\mu}{2}x(\eta(z+1)-1) \\ \omega x + \frac{\mu}{2}y(\eta(z+1)-1) \\ -\mu(1+z) + \frac{\mu}{2}\eta(1+z)^2 \end{pmatrix} + \sqrt{\eta\mu}x\hat{g}(q).$$

The corresponding deterministic control system is given by

$$\dot{q}(t) = \hat{f}_0(q(t)) + \hat{f}_1(q(t))u(t) + \hat{g}(q(t))v(t). \tag{10}$$

Firstly, we notice that the conclusion of Proposition 5 is still valid for the distribution generated by $\hat{f}_1, \hat{g}$. In particular, B is stratified by the manifolds $\mathcal{C}$ and $\mathcal{H}_\alpha$, $\alpha \in [-1, 1]$, and each stratum is an orbit of the family of vector fields $\{\hat{f}_1, \hat{g}\}$. Then, the controllability properties of the system can be deduced by studying the effect of the drift $\hat{f}_0$ on the manifolds $\mathcal{H}_\alpha$.

Theorem 12: The system (10) with control functions (u, v) taking values in $\mathbb{R}^2$ is controllable in the interior of B.

Proof: The proof follows the same ideas of the proof of Theorem 7.

First, by direct computation, it is easy to see that $\nabla\beta(q_\alpha)^\top \hat{f}_0(q_\alpha) < 0$ whenever $xy < 0$. On the other hand, for $\delta \in (0, 1)$ the curve $\gamma_\delta(t) = (t, t\delta, -\sqrt{1-2t^2})$ belongs to $\mathcal{H}_\delta$ if $t > 0$ and to $\mathcal{H}_{-\delta}$ if $t < 0$, and it is easy to check that

$$\lim_{t\to 0} \nabla\beta(\gamma_\delta(t))^\top \hat{f}_0(\gamma_\delta(t)) = 2\omega\delta(1-\delta^2) > 0.$$

We then deduce the existence of $q_\alpha \in \mathcal{H}_\alpha$ such that $\nabla\beta(q_\alpha)^\top \hat{f}_0(q_\alpha) > 0$ whenever $0 < |\alpha| < 1$. This proves that, for any such α, $\mathcal{H}_{\alpha'}$ can be reached from $\mathcal{H}_\alpha$, provided that α' is sufficiently close to α. Moreover, as above, it can be shown that $\mathcal{H}_\varepsilon$ can be reached from $\mathcal{H}_0$ if $|\varepsilon|$ is small enough.

Then, the closure of the reachable set from any point q in the interior of B coincides with B itself. By Krener's theorem, and due to the fact that the interior of B is invariant for the dynamics, we get the thesis. ■

Concerning the stochastic system (9), we deduce the following.

Corollary 13: For every open subset $\mathcal{O}$ of B and every $T > 0$ there exists $t > T$ and a piecewise-constant control function $u(\cdot)$ such that the solution of (9) visits $\mathcal{O}$ at time t with nonzero probability.

IV. Further perspectives

In this paper we present some preliminary results concerning the characterization of the support of the solutions of SDEs describing the evolution of quantum systems undergoing continuous-time measurement and in presence of a controlled Hamiltonian. In particular, for a model of qubit subject to fluorescence measurements, we show a controllability property corresponding to the fact that the support covers the whole space of density matrices.

A possible further direction concerns the algebraic characterization (at least for two-level systems) of the quantum channels and the controlled Hamiltonians for which the above mentioned controllability property is satisfied. This leads also to consider multiple dissipation channels, such as in the heterodyne measurement.

Another natural development of the results presented here concerns the design of a feedback control ensuring that the solution of (1) converges almost surely to some target invariant (for some choice of the control parameter) subset of the reachable set.

References

[1] A. Agrachev and Y. Sachkov. *Control theory from the geometric viewpoint*, volume 87 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004. Control Theory and Optimization, II.

[2] C. Altafini. Controllability properties for finite dimensional quantum Markovian master equations. *J. Math. Phys.*, 44(6):2357–2372, 2003.

[3] P. Campagne-Ibarcq, P. Six, L. Bretheau, A. Sarlette, M. Mirrahimi, P. Rouchon, and B. Huard. Observing quantum state diffusion by heterodyne detection of fluorescence. *Physical Review X*, 6(1):011002, 2016.

[4] D. D'Alessandro. *Introduction to quantum control and dynamics.* Applied Mathematics and Nonlinear Science Series. Boca Raton, FL: Chapman, Hall/CRC., 2008.

[5] G. Dirr, U. Helmke, I. Kurniawan, and T. Schulte-Herbrüggen. Lie-semigroup structures for reachability and control of open quantum systems: Kossakowski-Lindblad generators form lie wedge to Markovian channels. *Reports on Mathematical Physics*, 64(1):93–121, 2009.

[6] G. Dirr, F. vom Ende, and T. Schulte-Herbrüggen. Reachable sets from toy models to controlled markovian quantum systems. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 2322–2329, 2019.

[7] N. Ikeda and S. Watanabe. *Stochastic Differential Equations and Diffusion Processes*. ISSN. Elsevier Science, 2014.

[8] V. Jurdjevic. *Geometric control theory*, volume 52 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997.

[9] I. Kurniawan, G. Dirr, and U. Helmke. Controllability aspects of quantum dynamics: a unified approach for closed and open systems. *IEEE Transactions on Automatic Control*, 57(8):1984–1996, 2012.

[10] W. Liang, N. H. Amini, and P. Mason. On exponential stabilization of N-level quantum angular momentum systems. *SIAM Journal on Control and Optimization*, 57(6):3939–3960, 2019.

[11] W. Liang, N. H. Amini, and P. Mason. Robust feedback stabilization of N-level quantum spin systems. *SIAM Journal on Control and Optimization*, 59(1):669–692, 2021.

[12] L. Lokutsievskiy and A. Pechen. Reachable sets for two-level open quantum systems driven by coherent and incoherent controls. *Journal of Physics A: Mathematical and Theoretical*, 54(39):395304, 2021.

[13] M. Mirrahimi and R. Van Handel. Stabilizing feedback controls for quantum systems. *SIAM Journal on Control and Optimization*, 46(2):445–467, 2007.

[14] L. Pinna. *On the controllability of the quantum dynamics of closed and open systems*. Phd thesis, Université Paris Saclay (COmUE) ; Università degli studi La Sapienza (Rome), Jan. 2018.

[15] A. Sarlette and P. Rouchon. Deterministic submanifolds and analytic solution of the quantum stochastic differential master equation describing a monitored qubit. *Journal of Mathematical Physics*, 58(6):062106, 2017.

[16] D. Stroock and S. R. S. Varadhan. On the support of diffusion processes with applications to the strong maximum principle. In *Volume 3 Contributions to Probability Theory*, pages 333–360, Berkeley, 1972. University of California Press.

[17] H. Wiseman and G. Milburn. *Quantum Measurement and Control*. Cambridge University Press, 2010.