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On the existence and uniqueness of steady state solutions of a class of dynamic hydraulic networks via actuator placement

Mark Jeeninga[1], Juan E. Machado[2], Michele Cucuzzella[3], Giacomo Como[4], Jacquelien Scherpen[5]

***Abstract*— In this paper, using tools from graph theory we provide verifiable necessary and sufficient conditions for the existence of a unique hydraulic equilibrium in district heating systems of meshed topology and containing multiple heat sources. Even though numerous publications have addressed the design of efficient algorithms for numerically finding hydraulic equilibria in the general context of water distribution networks, this is not the case for the analysis of existence and uniqueness. Moreover, most of the existing work dealing with these aspects exploit the equivalence between the nonlinear algebraic equations describing the hydraulic equilibria and the KKT conditions of a suitably defined nonlinear convex optimization problem. Differently, this paper proposes necessary and sufficient graph-theoretic conditions on the actuator placement for the existence and uniqueness of a hydraulic equilibrium, independent of the actuators' control objective. An example based on a representative district heating network is considered to illustrate the key aspects of our contribution, and an explicit formulation of the steady state solution is given for the case in which pressure drops through pipes are linear with respect to the flow rate.**

I. INTRODUCTION

District heating systems distribute heat from heating plants towards clusters of consumers that are part of a neighborhood, or even a wider area, using a system of heat exchangers and a *closed* network of insulated water pipes [1]. The safe and efficient operation of district heating systems relies on control systems in charge of regulating the temperature, pressure, and flow rate of the stream of water through specific sections of the network (see, e.g., [2], [3], [4], [5], [6], [7], [8], [9], [10]).

An essential step to determine suitable setpoints of variables of interest, as well as the steady state of the system in general, refers to the (optimal) heat or power flow calculation, which usually aims at minimizing operational costs while satisfying physical constraints originating from energy, mass and momentum balance conditions imposed on the system [11], [12], [13].

In this work we focus on identifying necessary and sufficient conditions for the existence of a unique *hydraulic* equilibrium, which corresponds to a solution of a system of nonlinear algebraic equations, which are necessarily embedded into the heat flow calculation.

A vast amount of literature is proposing methods for rapidly and efficiently solving the hydraulic steady state equations of district heating systems, see, e.g., [12] and the references therein. However, our focus is on constructively identifying conditions based on graph theoretic elements to certificate the solvability of the considered nonlinear algebraic equations. This has also been the focus of [14], [15], [16], [17], [18], but in the context of water distribution networks, which are typically assumed to be *open*, i.e., with water inflows and outflows. Moreover, most of these references rely on reformulating the hydraulic steady state equations as the KKT conditions of a suitably defined convex optimization problem [19]. In particular, and in contrast to most publications, [14] also derives necessary and sufficient conditions in terms of the solvability of the Kirchhoff's relations of the network. However, it is still unclear how the placement of the actuators affect the existence and uniqueness of a hydraulic equilibrium. To fill this gap, we propose necessary and sufficient graph-theoretic conditions on the actuator placement for the existence and uniqueness a hydraulic equilibrium, independent of the actuators' control objective. In addition, we derive closed-form expressions of the steady state solution in the case of linear flow.

The remainder of the paper is structured as follows. Section II introduces the model for the hydraulic network and describes the problem of the existence of steady state solutions that is addressed in this paper. In Section III we explore the natural necessary conditions on the network that arise from the problem, and we subsequently state our main result, along with a brief discussion. An explicit formulation of the (unique) steady state solution for the linear case is

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[1]M. Jeeninga is with the Department of Automatic Control, Lund University, 223 63 Lund, Sweden. mark.jeeninga@control.lth.se

[2]J. E. Machado is with the Chair of Control Systems and Network Control Technology, Faculty 3 Mechanical Engineering, Electrical and Energy Systems, Brandenburg University of Technology Cottbus - Senftenberg, 03046 Cottbus, Germany. machadom@b-tu.de

[3]M. Cucuzzella is with the Department of Electrical, Computer Science and Biomedical Engineering, Univeristy of Pavia, 27100 Pavia, Italy. michele.cucuzzella@unipv.it

[4]G. Como is with the Department of Mathematical Science "G.L. Lagrange", Politecnico di Torino, 10129 Torino, Italy giacomo.como@polito.it. He is also with the Department of Automatic Control, Lund University. giacomo.como@polito.it

[5]J. M. A. Scherpen is with the Jan C. Willems Center for Systems and Control, ENTEG, Faculty of Science and Engineering, University of Groningen, 9747 AG Groningen, the Netherlands. j.m.a.scherpen@rug.nl

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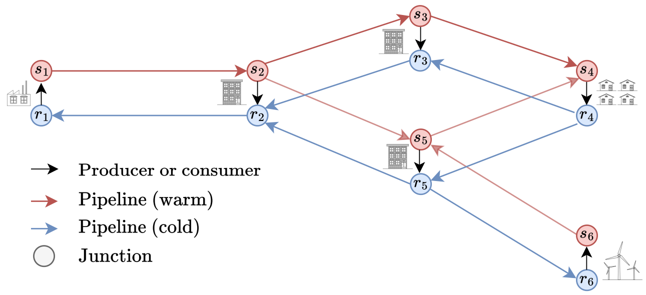

Fig. 1: Simplified representation inspired by [20] of the hydraulic network of a district heating system with multiple heat sources. Red (blue) arrows and circles respectively represent pipes and junctions of the supply (return) layer. The heat exchangers of producers or consumers are denoted by black arrows.

found in Section IV. The paper is concluded in Section V with a number of remarks and directions for future research.

Notation: $\mathbb{R}$ represents the set of real numbers and $\mathbb{R}^n$ is the set of n-dimensional vectors. The set of m by n real matrices is denoted by $\mathbb{R}^{m\times n}$. The ith and (i,j)th component of any $x \in \mathbb{R}^n$ and any $A \in \mathbb{R}^{m\times n}$ are respectively denoted by x_i and A_{ij}. For any $A \in \mathbb{R}^{m\times n}$ and any ordered index sets $\mathcal{I}, \mathcal{J}$, we denote by $A_{\mathcal{I},\mathcal{J}}$ the submatrix formed with the rows of A in $\mathcal{I}$ and the columns in $\mathcal{J}$. By 0_n and 1_n we represent the n-dimensional vectors of only zero and only one entries, respectively. The Euclidean norm of any vector $v \in \mathbb{R}^n$ is denoted by $\|v\|$. For any function $t \mapsto \gamma(t)$, we denote by $\dot\gamma$ its derivative with respect to t.

II. MODEL AND PROBLEM FORMULATION

In this section, we describe the setup of the considered hydraulic system and present its model. In addition, we formulate the problem we address in the paper and briefly describe the approach we take for its solution.

A. Model

We focus on the hydraulic subsystem of district heating networks, which as mentioned are closed networks for the distribution of heat from production centers to clusters of consumers via heat exchangers and a set of hydraulic pumps that move (typically) water throughout the system (see Fig. 1 for a simplified depiction of a district heating system's hydraulic network).

Following [3], [21], [22], [20], we represent the plant as a connected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with no self-loops. The set of e edges $\mathcal{E}$ is comprised by the secondary (primary) sides of the producers (consumers) heat exchangers and all the pipes of the system. The set of n nodes $\mathcal{N}$ are the points where two or more edges physically interconnect. For the edges, we fix an arbitrary orientation, i.e., for any $i \in \mathcal{E}$ with end nodes $j, k \in \mathcal{N}$, $j \neq k$, we can either say that j is the head and k is the tail of i, or conversely, that j is the tail and k is the head of i; the orientation of the edges is the reference direction for positive flows.

Let $q_i(t)$ denote the volumetric flow rate through a given edge i, and let $p_j(t)$ denote the pressure of a given node j. Also, for any $j \in \mathcal{N}$, let $\mathcal{I}_j$ be the set of edges that are incident to j, and, based on [20], [8], [23], let us also define for any $i \in \mathcal{E}$ the sets $\mathcal{N}_i^-$ and $\mathcal{N}_i^+$, which respectively represent the tail and head of i. Moreover, throughout this paper we assume that the hydraulic network is actuated by pumps that are connected in series with some of the pipes or heat exchangers. The control variable u_i represents the pressure difference produced by a hydraulic pump in series with edge i. If there is no pump in series with i, we simply fix $u_i = 0$.

With the above considerations, the system's model is the following (see, e.g., [3], [21], [24]):

$$J_i\dot q_i = -f_i(q_i) + u_i + p_j - p_k,\ \forall i \in \mathcal{E},\ j \in \mathcal{N}_i^-,\ k \in \mathcal{N}_i^+ \tag{1a}$$

$$0 = \sum_{i\in\mathcal{I}_\ell} q_i,\ \forall \ell \in \mathcal{N}, \tag{1b}$$

where $J_i > 0$ is the inertia of the stream through i, and f_i models the pressure drop across i due to frictional forces and/or the presence of a hydraulic valve in series with i.

Equation (1a) represents the momentum balance at any edge i and equation (1b) is the volume balance at each node ℓ; for a more detailed hydraulic model based on PDEs we refer the reader to [20]. Underlying assumptions behind the model (1) are that the streams through pipes and heat exchangers are one dimensional and that water is incompressible; these are typical considerations when modeling this type of systems (see e.g., [3], [21], [20]). Another underlying assumption is that the amount of water stored at any node is constant; for the purposes of this paper, such amount can be assumed to be zero without loss of generality.

For the developments in the coming sections, we find it convenient to represent the system (1) in vector form as follows:

$$J\dot q = -f(q) + u - \mathcal{B}^\top p, \tag{2a}$$

$$0 = \mathcal{B}q, \tag{2b}$$

where $q \in \mathbb{R}^e$, $p \in \mathbb{R}^n$, $J = \mathrm{diag}(J_i)$, $f(q) = \mathrm{col}(f_i(q_i))$, $u \in \mathbb{R}^e$ and $\mathcal{B}$ is the incidence matrix of $\mathcal{G}$, defined as

$$\mathcal{B}_{i,j} = \begin{cases} 1, & \text{if } i \in \mathcal{N} \text{ is the head of } j \in \mathcal{E}, \\ \text{-1}, & \text{if } i \in \mathcal{N} \text{ is the tail of } j \in \mathcal{E}, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

The assumption that $\mathcal{G}$ is connected is reflected by the fact that the kernel of $\mathcal{B}^\top$ is spanned by 1_n.

We also define $\Delta_i := p_k - p_j$ with $j \in \mathcal{N}_i^-$ and $k \in \mathcal{N}_i^+$ as the pressure difference over edge i. By definition of $\mathcal{B}$ we have that $\Delta = \mathcal{B}^\top p$.

Before moving on to the next section, we introduce further standing assumptions about the plant's model (c.f., [3], [25]):

Assumption 1.

(i) Each f_i is a continuous and strictly monotonically increasing function that satisfies $f_i(0) = 0$.

(ii) Pumps are represented as ideal pressure difference ("voltage") sources.

The following remarks about the considered model and associated assumptions are in order:

Remark 1. For any $i \in \mathcal{E}$ for which f_i models the pressure drop due to a valve, f_i will additionally depend on the valve's stem position $s_i \in [s_i^{\min}, 1]$, $s_i^{\min} > 0$, which in many practical scenarios it also represents a control variable. For a number of standard valve models (see, e.g., [22] and [23]) it is possible to factorize f_i as $f_i(s_i, q_i) = \gamma_i(s_i)\hat{f}_i(q_i)$, where γ_i is a surjective and positive function of the valve stem's position s_i and $\hat{f}_i$ is a strictly monotonically increasing, surjective function of q_i that satisfies $\hat{f}_i(0) = 0$.[1] For simplicity, in this paper we assume that actuation is only provided through pumps and if control valves are present then their stem position will be fixed to a constant value.

Remark 2. In [3] and [4] a new paradigm for the actuation of hydraulic networks of district heating systems is discussed. It consists in placing variable speed pumps in a distributed manner throughout the system. The aim is to better counter the increased pressure drops due to the presence of pipes with reduced diameters, which is a known design rule for reducing heat losses and improving the overall operational efficiency. In [26], a comparison of the energetic performance of this type of hydraulic configuration with respect to traditional valve-actuated hydraulic networks is presented. Through case studies, it is shown there that the former pumping scheme has the potential for substantially reducing the energy consumption associated to the operation of the system pumps. Extending our current results to also consider the presence of actuated valves is part of our ongoing research efforts.

Remark 3. (Linear) Dynamic pump models are considered in [27] and [25]. For each of these models it is possible to show that, at steady state, there exists a one-to-one correspondence between the control variable and the pressure difference produced across the pump. Hence, our results would also hold if we consider such dynamic models.

B. Problem formulation

The main problem addressed in this paper refers to finding conditions on the topology of the graph $\mathcal{G}$, on the functions f_i and on the placement of the system's actuators (pumps) such that (1) admits a unique equilibrium point. Due to the relevance in the control of hydraulic networks described by $\mathcal{G}$ and (1), we would like to cast this problem as an assignable equilibria problem. That is, we are particularly interested in describing the (disjoint) subsets $\alpha, \beta \subseteq \mathcal{E}$ such that, for desired equilibrium values $q_i = q_i^*$, $i \in \alpha$, and $p_k - p_j = \Delta_i^*$, $(j,k) = i \in \beta$, an equilibrium point of (2) exists and is unique; henceforth we assume that every q_i, p_i, u_i and Δ_i are at steady state. In practice this means that for given such sets α or β, we can check *a priori* if an equilibrium exists and is unique. Alternatively, when designing the actuator placement, this gives a description of the α and β for which an equilibrium to (2) exists and is unique.

The set α is associated to those edges for which it is of interest to regulate their flows, as is the case for the consumers and a number of producers of a district heating network. The set β corresponds to those edges for which it is relevant to regulate the pressure drop across them, as is the case for at least one producer in district heating systems [28], but also for (potentially lengthy) pipes with supporting *booster* pumps [3]. Note that α or β may be empty, which corresponds to the case where there are no flow-regulating pumps or pressure-regulating pumps in the network, respectively. We additionally introduce the set $\gamma \subseteq \mathcal{E}$ of unactuated edges. Naturally, the sets α, β and γ are disjoint and form a partition of the edge set $\mathcal{E}$.

We assume that each $i \in \alpha \cup \beta$ has an independently actuated pump in series with it. This pump is to provide the necessary pressure difference to achieve in steady state the desired flow or pressure drop through/across i. With these considerations, the hydraulic dynamics (2) admits an equilibrium point if and only if the following system is solvable for Δ_i, $i \in \alpha \cup \gamma$, q_i, $i \in \beta \cup \gamma$ and u_i, $i \in \alpha \cup \beta$:

$$0 = -f_i(q_i^*) + u_i - \Delta_i, \ i \in \alpha, \tag{4a}$$

$$0 = -f_i(q_i) + u_i - \Delta_i^*, \ i \in \beta \tag{4b}$$

$$0 = -f_i(q_i) - \Delta_i, \ i \in \gamma, \tag{4c}$$

$$0 = \sum_{i \in \mathcal{I}_\ell} q_i, \ \forall \ell \in \mathcal{N}, \tag{4d}$$

where we have used $\Delta_i = p_k - p_j$ for any $(j,k) = i \in \mathcal{E}$.

An observation that will be useful in the sequel is that for any solution of (4) the steady state input u_i can be expressed for each $i \in \mathcal{E}$ as follows:

$$u_i = \begin{cases} f_i(q_i^*) + \Delta_i & \text{if } i \in \alpha, \\ f_i(q_i) + \Delta_i^* & \text{if } i \in \beta, \\ 0 & \text{if } i \in \gamma, \end{cases} \tag{5}$$

where $u_i = 0$ for $i \in \gamma$ reflects the unactuated nature of these edges.

We would like to emphasize our desire to determine the placement of the actuator pumps that would guarantee the existence of steady state solution of (1) in such a way that a hydraulic equilibrium exists independently of the values of the desired setpoints q_i^*, $i \in \alpha$ and Δ_i^*, $i \in \beta$. Then, for precision and conciseness, we summarize the above problem formulation as follows:

Problem 1. Given a closed hydraulic network that is represented by the graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ and dynamics (1), subject to Assumption 1. Determine for which placements of flow-regulating pumps $\alpha \subseteq \mathcal{E}$ and pressure-regulating pumps $\beta \subseteq \mathcal{E}$ there exists a steady state solution to (1), for any choice of the desired equilibria q_i^*, $i \in \alpha$ and Δ_i^*, $i \in \beta$, and under which conditions it is unique.

[1] This observation originates from an ongoing research of some the authors with F. Strehle, A. Malan and S. Hohmann from Karlsruhe Institute of Technology (KIT).

III. MAIN RESULT

In this section we explore two necessary conditions for the placement of flow- and pressure-regulating pumps, which inspires the notion of a properly actuated network. We then state the main result of this paper, followed by a short discussion on how our main result is related to the literature.

A. Necessary conditions for the existence of a steady state solution

We recall some further notions from graph theory as reported in [29, Ch. 11] and [30] (see also [3]). In this work we associate each subset $S \subseteq \mathcal{E}$ to the subgraph of $\mathcal{G}$ obtained by selecting all edges in S and its incident nodes. A subgraph $\mathcal{G}' = (\mathcal{N}', \mathcal{E}')$ of $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ is said to be *spanning* if it contains all the nodes of $\mathcal{G}$, by which we mean that $\mathcal{N}' = \mathcal{N}$. If $\mathcal{G}'$ is connected and does not have loops, then it is a *tree* of $\mathcal{G}$. A *spanning tree* is a tree that is also a spanning subgraph. Each connected graph has at least one spanning tree [30, Cor. 7]. A *cut set* is a minimal set of edges $S \subseteq \mathcal{E}$ such that the subgraph obtained by removing the edges in S from $\mathcal{G}$, denoted by $\mathcal{G} \setminus S$, is disconnected and has exactly two connected components. Note that we do not remove any nodes to obtain $\mathcal{G} \setminus S$, and hence $\mathcal{G} \setminus S$ is always spanning.

Cut sets represent sets of edges for which the flow cannot be independently assigned. If $S \subseteq \mathcal{E}$ is a cut set of the graph, then let U and V represent the nodes of the two connected component of $\mathcal{G} \setminus S$. The minimality of the cut set ensures that each edge in S is an edge between a node in U and a node in V. Since we consider a closed network, the flow that leaves component U must enter component V, and vice versa. The total flow that U exchanges with V is given by $\sum_{l \in U} \sum_{i \in I_l} q_i$. However, due to the nodal conservation of flow (1b), which corresponds to the Kirchhoff current law, this total flow is equal to zero. Since the flow between U and V is transported through the edges in S, the flow of the edges in S must sum to zero. Consequently, if α were to contain a cut set, then the desired equilibrium values q_i^* for $i \in \alpha$ cannot be independently assigned. Intuitively, the set of q^* for which the steady state equations (4) are solvable has measure zero. The above is a restriction on the placement of flow-regulating pumps in the network. Figure 2 illustrates this restriction by an example.

Analogous to cut sets, loops in the graph correspond to sets of edges for which the pressure drop cannot be independently assigned. Since the pressure drop Δ_i over each edge i is the difference between the nodal pressures at the head and tail of the edge, the difference of the nodal pressure of the start and end node of a walk coincides with the signed sum of the pressure drops over each edge in the walk. The sign in the sum respects orientation of the edge relative to the direction of the walk. If $S \subseteq \mathcal{E}$ forms a loop in the graph, the signed sum of the pressure drops in the loop therefore corresponds to a walk along the loop from any node in the loop to itself. Since the start and end point of the walk are the same node, the pressure drops over the edges in S must sum to zero. Hence, if β were to contain a loop, then the desired equilibrium values Δ_i^* for $i \in \beta$ cannot be independently assigned. Intuitively, the set of Δ^* for which the steady state equations (4) are solvable has measure zero. The above is a restriction on the placement of pressure-regulating pumps in the network. Figure 3 illustrates this restriction by an example.

We conclude that, if α contains a cut set and/or β contains a loop, then the choices of q_i^* and Δ_i^* for which (1) has an equilibrium are constrained. As such, the absence of cut sets in α and loops in β is a necessary condition for the placements of the actuator pumps that solves Problem 1. Inspired by this observation, we introduce the following definition.

Definition 1. The network represented by the graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ for which the edges in α are flow-regulated and the edges in β are pressure-regulated is called *properly actuated* if there exists a spanning tree T of $\mathcal{G}$ such that $\alpha \subseteq \mathcal{E} \setminus T$ and $\beta \subseteq T$.

Although Definition 1 appears to be unrelated to the previous discussion, the "properly actuated"-property is a graph-theoretic reformulation of the condition that α does not contain a cut set and β does not contain a loop. Their equivalence is shown in the following lemma.

Lemma 1. The set $\alpha \subseteq \mathcal{E}$ does not contain a cut set and the set $\beta \subseteq \mathcal{E}$ does not contain a loop if and only if α and β are such that $\mathcal{G}$ is properly actuated.

Proof. ($\Rightarrow$): Since α does not contain a cut set, it means that removing the edges in α from $\mathcal{G}$, denoted as $\mathcal{G} \setminus \alpha$, results in a spanning subgraph of $\mathcal{G}$. Since β contains no loops, each connected component of β is a tree. We let $(\mathcal{G} \setminus \alpha)/\beta$ denote the graph obtained from $\mathcal{G} \setminus \alpha$ by contracting over the edges from β (*cf.* [30, p. 24]). Consequently, each connected component of β in $\mathcal{G} \setminus \alpha$ corresponds to a single node in $(\mathcal{G} \setminus \alpha)/\beta$. Since $\mathcal{G} \setminus \alpha$ is connected, so is $(\mathcal{G} \setminus \alpha)/\beta$, and hence $(\mathcal{G} \setminus \alpha)/\beta$ has a spanning tree [30, Cor. 7]. Let $T' \subseteq \mathcal{E} \setminus (\alpha \cup \beta)$ represent the edges in such a spanning tree. Since each connected component of β in $\mathcal{G} \setminus \alpha$ was a tree, we have that $T := \beta \cup T'$ forms a spanning tree of $\mathcal{G} \setminus \alpha$. Since $\mathcal{G} \setminus \alpha$ is a spanning subgraph, T is also a spanning tree of $\mathcal{G}$.

($\Leftarrow$): If β is contained in a tree, it cannot contain a loop. If the subgraph $\mathcal{E} \setminus \alpha$ contains a spanning tree, then $\mathcal{E} \setminus \alpha$ is connected and hence α cannot contain a cut set for $\mathcal{G}$. □

Figure 4 illustrates an example of a network which is properly actuated. Due to Lemma 1 and the discussion above, it is necessary for a network to be properly actuated in order to solve Problem 1.

Lemma 2. An equilibrium (p, q) to (1) exists for any choice of q_i^*, $i \in \alpha$ and Δ_i^*, $i \in \beta$ only if the network is properly actuated.

B. Statement of the main result

Lemma 2 shows that the "properly actuated"-property is a necessary condition for the actuator pump placement problem of Problem 1. A natural continuation is to ask if

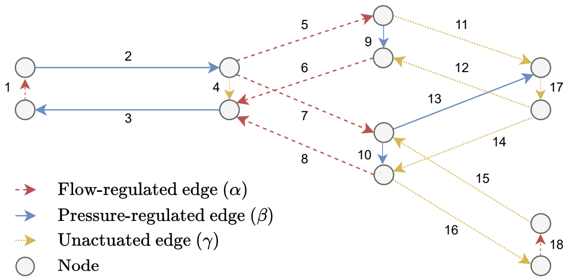

Fig. 2: A hypothetical placement for the flow-regulating pumps α (dashed red edges) and pressure-regulating pumps β (solid blue edges) for the graph associated to the district heating diagram in Fig. 1. The edges $\{5, 6, 7, 8\} \subseteq \alpha$ form a cut set. Consequently, the flow exchanged between the subsystems $\{1, \ldots, 4\}$ and $\{9, \ldots, 18\}$ is fully determined by the flow through the edges $\{5, 6, 7, 8\}$. Since the network is closed, there is a conservation of mass in the network, and the mass that enters each subsystem equals the mass that leaves it. Hence the volumetric flow over the edges $\{5, 6, 7, 8\}$ must sum to zero, meaning that the q_i^* with $i \in \alpha$ cannot be independently assigned.

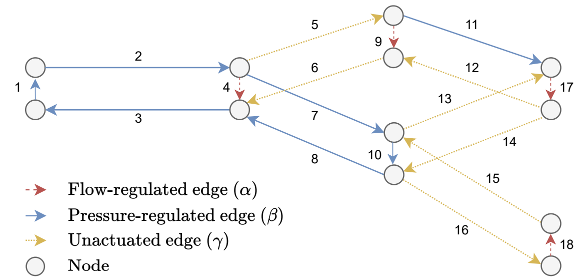

Fig. 3: A hypothetical placement for the flow-regulating pumps α (dashed red edges) and pressure-regulating pumps β (solid blue edges) for the graph associated to the district heating diagram in Fig. 1. The edges $\{1, 2, 3, 7, 8, 10\} \subseteq \beta$ form a loop. Consequently, the pressure drops over the loop must sum to zero. This means that the pressure drops Δ_i^* with $i \in \beta$ cannot be independently assigned.

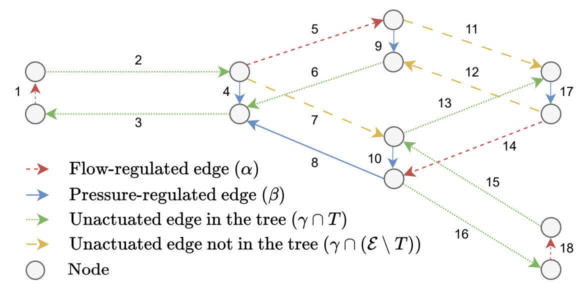

Fig. 4: A hypothetical placement for the flow-regulating pumps α (dashed red edges) and pressure-regulating pumps β (solid blue edges) for the graph associated to the district heating diagram in Fig. 1. The edges $T := \{2, 3, 4, 6, 8, 9, 10, 13, 15, 16, 17\}$ (solid blue and dotted green edges) form a spanning tree of the graph such that $\alpha \subseteq \mathcal{E} \setminus T$ and $\beta \subseteq T$, meaning that the network is properly actuated (Definition 1). Note that the graph remains connected upon removing the edges in α, which means that α does not contain a cut set. Also note that β does not contain any loops. According to Lemma 1, the absence of cut sets in α and loops in β is equivalent to the network being properly actuated.

this property is also sufficient, and what restrictions arise if it is not. Since Lemma 2 does not take into account the functions f_i, such restrictions may come from the properties of f_i. However, by virtue of the properties of f_i, which were summarized in Assumption 1, we have that the "properly actuated"-property is both necessary and sufficient for solving Problem 1.

Theorem 1. Consider a closed hydraulic network that is represented by the graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ and dynamics (1), subject to Assumption 1. The placements of flow-regulating pumps $\alpha \subseteq \mathcal{E}$ and pressure-regulating pumps $\beta \subseteq \mathcal{E}$ is such that for any choice of the desired equilibria q_i^* $i \in \alpha$ and Δ_i^*, $i \in \beta$ there exists a steady state solution (p, q) to (1) if and only if the graph $\mathcal{G}$ is properly actuated. Moreover, for each choice of q_i^* $i \in \alpha$ and Δ_i^*, $i \in \beta$, the flow vector q is unique, and pressure vector p is unique up to addition by the vector 1_n.

The proof of Theorem (1) is omitted for the sake of brevity. To illustrate to result, we note that it follows from Theorem 1 that the actuator pump placement in Figure 4 guarantees the existence of a steady state solution, independent of the choice of q_i^*, $i \in \alpha$ and Δ_i^*, $i \in \beta$.

Some special use-cases of Theorem 1 may be highlighted. We recall the set α of flow-regulating pumps or the set β of pressure-regulating pumps may be empty. The case where $\mathcal{E} \setminus \alpha$ forms a spanning tree and $\beta = \emptyset$ was considered in [3] and [4] and corresponds to the typical mesh analysis in electric networks [29]. Similarly, the case where $\alpha = \emptyset$ and the edges in β form a spanning tree corresponds to the typical cut-set analysis in electric networks.

IV. CLOSED-FORM EXPRESSIONS FOR LINEAR FLOW

Whenever all the functions f_i are linear, the (unique) steady state solution in Theorem 1 of (1) can be expressed explicitly. In order to do so, we recall several more graph-theoretic notions from [29]. For a given spanning tree T of $\mathcal{G}$, any edge of T is referred to as a *twig* and any edge of $\mathcal{G}$ that is not in T is referred to as a *chord*. We let $\mathcal{C} := \mathcal{E} \setminus T$ denote the set of chords. There are $n-1$ twigs and $e-n+1$ chords. Each chord corresponds to a unique loop, which is obtained by adding the chord to the tree. This results in $e - n + 1$ distinct loops. By this association, we identify each chord with a loop in the graph, which is commonly referred to as a fundamental loop. As was shown in [29], all other loops may be expressed as a combination of fundamental loops.

Let us assign a reference orientation of every fundamental loop of a tree T such that it agrees with the orientation of the associated chord. Then a *fundamental loop matrix*, denoted by $\mathcal{F} \in \mathbb{R}^{(e-n+1) \times e}$, is defined as follows [29]:

$$\mathcal{F}_{i,j} = \begin{cases} 1 & \text{if edge } j \text{ is in loop } i \text{ and the reference directions agree,} \\ \text{-1} & \text{if edge } j \text{ is in loop } i \text{ and the reference directions disagree,} \\ 0 & \text{if edge } j \text{ is not in loop } i. \end{cases}$$

We let the rows of F be indexed by $\mathcal{C}$. By construction of F it follows that $F_{\mathcal{C},\mathcal{C}} = I$. We let $\gamma_{\mathcal{C}} := \gamma \cap \mathcal{C}$ and $\gamma_T := \gamma \cap T$ be a partition of the unactuated edges γ. When the network is properly actuated, its equilibria may be expressed explicitly as follows. The result is again stated without proof for the sake of brevity.

Theorem 2. Consider a closed-loop hydraulic network as in Theorem 1 that is properly actuated. In the case where all f_i are linear, say $f_i(q_i) = R_{ii}q_i$, where $R \in \mathbb{R}^{e\times e}$ is a diagonal matrix with positive diagonal entries, we have that

$$q_{\gamma c} = (F_{\gamma c,\gamma}R_{\gamma,\gamma}F_{\gamma c,\gamma}^\top)^{-1}(F_{\gamma c,\beta}\Delta_\beta^* - F_{\gamma c,\gamma}R_{\gamma,\gamma}F_{\alpha,\gamma}^\top q_\alpha^*).$$

Consequently, $q = F_{\gamma c,\mathcal{E}}^\top q_{\gamma c} + F_{\alpha,\mathcal{E}}^\top q_\alpha^*$ and Δ satisfies

$$\begin{aligned}
&\Delta_\alpha = F_{\alpha,\gamma_T}R_{\gamma_T,\gamma_T}(F_{\gamma c,\gamma_T}^\top q_{\gamma c} + F_{\alpha,\gamma_T}^\top q_\alpha^*) - F_{\alpha,\beta}\Delta_\beta^*;\\
&\Delta_\beta = \Delta_\beta^*;\\
&\Delta_\gamma = -R_{\gamma,\gamma}(F_{\gamma c,\gamma}^\top q_{\gamma c} + F_{\alpha,\gamma}^\top q_\alpha^*),
\end{aligned}$$

and the associated nodal pressures p are given by $p = (\mathcal{B}^\top)^+\Delta + c1_n$, where $c \in \mathbb{R}$ is any scalar and $(\mathcal{B}^\top)^+$ is the Moore-Penrose pseudo-inverse of $\mathcal{B}^\top$.

V. CONCLUSIONS

Using tools from graph theory, we have presented necessary and sufficient conditions for the existence and uniqueness of a hydraulic equilibrium in district heating systems featuring multiple heat sources and meshed topologies. Our results have been established under the assumption that any edge for which it is desired to fix the steady state pressure drop or flow rate is actuated by an independent variable speed pump in series with it; with any pump being considered as an ideal pressure difference ("voltage") source. Despite this simplifying assumption, our contribution is distinguished from most of existing results in the literature of water distribution networks for the constructive nature of the proof of our claims, which do not rely on establishing an equivalence between the hydraulic equilibrium equations of interest to a convex optimization problem and its associated KKT conditions. Nonetheless, further work is currently being carried out to efficiently compute the singled-out equilibrium, under the circumstances in which it is guaranteed to exist. Moreover, we are interested in extending our results to consider more complex pump models, to allow the presence of actuated valves, and to extend our model with the incorporation of the thermal dynamics of the district heating system to address in full the heat flow calculation.

REFERENCES

[1] H. Lund, S. Werner, R. Wiltshire, S. Svendsen, J. E. Thorsen, F. Hvelplund, and B. V. Mathiesen, "4th generation district heating (4GDH): Integrating smart thermal grids into future sustainable energy systems," *Energy*, vol. 68, pp. 1–11, 2014.

[2] G. Sandou, S. Font, S. Tebbani, A. Hiret, and C. Mondon, "Predictive control of a complex district heating network," *Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference, CDC-ECC '05*, vol. 2005, pp. 7372–7377, 2005.

[3] C. De Persis and C. S. Kallesøe, "Pressure regulation in nonlinear hydraulic networks by positive and quantized controls," *IEEE Transactions on Control Systems Technology*, vol. 19, no. 6, pp. 1371–1383, 2011.

[4] C. De Persis, T. Jensen, R. Ortega, and R. Wisniewski, "Output regulation of large-scale hydraulic networks," *IEEE Transactions on Control Systems Technology*, vol. 22, no. 1, pp. 238–245, 2014.

[5] T. Scholten, C. De Persis, and P. Tesi, "Modeling and control of heat networks with storage: The single-producer multiple-consumer case," *IEEE Transactions on Control Systems Technology*, vol. 25, no. 2, pp. 414–427, 2015.

[6] S. Trip, T. Scholten, and C. De Persis, "Optimal regulation of flow networks with transient constraints," *Automatica*, vol. 104, pp. 141–153, 2019.

[7] J. Bendtsen, J. Val, C. Kallesøe, and M. Krstic, "Control of district heating system with flow-dependent delays," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 13 612–13 617, 2017.

[8] R. Krug, V. Mehrmann, and M. Schmidt, "Nonlinear optimization of district heating networks," *Optimization and Engineering*, vol. 22, pp. 783–819, 2021.

[9] J. E. Machado, J. Ferguson, M. Cucuzzella, and J. M. A. Scherpen, "Decentralized temperature and storage volume control in multi-producer district heating," *IEEE Control Systems Letters*, vol. 7, pp. 413–418, 2022.

[10] J. E. Machado, M. Cucuzzella, N. Pronk, and J. M. Scherpen, "Adaptive control for flow and volume regulation in multi-producer district heating systems," *IEEE Control Systems Letters*, vol. 6, pp. 794–799, 2021.

[11] M. Geidl and G. Andersson, "Optimal power flow of multiple energy carriers," *IEEE Transactions on power systems*, vol. 22, no. 1, pp. 145–155, 2007.

[12] V. D. Stevanovic, S. Prica, B. Maslovaric, B. Zivkovic, and S. Nikodijevic, "Efficient numerical method for district heating system hydraulics," *Energy Conversion and Management*, vol. 48, no. 5, pp. 1536–1543, 2007.

[13] J. Dancker and M. Wolter, "Improved quasi-steady-state power flow calculation for district heating systems: A coupled newton-raphson approach," *Applied Energy*, vol. 295, p. 116930, 2021.

[14] M. Hall, "Hydraulic network analysis using (generalized) geometric programming," *Networks*, vol. 6, no. 2, pp. 105–130, 1976.

[15] M. Collins, L. Cooper, R. Helgason, J. Kennington, and L. LeBlanc, "Solving the pipe network analysis problem using optimization techniques," *Management science*, vol. 24, no. 7, pp. 747–760, 1978.

[16] B. L. Berghout and G. Kuczera, "Network linear programming as pipe network hydraulic analysis tool," *Journal of Hydraulic Engineering*, vol. 123, no. 6, pp. 549–559, 1997.

[17] A. Fügenschuh and J. Humpola, "A unified view on relaxations for a nonlinear network flow problem," Zuse-Institut Berlin, Tech. Rep., 2013.

[18] M. K. Singh and V. Kekatos, "On the flow problem in water distribution networks: Uniqueness and solvers," *IEEE Transactions on Control of Network Systems*, vol. 8, no. 1, pp. 462–474, 2020.

[19] S. Boyd, S. P. Boyd, and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2004.

[20] S.-A. Hauschild, N. Marheineke, V. Mehrmann, J. Mohring, A. M. Badlyan, M. Rein, and M. Schmidt, "Port-Hamiltonian modeling of district heating networks," in *Progress in Differential-Algebraic Equations II*. Springer, Cham., 2020, pp. 333–355.

[21] J. E. Machado, M. Cucuzzella, and J. M. A. Scherpen, "Modeling and passivity properties of multi-producer district heating systems," *Automatica*, vol. 142, 2022.

[22] Y. Wang, S. You, H. Zhang, W. Zheng, X. Zheng, and Q. Miao, "Hydraulic performance optimization of meshed district heating network with multiple heat sources," *Energy*, vol. 126, pp. 603–621, 2017.

[23] P. Vladimarsson, "District heat distribution networks," *United Nations University Geothermal Training Programme*, vol. 30, no. 9, pp. 239–240, 2014.

[24] A. van der Schaft and D. Jeltsema, "Port-hamiltonian systems theory: An introductory overview," *Foundations and Trends® in Systems and Control*, no. 2-3, pp. 173–378, 2014.

[25] R. Perryman, J. A. Taylor, and B. Karney, "Port-hamiltonian based control of water distribution networks," *Systems & Control Letters*, vol. 170, p. 105402, 2022.

[26] A. Yan, J. Zhao, Q. An, Y. Zhao, H. Li, and Y. J. Huang, "Hydraulic performance of a new district heating systems with distributed variable speed pumps," *Applied energy*, vol. 112, pp. 876–885, 2013.

[27] F. Strehle, J. E. Machado, M. Cucuzzella, A. J. Malan, J. M. Scherpen, and S. Hohmann, "Port-hamiltonian modeling of hydraulics in 4th generation district heating networks," in *2022 IEEE 61st Conference on Decision and Control (CDC)*. IEEE, 2022, pp. 1182–1189.

[28] S. Buffa, M. H. Fouladfar, G. Franchini, I. Lozano Gabarre, and M. Andrés Chicote, "Advanced control and fault detection strategies for district heating and cooling systems—a review," *Applied Sciences*, vol. 11, no. 1, p. 455, 2021.

[29] C. Desoer and E. Kuh, *Basic Circuit Theory*. McGraw-Hill Book Company, 1969.

[30] B. Bollobás, *Modern Graph Theory*. Springer, 1998.