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Repetitive T-S Fuzzy Model-Based Iterative Learning Control Law Design

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***Abstract*— This paper develops an iterative learning control law for a class of nonlinear systems. The approach used to represent the nonlinear system dynamics is a Takagi-Sugeno fuzzy repetitive process that considers the two directions of information propagation. Then, the control action investigated is a state feedback control law combined with a PD-type feed-forward learning control law. Consequently, linear matrix inequality techniques can be used for control design. Furthermore, this approach allows the design of control action to satisfy the requirements on both the error convergence and the transient dynamics. Finally, an example demonstrates the properties of the new design.**

I. Introduction

Many applications, such as various forms of robotics and chemical batch systems, execute the same finite-duration task repeatedly. Moreover, the task is to force each execution, termed a trial in this paper, to follow a specified reference trajectory, where each trial's duration is finite. Given this trajectory, the error on any trial is the difference between the reference trajectory and the trial output. Hence, a sequence of errors, where each entry is the error on the corresponding trial, can be formed, and the control design objective is to force this sequence to converge as the number of trials completed increases. Also, it is necessary to regulate the dynamics along the trial direction.

Once a trial is complete, the system resets to the starting location before the start of the subsequent trial. Also, all information generated on the previous trial is available to update the control input for the ensuing trial. This availability of previous trial data is the core feature of iterative learning control (ILC), where, in contrast to other control laws, it is the input, a signal, that is updated rather than a controller, which is a system. Let the variable p for discrete dynamics represent the dynamics produced along a trial, denoted by the nonnegative integer k. Then, at sample instant p on trial k information at, as one example, sample instant $p+\lambda$, $\lambda > 0$ from the previous trial can be used. Using such information leads to the terminology 'noncausal' in some literature. Moreover, data from more than the previous trial can be used, termed higher order, and also ILC is not restricted to applications with resetting; it is enough that there is a stoppage between the completion of one trial and the beginning of the next. Hence, ILC can be applied to batch processing systems.

Since the original work, widely credited to [1], ILC has remained a significant area of control systems research with many designs experimentally verified in the research laboratory and applied in industrial applications. An overview of the early developments can be found in, e.g., the survey papers [2], [3]. The survey paper [4] focuses on run-to-run control in the chemical process industries. This area is one where there is a stoppage time between one trial and the next instead of resetting. Applications areas include industrial robotics, for early application see, e.g., [5], nano-positioning, for recent progress see, e.g., [6] and optimizing broiler weight in agriculture [7]. Also, there has been productive work on using ILC in healthcare. For examples of recent progress in this last area, see, e.g., [8].

In the above-cited references (and others), ILC designs use linear models of the dynamics. However, many physical systems are nonlinear, and linear models for design cannot adequately approximate their dynamics. Moreover, a general method to enable design for all possible nonlinearities is not feasible. Therefore, such designs have to be developed for subclasses of nonlinear dynamics. This paper considers the use of Takagi-Sugeno (T-S) fuzzy models [9], coupled with a linear dynamic model. The key feature of the T-S fuzzy model resides in its representation of nonlinear dynamics by a family of local linear models interpolated through nonlinear membership functions. Such an approach to ILC design has previously been considered; see, e.g., [10].

This paper develops a robust (proportional plus derivative) PD-type ILC law for a class of nonlinear systems. The local-sector nonlinearity method transforms the nonlinear ILC dynamics into uncertain T-S fuzzy models. The resulting design method can be computed using linear matrix inequalities (LMIs). In particular, sufficient LMI-based conditions for the existence of a robust ILC law are derived, together with the design algorithms to compute the associated matrices. An illustrative example is given.

The following notation is used in this paper; the identity and null matrices of compatible dimensions are denoted, respectively, by I and 0. Also, the notation $R \succ 0$ (respectively $R \prec 0$) denotes a symmetric positive definite (respectively negative definite) matrix. The symbol $\mathrm{sym}\{R\}$ denotes

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the matrix $R + R^T$. The symbol $\mathrm{diag}\{R_1, R_2, \cdots, R_n\}$ denotes a block diagonal matrix with diagonal blocks $R_1, R_2, \cdots, R_n$. Finally, $\otimes$ denotes the matrix Kronecker product, and $(\star)$ denotes block entries in symmetric matrices.

The development of the new results in this paper uses the following results.

Lemma 1: [11] Given matrices $\Gamma = \Gamma^T \in \mathbb{R}^{p\times p}$ and two matrices Λ, Σ of column dimension p, there exists an unstructured matrix W that satisfies

$$\Gamma + \mathrm{sym}\{\Lambda^T W \Sigma\} \prec 0, \tag{1}$$

if, and only if

$$\Lambda_\perp{}^T \Gamma \Lambda_\perp \prec 0, \text{ and } \Sigma_\perp{}^T \Gamma \Sigma_\perp \prec 0, \tag{2}$$

where $\Lambda_\perp$ and $\Sigma_\perp$ are arbitrary matrices whose columns form a basis of null spaces of Λ and Σ, respectively, Hence $\Lambda\Lambda_\perp = 0$ and $\Sigma\Sigma_\perp = 0$.

Lemma 2: [12] Given matrices X, Y, $\Phi = \Phi^{\mathrm{T}}$ and $\mathcal{F}(t)$ of compatible dimensions

$$\Phi + \mathrm{sym}\{X\mathcal{F}(t)Y\} \prec 0,$$

for all $\mathcal{F}(t)$ satisfying $\mathcal{F}(t)^T\mathcal{F}(t) \preceq I$ if and only if there exists $\varepsilon > 0$ such that

$$\Phi + \varepsilon X X^{\mathrm{T}} + \varepsilon^{-1} Y^{\mathrm{T}} Y \prec 0.$$

II. Problem formulation

Consider the class of nonlinear continuous-time systems described in the ILC setting by the state-space model

$$\begin{cases} \dot{x}_k(t) = f\left(x_k(t), u_k(t)\right), \\ y_k(t) = g\left(x_k(t)\right), \forall k \geq 1,\ t \in [0, \alpha], \end{cases} \tag{3}$$

where $x_k(t) \in \mathbb{R}^n$, $u_k(t) \in \mathbb{R}^m$, and $y_k(t) \in \mathbb{R}^l$ denote, respectively, the state, input and output vectors at time instant t on trial k; α is the trial length; $f(\cdot)$ and $g(\cdot)$ are nonlinear functions with compatible dimensions. Application of the local sector nonlinear method [10] to (3), and discretizing the resulting dynamics with an appropriate sampling time results in a discrete T-S fuzzy model with the IF-THEN rules of the form

Rule i: IF $\vartheta_{k_1}(t)$ is $\mathcal{M}_{i1}$ and $\vartheta_{k_j}(t)$ is $\mathcal{M}_{ij}$, $\ldots$, $\vartheta_{k_p}(t)$ is $\mathcal{M}_{ip}$, THEN

$$\begin{cases} x_k(t+1) = (A_i + \Delta A_i(t))x_k(t) \\ \qquad\qquad + (B_i + \Delta B_i(t))u_k(t) + w_k(t), \\ \quad y_k(t) = C_i x_k(t),\ i = 1, 2, \ldots, r;\ j = 1, 2, \ldots, p, \end{cases} \tag{4}$$

where $\vartheta_{k_1}(t), \ldots, \vartheta_{k_p}(t)$ are known premise variables, $\mathcal{M}_{ij}$ is the fuzzy set, r is the number of IF-THEN rules, p denotes the number of premise variables. $w(t, k)$ denotes disturbances that are assumed to belong to the $\mathcal{L}_2$ space. Also, A_i, B_i, and C_i denote known constant matrices of compatible dimensions, $\Delta A_i(t)$, and $\Delta B_i(t)$ denote time-varying uncertainties which are assumed to satisfy $\forall i = 1, 2, \ldots, r$

$$\begin{bmatrix} \Delta A_i(t) & \Delta B_i(t) \end{bmatrix} = E\mathcal{F}(t) \begin{bmatrix} F_{Ai} & F_{Bi} \end{bmatrix}, \tag{5}$$

where E, F_{Ai}, and F_{Bi} are known real matrices that characterize the structure of the uncertainty. Finally, $\mathcal{F}(t)$ is an unknown time-varying matrix that also satisfies

$$\mathcal{F}^T(t)\mathcal{F}(t) \preceq I. \tag{6}$$

Remark 1: The results in this paper assume that all fuzzy subsystems have the same output matrix C, i.e., $C = C_1 = C_2 = \cdots = C_r$. This assumption reduces the complexity of design and the required computations, but it introduces some level of conservativeness, see [13].

The fuzzy basis functions used in this paper are

$$\mu_i(\vartheta_k(t)) = \frac{\prod_{j=1}^{p} \mathcal{M}_{ij}(\vartheta_{k_j}(t))}{\sum_{i=1}^{r}\prod_{j=1}^{p} \mathcal{M}_{ij}(\vartheta_{k_j}(t))}, \tag{7}$$

where $\mathcal{M}_{ij}(\vartheta_{k_j}(t))$ is the grade of membership of $\vartheta_{k_j}(t)$ in $\mathcal{M}_{ij}$. Moreover,

$$\mu_i(\vartheta_k(t)) \geq 0,\ \sum_{i=1}^{r} \mu_i(\vartheta_k(t)) = 1. \tag{8}$$

Without loss of generality, it is assumed that $x_k(0) = x_0$, i.e., the same initial state vector at the beginning of each trial. Application of the common center-average defuzzifier approach gives an uncertain T-S fuzzy repetitive process, which can be written as

$$\begin{cases} x_k(t+1) = (A(\mu) + \Delta A(\mu))x_k(t) \\ \qquad\qquad + (B(\mu) + \Delta B(\mu))u_k(t) + w_k(t), \\ \quad y_k(t) = Cx_k(t), \end{cases} \tag{9}$$

with

$$\begin{aligned} A(\mu) &= \sum_{i=1}^{r} \mu_i(\vartheta_k(t))A_i,\ \Delta A(\mu) = \sum_{i=1}^{r} \mu_i(\vartheta_k(t))\Delta A_i, \\ B(\mu) &= \sum_{i=1}^{r} \mu_i(\vartheta_k(t))B_i,\ \Delta B(\mu) = \sum_{i=1}^{r} \mu_i(\vartheta_k(t))\Delta B_i. \end{aligned} \tag{10}$$

Let $y_d(t)$ denote the specified reference (or output) trajectory. Then the tracking error on trial k is

$$e_k(t) = y_d(t) - y_k(t). \tag{11}$$

A commonly used ILC law has the form

$$u_k(t) = u_{k-1}(t) + \Delta u_k(t), \tag{12}$$

i.e., the sum of the control input on the previous trial and a term, $\Delta u_k(t)$, often referred to as the control update, where the previous trial data is used. Also, $u_0(t)$ is the initial control input commonly set to zero for implementation. Introduce $\delta x_k(t) = x_k(t) - x_{k-1}(t)$ and then, using (4)-(12),

$$\begin{cases} \delta x_k(t+1) = (A(\mu) + \Delta A(\mu))\delta x_k(t) \\ \qquad\qquad + (B(\mu) + \Delta B(\mu))\Delta u_k(t) + \bar{w}_k(t), \\ e_k(t+1) = e_{k-1}(t+1) - C(A(\mu) + \Delta A(\mu)\delta x_k(t) \\ \qquad\qquad - C(B(\mu) + \Delta B(\mu))\Delta u_k(t) - C\bar{w}_k(t), \end{cases} \tag{13}$$

where

$$\begin{aligned}\bar{w}_k(t) =& w_{k_1}(t) + w_k(t) - w_{k-1}(t),\\ w_{k_1}(t) =& (A(\delta\mu) + \Delta A(\delta\mu))x_{k-1}(t)\\ &+ (B(\delta\mu) + \Delta B(\delta\mu))u_{k-1}(t),\\ A(\delta\mu) =& \sum_{i=1}^{r}(\mu_i(\vartheta_k(t)) - \mu_i(\vartheta_{k-1}(t)))A_i,\\ \Delta A(\delta\mu) =& \sum_{i=1}^{r}(\mu_i(\vartheta_k(t)) - \mu_i(\vartheta_{k-1}(t)))\Delta A_i,\\ B(\delta\mu) =& \sum_{i=1}^{r}(\mu_i(\vartheta_k(t)) - \mu_i(\vartheta_{k-1}(t)))B_i,\\ \Delta B(\delta\mu) =& \sum_{i=1}^{r}(\mu_i(\vartheta_k(t)) - \mu_i(\vartheta_{k-1}(t)))\Delta B_i.\end{aligned}$$

Also the control update considered has the following description

Rule i: IF $\vartheta_{k_1}(t)$ is $\mathcal{M}_{i1}$ and $\vartheta_{k_j}(t)$ is $\mathcal{M}_{ij}$, ..., $\vartheta_{k_p}(t)$ is $\mathcal{M}_{ip}$, THEN

$$\Delta u_k(t) = K_{1i}\delta x_k(t) + K_{3i}e_{k-1}(t) + L_i e_{k-1}(t+1), \quad (14)$$

where $L_i = K_{2i} - K_{3i}$, and K_{1i}, K_{2i} and K_{3i} are the control law matrices to be designed. Hence this correction term consists of a state feedback control action on the current trial and a PD-type feed-forward learning term depending on the previous trial error. Also, when $K_{2i} = K_{3i}$, a P-type ILC law results. By fuzzy blending,

$$\Delta u_k(t) = K_1(\mu)\delta x_k(t) + K_3(\mu)e_{k-1}(t) + L(\mu)e_{k-1}(t+1), \quad (15)$$

where

$$\begin{aligned}L(\mu) &= \sum_{i=1}^{r}\mu_i L_i, K_1(\mu) = \sum_{i=1}^{r}\mu_i K_{1i},\\ K_2(\mu) &= \sum_{i=1}^{r}\mu_i K_{2i}, K_3(\mu) = \sum_{i=1}^{r}\mu_i K_{3i}\end{aligned}$$

and μ_i is used to replace $\mu_i(\vartheta(t,k))$ for notational simplicity.

Application of the control law defined by (12) and (15) gives the following description of the controlled dynamics

$$\begin{aligned}\begin{bmatrix}\delta x_k(t+1)\\ e_{k-1}(t+1)\end{bmatrix} &= \mathbb{A}(\mu)\begin{bmatrix}\delta x_k(t)\\ e_{k-1}(t)\end{bmatrix} + \mathbb{B}_0(\mu)e_{k-1}(t+1) + \mathbb{B}_1\bar{w}_k(t),\\ e_k(t+1) &= \mathbb{C}(\mu)\begin{bmatrix}\delta x_k(t)\\ e_{k-1}(t)\end{bmatrix} + \mathbb{D}_0(\mu)e_{k-1}(t+1) + \mathbb{D}_1\bar{w}_k(t),\end{aligned} \quad (16)$$

where

$$\begin{aligned}\left[\begin{array}{c|c}\mathbb{A}(\mu) & \mathbb{B}_0(\mu)\\ \hline \mathbb{C}(\mu) & \mathbb{D}_0(\mu)\end{array}\right] &= (\bar{A}(\mu) + \Delta\bar{A}(\mu)) + (\bar{B}(\mu) + \Delta\bar{B}(\mu))K(\mu),\\ &= \left(\left[\begin{array}{cc|c} A(\mu) & 0 & 0\\ 0 & 0 & I\\ \hline -CA(\mu) & 0 & I\end{array}\right] + \left[\begin{array}{cc|c}\Delta A(\mu) & 0 & 0\\ 0 & 0 & 0\\ \hline -C\Delta A(\mu) & 0 & 0\end{array}\right]\right)\\ &+ \left(\left[\begin{array}{c} B(\mu)\\ 0\\ \hline -CB(\mu)\end{array}\right] + \left[\begin{array}{c}\Delta B(\mu)\\ 0\\ \hline -C\Delta B(\mu)\end{array}\right]\right)K(\mu),\\ K(\mu) &= \left[\begin{array}{cc|c} K_1(\mu) & K_3(\mu) & L(\mu)\end{array}\right], \mathbb{D}_1 = -C, \mathbb{B}_1 = \begin{bmatrix} I\\ 0\end{bmatrix}.\end{aligned}$$

Also,

$$\begin{aligned}\Delta\bar{A}(\mu) &= \begin{bmatrix} E\\ 0\\ -CE\end{bmatrix}\mathcal{F}(t)\begin{bmatrix}F_A(\mu) & 0 & 0\end{bmatrix} = \hat{E}\mathcal{F}(t)\hat{F}_A(\mu),\\ \Delta\bar{B}(\mu) &= \begin{bmatrix} E\\ 0\\ -CE\end{bmatrix}\mathcal{F}(t)F_B(\mu) = \hat{E}\mathcal{F}(t)F_B(\mu).\end{aligned}$$

The state space model (16) has the structure of a discrete linear repetitive process. These processes make a series of trials (also termed passes in the literature) through dynamics defined over a finite interval. Once a trial is complete, resetting to the starting point takes place, and during the subsequent trial, the previous trial output acts as a forcing function and hence contributes to the dynamics of this profile. The result can be oscillations that increase in amplitude from trial to trial; standard action cannot control them. Instead, a stability and control law design theory for the linear dynamics case has been developed [14].

Next, the repetitive process theory is applied to the ILC design problem. Repetitive processes are a class of 2D systems with, in the ILC setting, information propagation from trial to trial (k) and along the trials (p). Moreover, the trial length is finite and, hence, matches ILC dynamics more closely than other 2D systems models. Note also that the model (16) includes disturbances resulting from non-repetitive behavior. However, due to space limitations, the disturbance attenuation problem is not considered here and is left as a topic for future research.

III. Main Results

The stability theory for repetitive processes requires that a bounded initial trial profile produces a bounded sequence of trial profiles, i.e., in the k direction, where the bounded property is defined in terms of the norm on the underlying function space. Moreover, this property can be applied over the finite and fixed trial length, or uniformly, i.e., for all possible values of the finite trial length. The former property is termed asymptotic stability, which is distinct from the latter, known as stability along the trial. As the trial length is finite, it is possible to ensure asymptotic stability even for examples that are unstable, i.e., for linear dynamics not all eigenvalues of the state matrix have strictly negative real parts. Hence stability along the trial is required.

The following result, first developed in [15], gives a condition for robust stability along the trial of discrete repetitive processes described by (16).

Lemma 3: A discrete repetitive process described by (16) is stable along the trial if there exist compatibly dimensioned matrices $P_1 \succ 0$, $P_2 \succ 0$ such that

$$\begin{aligned}&\begin{bmatrix}\mathbb{A}(\mu) & I\\ \mathbb{C}(\mu) & 0\end{bmatrix}(\Phi\otimes P_1)\begin{bmatrix}\mathbb{A}(\mu) & I\\ \mathbb{C}(\mu) & 0\end{bmatrix}^T\\ &+\begin{bmatrix}\mathbb{B}_0(\mu) & 0\\ \mathbb{D}_0(\mu) & I\end{bmatrix}(\Pi\otimes P_2)\begin{bmatrix}\mathbb{B}_0(\mu) & 0\\ \mathbb{D}_0(\mu) & I\end{bmatrix}^T \prec 0\end{aligned} \quad (17)$$

is feasible, where $\Phi = \mathrm{diag}\{1, -1\}$, $\Pi = \mathrm{diag}\{1, -\gamma^2\}$, where γ is a given scalar satisfying $0 < \gamma \leq 1$ and allows the specification of a gain bound for the controlled dynamics.

Clearly, the result of Lemma 3 cannot be directly applied to the considered ILC design since there exist product terms between control law matrices and the matrix variables P_1 and P_2. The unknown matrix $\mathcal{F}(t)$ is also coupled. The application of transformations now leads to the following result.

Theorem 1: Let γ be a positive scalar satisfying $0 < \gamma \leq 1$. Then, the controlled dynamics represented as a discrete linear repetitive process of the form (16) is robustly stable along the trial, and hence trial-to-trial error convergence occurs for all admissible uncertainties if there exist compatibly dimensioned matrices $P_1 \succ 0$, $P_2 \succ 0$, W, F_1, F_2, F_3, $Y(\mu)$ and scalars $\beta \in (-1, 1)$ and $\epsilon_1 > 0$ such that the following inequalities hold for all μ

$$\begin{bmatrix} \Upsilon_1 - \mathrm{sym}\{W\} & (\star) & (\star) & (\star) & (\star) \\ \Upsilon_3 + \Omega_1(\mu) - \beta W & \Upsilon_2 + \beta \mathrm{sym}\{\Omega_1(\mu)\} & (\star) & (\star) & (\star) \\ G_b - [0\ I]W & -G_a^T + [0\ I]\Omega_1^T(\mu) & P_2 - \mathrm{sym}\{F_3\} & (\star) & (\star) \\ 0 & \epsilon_1 \hat{E}^T & 0 & -\epsilon_1 I & (\star) \\ (\hat{F}_A(\mu)W + F_B(\mu)Y(\mu)) & (\hat{F}_A(\mu)W + F_B(\mu)Y(\mu)) & (\hat{F}_A(\mu)W + F_B(\mu)Y(\mu)) \begin{bmatrix} 0 \\ I \end{bmatrix} & 0 & -\epsilon_1 I \end{bmatrix} \prec 0, \tag{18}$$

where $\Omega_1(\mu) = \bar{A}(\mu)W^T + \bar{B}(\mu)Y(\mu)$ and

$$\Upsilon_1 = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}, \Upsilon_2 = \begin{bmatrix} -P_1 & 0 \\ 0 & -\gamma^2 P_2 \end{bmatrix},$$
$$\Upsilon_3 = \begin{bmatrix} 0 & F_1 \\ 0 & F_2 \end{bmatrix}, G_a = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, G_b = \begin{bmatrix} 0 & F_3 \end{bmatrix}.$$

Proof: Assume that the inequality of (18) is feasible. Then, application of the Schur's complement formula to this inequality gives

$$\Gamma_1(\mu) + \epsilon_1 \mathcal{E}_1 \mathcal{E}_1{}^T + \epsilon_1{}^{-1} \mathcal{H}_1(\mu)^T \mathcal{H}_1(\mu) \prec 0,$$

where

$$\Gamma_1(\mu) = \begin{bmatrix} \Upsilon_1 - \mathrm{sym}\{W\} & (\star) & (\star) \\ \Upsilon_3 + \Omega_1(\mu) - \beta W & \Upsilon_2 + \beta \mathrm{sym}\{\Omega_1(\mu)\} & (\star) \\ G_b - [0\ I]W & -G_a^T + [0\ I]\Omega_1^T(\mu) & P_2 - \mathrm{sym}\{F_3\} \end{bmatrix}, \ \mathcal{E}_1 = \begin{bmatrix} 0 \\ \hat{E}^T \\ 0 \end{bmatrix},$$
$$\mathcal{H}_1 = \left[\hat{F}_A(\mu)W + F_B(\mu)Y(\mu) \ \ \hat{F}_A(\mu)W + F_B(\mu)Y(\mu) \ \ (\hat{F}_A(\mu)W + F_B(\mu)Y(\mu)) \begin{bmatrix} 0 \\ I \end{bmatrix} \right].$$

Also, given Lemma 2, this last inequality is feasible if and only if

$$\Gamma_1(\mu) + \mathrm{sym}\{\mathcal{E}_1 \mathcal{F}(t) \mathcal{H}_1(\mu)\} \prec 0.$$

Moreover, by introducing $\Omega_2(\mu) = (\bar{A}(\mu) + \Delta\bar{A}(\mu)) + (\bar{B}(\mu) + \Delta\bar{B}(\mu))K(\mu)$ the above inequality is equivalent to (1) with

$$\Gamma = \begin{bmatrix} \Upsilon_1 & \Upsilon_3^T & G_b^T \\ \Upsilon_3 & \Upsilon_2 & -G_a \\ G_b & -G_a^T & P_2 - \mathrm{sym}\{F_3\} \end{bmatrix}, \Lambda^T = \begin{bmatrix} I & 0 \\ \beta I & 0 \\ 0 & I \end{bmatrix},$$
$$\mathcal{W} = \begin{bmatrix} W \\ [0\ I]W \end{bmatrix}, \Sigma = \begin{bmatrix} -I & \Omega_2^T(\mu) & 0 \end{bmatrix}. \tag{19}$$

Next, by Lemma 1, the inequality (18) is feasible if and only if the inequality (1) holds for matrices chosen as in (19). Also, $\Sigma^\perp$ and $\Lambda^\perp$ can be chosen as

$$\Sigma^\perp = \begin{bmatrix} \Omega_2^T(\mu) & 0 \\ I & 0 \\ 0 & I \end{bmatrix}, \ \Lambda^\perp = \begin{bmatrix} \beta I \\ -I \\ 0 \end{bmatrix}.$$

It now follows immediately that the first inequality in (2) can be is transformed to the following form

$$\begin{bmatrix} (\beta^2 - 1)P_1 & 0 \\ 0 & -\gamma^2 P_2 \end{bmatrix} + \mathrm{sym}\left\{ \begin{bmatrix} 0 \\ -\beta I \end{bmatrix} \begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \right\} \prec 0. \tag{20}$$

Also, making the substitutions

$$\Gamma \leftarrow \begin{bmatrix} (\beta^2 - 1)P_1 & 0 \\ 0 & -\gamma^2 P_2 \end{bmatrix}, \Lambda^T \leftarrow \begin{bmatrix} 0 \\ -\beta I \end{bmatrix},$$
$$\mathcal{W} \leftarrow \begin{bmatrix} F_1 & F_2 \end{bmatrix}, \Sigma \leftarrow \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

and applying Lemma 1 with $\Lambda^\perp = [I\ 0]^T$, it follows that the inequality (20) holds if and only if $(\beta^2 - 1)P_1 \prec 0$. Also, on setting $\beta \in (-1, 1)$ and $P_1 \succ 0$ (20) holds. Additionally, using the notation in (19) the second inequality in (2) gives

$$\begin{bmatrix} \mathbb{A}(\mu)P_1\mathbb{A}^T(\mu) - P_1 & \mathbb{A}(\mu)P_1\mathbb{C}^T(\mu) & 0 \\ \mathbb{C}(\mu)P_1\mathbb{A}^T(\mu) & \mathbb{C}(\mu)P_1\mathbb{C}^T(\mu) - \gamma^2 P_2 & 0 \\ 0 & 0 & P_2 \end{bmatrix} + \mathrm{sym}\left\{ \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \begin{bmatrix} \mathcal{B}_0^T(\mu) & \mathcal{D}_0^T(\mu) & -I \end{bmatrix} \right\} \prec 0. \tag{21}$$

Finally, by Lemma 1, the inequality (21) can be directly transformed into (17). This last fact implies robust stability of the resulting repetitive process (16); hence, trial-to-trial error convergence occurs, and the proof is complete. ■

A. LMI-based design

The result in Theorem 1 establishes sufficient conditions for stability along the trial for repetitive processes described by (16). Unfortunately, the control law design problem cannot be directly solved since the conditions are not LMIs. To achieve LMI-based design, any products involving unknown matrix variables must be decoupled, which is achieved by the following result.

Theorem 2: Let γ be a positive scalar satisfying $0 < \gamma \leq 1$. Then the controlled dynamics represented as a discrete linear repetitive process of the form (16) is robustly stable along the trial, and hence trial-to-trial error convergence occurs for all admissible uncertainties, if there exist compatibly dimensioned matrices $P_1 \succ 0$, $P_2 \succ 0$, W, F_1, F_2, F_3, Y_i and scalars $\beta \in (-1,1)$ and $\epsilon_1 > 0$ such that such that the following LMIs are feasible for all $i,j=1,2,\dots,r$

$$\begin{aligned} &\Upsilon_{ii} \prec 0, \\ &\Upsilon_{ij} + \Upsilon_{ji} \prec 0, i < j, \end{aligned} \tag{22}$$

where

$$\Upsilon_{ii}=\begin{bmatrix} \Upsilon_1-\mathrm{sym}\{W\} & (\star) & (\star) & (\star) & (\star) \\ \Upsilon_3+\Omega_{1_{ii}}-\beta W & \Upsilon_2+\beta\mathrm{sym}\{\Omega_{1_{ii}}\} & (\star) & (\star) & (\star) \\ G_b-[0\ I]W & -G_a^T+[0\ I]\Omega_{1_{ii}}^T & P_2-\mathrm{sym}\{F_3\} & (\star) & (\star) \\ 0 & \epsilon_1\hat{E}^T & 0 & -\epsilon_1 I & (\star) \\ (\hat{F}_{Ai}W+F_{Bi}Y_i) & (\hat{F}_{Ai}W+F_{Bi}Y_i) & (\hat{F}_{Ai}W+F_{Bi}Y_i)\begin{bmatrix}0\\I\end{bmatrix} & 0 & -\epsilon_1 I \end{bmatrix},$$

$$\Upsilon_{ij}=\begin{bmatrix} \Upsilon_1-\mathrm{sym}\{W\} & (\star) & (\star) & (\star) & (\star) \\ \Upsilon_3+0.5\Omega_{1_{ij}}-\beta W & \Upsilon_2+0.5\beta\mathrm{sym}\{\Omega_{1_{ij}}\} & (\star) & (\star) & (\star) \\ G_b-[0\ I]W & -G_a^T+0.5[0\ I]\Omega_{1_{ij}}^T & P_2-\mathrm{sym}\{F_3\} & (\star) & (\star) \\ 0 & \epsilon_1\hat{E}^T & 0 & -\epsilon_1 I & (\star) \\ 0.5\Omega_{2_{ij}} & 0.5\Omega_{2_{ij}} & 0.5\Omega_{2_{ij}}\begin{bmatrix}0\\I\end{bmatrix} & 0 & -\epsilon_1 I \end{bmatrix} \tag{23}$$

and

$$\begin{aligned} \Omega_{1_{ii}} &= \bar{A}_iW^T+\bar{B}_iY_i, \quad \Omega_{1_{ij}} = \bar{A}_iW^T+\bar{B}_iY_j, \\ \Omega_{2_{ij}} &= \hat{F}_{Ai}W+F_{Bi}Y_j, \\ \bar{A}_i &= \left[\begin{array}{cc|c} A_i & 0 & 0 \\ 0 & 0 & I \\ \hline -CA_i & 0 & I \end{array}\right], \ \bar{B}_i = \left[\begin{array}{c} B_i \\ 0 \\ \hline -CB_i \end{array}\right]. \end{aligned}$$

Also, if these last LMIs are feasible for all $i=1,2,\dots,r$ the corresponding ILC law matrices of (14) are given by

$$\left[\, K_{1i}\ K_{3i} \,\middle|\, L_i \,\right] = Y_iW^{-T}, \ K_{2i} = L_i + K_{3i}. \tag{24}$$

Proof: Assume that the inequality of (22) is feasible. Furthermore, by the nonnegative property of the membership functions, feasibility of (22) immediately implies that

$$\sum_{i=1}^{r}\mu_i^2\Upsilon_{ii} + 2\sum_{i=1}^{r}\sum_{i<j}\mu_i\mu_j\Upsilon_{ij} \prec 0.$$

This last inequality can be equivalently rewritten as (18). Therefore, given Theorem 1, the controlled dynamics represented as a discrete linear repetitive process of the form (16) is robustly stable along the trial. Hence, trial-to-trial error convergence occurs for all admissible uncertainties, and the proof is complete. ■

Remark 2: The result in Theorem 2 enables the introduction of fuzzy-basis-dependent matrix variables, i.e. $P_1(\mu) \succ 0$, $P_2(\mu) \succ 0$, $F_1(\mu)$, $F_2(\mu)$ and $F_3(\mu)$ since there are no products with other fuzzy-basis-dependent variables. This fact could result in a less conservative design condition with further research.

IV. Case study

As a numerical example, the ILC design of the previous section is applied in simulation to a single-link rigid robot system [16] whose dynamics are described by

$$\mathrm{J}\ddot{\eta} = -(0.5m_0gl + M_0gl)\sin(\eta) + u, \tag{25}$$

where η denotes the joint rotation angle in radians, $m_0 = 1.5$kg, $M_0 = 3$kg, $g = 9.8\mathrm{m/s}^2$ and $l = 0.5$m denote the mass of the load, the rigid link, the gravity constant and the robot link length, respectively. Also, the moment of inertia is

$$\mathrm{J} = M_0l^2 + (1/3)m_0l^2 = 0.875\mathrm{kg{\cdot}m}^2$$

and the control torque applied to the joint u in Nm is denoted by u.

Introduce the state vectors

$$x_k(t) = [x_{1k}(t)\ \ x_{2k}(t)]^T = [\eta\ \ \dot{\eta}]^T,$$

and take the output as $y_k(t) = \dot{\eta}$. Similar to [17], the nonlinear-sector approach leads to a T-S fuzzy model for the nonlinear system (25) as

Plant Rule 1: IF $x_{1k}(t)$ is about 0, THEN
$$\dot{x}_k(k) = A_1x_k(t) + B_1u_k(t)$$
Plant Rule 2: IF $x_{1k}(t)$ is about 0.5π, THEN
$$\dot{x}_k(k) = A_2x_k(t) + B_2u_k(t)$$

where

$$A_1=\begin{bmatrix} 0 & 1 \\ \frac{-m_0gl-2M_0gl}{2\mathrm{J}} & 0 \end{bmatrix}, A_2=\begin{bmatrix} 0 & 1 \\ \frac{-m_0gl-2M_0gl}{\pi\mathrm{J}} & 0 \end{bmatrix}, B_1=B_2=\begin{bmatrix} 0 \\ \frac{1}{\mathrm{J}} \end{bmatrix}.$$

Also, see [16],

$$\begin{aligned} &\mu_1(x_{1k}(t))=(0.5\pi - |x_{1k}(t)|)/(0.5\pi), \\ &\mu_2(x_{1k}(t))=|x_{1k}(t)|/(0.5\pi). \end{aligned}$$

Sampling the system with sampling period $T_0 = 0.02$s and assuming that the system model is subject to uncertainty, results in the two-rule discrete-time T-S fuzzy model

$$\begin{cases} x_k(t+1)=\sum_{i=1}^{2}\mu_i((A_i+\Delta A_i(t))x_k(t)+(B_i+\Delta B_i(t))u_k(t)), \\ y_k(t)=Cx_k(t), \end{cases}$$

where

$$\begin{aligned} &A_1=\begin{bmatrix} 0.9958 & 0.0200 \\ -0.4194 & 0.9958 \end{bmatrix}, A_2=\begin{bmatrix} 0.9973 & 0.0200 \\ -0.2671 & 0.9973 \end{bmatrix}, \\ &B_1=B_2=\begin{bmatrix} 0.0002 \\ 0.0228 \end{bmatrix}, E=\begin{bmatrix} 0 & 0 \\ 0.012 & 0.01 \end{bmatrix}, C=[0\ \ 1], \\ &F_{A1}=\begin{bmatrix} 0 & 0.21 \\ -0.05 & 0.41 \end{bmatrix}, F_{A2}=\begin{bmatrix} 0 & 0.13 \\ -0.06 & 0.37 \end{bmatrix}, \\ &F_{B1}=\begin{bmatrix} 0 \\ 0.024 \end{bmatrix}, F_{B2}=\begin{bmatrix} 0 \\ 0.016 \end{bmatrix}, \mathcal{F}(t)=\begin{bmatrix} -\sin(2t) & 0 \\ 0 & 0.5(1-e^{-t}) \end{bmatrix}. \end{aligned}$$

Consider the reference trajectory $y_d(t) = 2.236\pi(t - 0.5t^2)\sin(\pi t)$, $t \in [0,\ 2]$ s. Then solving the LMIs of

Theorem 2 with $\gamma = 0.9$ and $\beta = 0.8$ the matrices in (14) are

$$K_{11}=\begin{bmatrix}9.4223 & -43.4399\end{bmatrix}, K_{21}=33.4405, K_{31}=-1.9622\cdot 10^{-15};$$
$$K_{12}=\begin{bmatrix}4.3958 & -44.6048\end{bmatrix}, K_{22}=31.8862, K_{32}=-1.7876\cdot 10^{-15}.$$

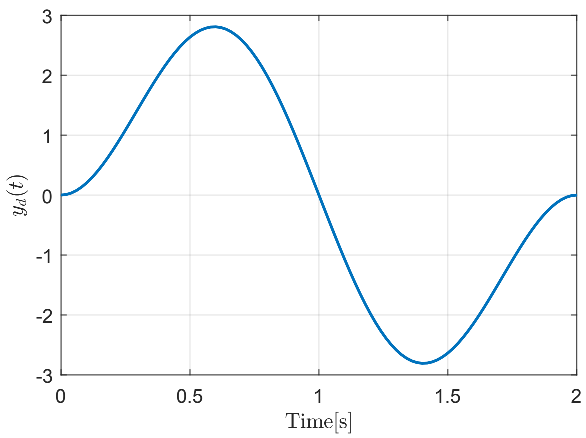

Fig. 1: The reference trajectory.

The resulting controlled system is stable along the trial; hence, trial-to-trial error convergence occurs. To assess ILC performance, a measure of the trial-to-trial error progress is needed. One measure is the root mean square error for each trial, i.e.,

$$\mathrm{RMSE} = \sqrt{\frac{1}{N}\sum_{t=0}^{N} |e_k(t)|^2} \tag{26}$$

plotted against the trial number k, where N denotes the number of samples along a trial. Performance along a trial (in t) can be assessed by standard measures. The RMSE plot for this example is shown over 100 trials in Fig 2, where noise with root mean square value 10^{-8} was added in the simulation and the convergence curve stays at this level after 16 trials.

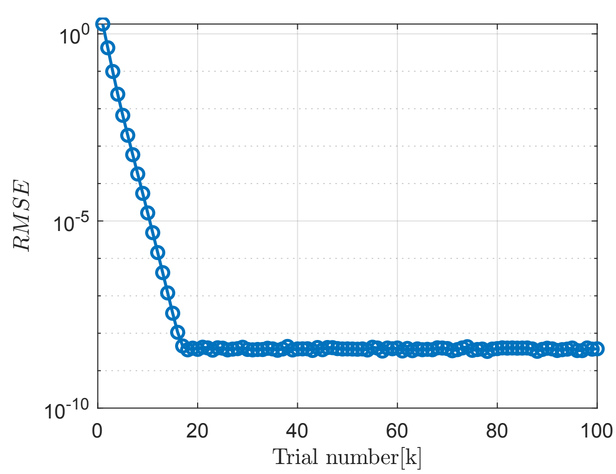

Fig. 2: RMSE values of the output tracking error.

V. Conclusions and Further Research

In this paper, new results on ILC design for a class of nonlinear systems have been developed. The approach is based on applying the local linearization method to establish a repetitive T-S fuzzy model of the controlled dynamics. Then, the stability theory for linear repetitive processes has been used to design the ILC law using LMIs. A numerical example illustrates the effectiveness of the design. Topics for future research include a detailed investigation into the attenuation of non-repetitive disturbances and control laws that use only measured outputs. Finally, a longer-term aim is to validate the design experimentally.

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