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Design of Controls for Boundedness of Trajectories of Multistable State Periodic Systems

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Abstract— **In this paper, the concept of control Leonov functions is introduced, and it is shown that their information is enough to design continuous and periodic controllers that provide boundedness of the state for a class of multistable state-periodic systems. These feedback control laws are based on a mild adaptation of Sontag's universal formula, and a kind of small control property. The proposed method is illustrated via application in a microgrid.**

I. Introduction

The control Lyapunov function framework has been found very useful for analysis of stabilizability and stabilization of nonlinear dynamical systems with a unique equilibrium at the origin [1]–[3]. Merely, in many cases the existence of a control Lyapunov function is a necessary and sufficient condition to conclude stabilizability of the system under study. Then, explicit expressions of control feedback laws are determined by using the so-called universal Sontag's formula [2]. These approaches have been extended for systems with exogenous inputs by means of the concepts of ISS- and integral ISS-control Lyapunov functions [4].

Many engineering applications concern systems with multiple equilibria or invariant sets, e.g., power or biological systems. However, the study of stability and boundedness of solutions for multistable systems requires significant modifications and relaxations of the employed Lyapunov stability notions, see for example [5]–[11]. In this context, extensions of the control Lyapunov function framework for multistable systems have been presented in [12], [13].

In this note, we are interested in a special class of such multistable systems being periodic with respect to a part of the state. Some examples of such a class of systems include the forced nonlinear pendulum [14], [15], power systems [16], [17], microgrids [18], [19], and phase-locked loops [20]. Recently, a fitting tool for the analysis of boundedness of solutions for multistable state-periodic systems has been proposed in [21], [22] through the concept of Leonov functions. These Leonov functions are useful to establish attractiveness of certain sets of the state space, and boundedness of solutions is concluded by exploiting the periodicity of the system's dynamics [21]–[23]. These results are based on the ideas of [24] (see also [7], [25], [26]), who have proposed the so-called *cell structure* approach. Moreover, the concepts of ISS-Leonov functions and iISS-Leonov functions have been presented in [27] and [28], respectively, as relaxed characterizations of the corresponding properties.

An extension of the Leonov function framework for ISS and iISS stabilization of multistable state-periodic systems has been presented in [29] through the concepts of ISS- and iISS-control Leonov functions. Unfortunately, the latter do not imply also boundedness of periodic variables by its definition [8], [9]. Therefore, in this paper we further develop the results of [29] in order to design continuous and periodic feedback laws providing the boundedness of all states for multistable state-periodic systems. To this end, we introduce the concept of control Leonov function (CLeF). The main advantage of the CLeF's approach is that the designed controllers are capable to ensure global boundedness of the trajectories of the controlled plant, while the most of the reported methods provide just local results, see for example [16], [19]. This is illustrated by applying the proposed method for the design of a controller for global boundedness of solutions in a small microgrid.

The remainder of the paper is structured as follows. Section II presents the considered class of systems and some useful definitions. Section III contains our main results. Then, Section IV presents its application in a small microgrid. Finally, the conclusions are given in Section V.

II. Preliminaries

A. Notation

$\mathbb{Z}$ and $\mathbb{R}$ stand for the sets of integer and real numbers, respectively. Moreover, $\mathbb{R}_+$ represents the set of non-negative real numbers, i.e., $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. Similarly, $\mathbb{Z}_+ = \{x \in \mathbb{Z} : x \geq 0\}$

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if, for each fixed $t \in \mathbb{R}_+$, $\beta(\cdot, t) \in \mathcal{K}_\infty$ and, for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is non-increasing and it tends to zero for $t \to \infty$.

The distance from a point $x \in \mathbb{R}^n$ to the set $S \subset \mathbb{R}^n$ is defined as $|x|_S = \inf_{a \in S} |x - a|$, where $|x| = |x|_{\{0\}}$ for $x \in \mathbb{R}^n$ is the usual Euclidean norm for a vector $x \in \mathbb{R}^n$. Furthermore, we introduce the vector norm $|x|_\infty =$

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$\max_{1\le i\le n}|x_i|$. Then, $|x|_\infty \le |x| \le \sqrt{n}|x|_\infty$. Besides, for a signal $d : \mathbb{R} \to \mathbb{R}^m$ the essential supremum norm is defined as $\|d\|_\infty = \text{ess}\ \sup_{t\ge0}|d(t)|$, and the set of such inputs with a finite norm is denoted by $\mathcal{L}_\infty^m$.

B. Multistable state periodic system

Consider the system

$$\dot{x} = f(x), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a map of class $\mathcal{C}^1$ (continuously differentiable with respect to its arguments), satisfying $f(0) = 0$. For any $x \in \mathbb{R}^n$, we denote by $X(t, x)$ the uniquely defined solution of (1) at time t fulfilling $X(0, x) = x$.

A set $S \subset \mathbb{R}^n$ is invariant for (1) if $X(t, x) \in S$ for all $t \in \mathbb{R}$ and for all $x \in S$; for $x \in \mathbb{R}^n$ the point $y \in \mathbb{R}^n$ belongs to its ω-limit (α-limit) set if there is a sequence t_i, $\lim_{i\to+\infty} t_i = +\infty$, such that $\lim_{i\to+\infty} x(t_i, x) = y$ ($\lim_{i\to+\infty} x(-t_i, x) = y$); for any $x \in \mathbb{R}^n$ its $\alpha-$ and $\omega-$limit sets are invariant [30].

Now, we say that the system (1) is periodic in a part of the state [7], [26] if it satisfies the following assumption.

Assumption 1: Let $x = (z, \theta) \in \mathbb{R}^n$, where $z \in \mathbb{R}^k$ and $\theta \in \mathbb{R}^q$ are two subsets of the state vector, $n = k + q$, $k > 0$ and $q > 0$. The vector field f in (1) is $2\pi-$periodic with respect to θ, i.e., for all $x = (z, \theta) \in \mathbb{R}^n$

$$f(x) = f(x + \xi_j), \quad \xi_j = [0_k^\top\ 2\pi j^\top]^\top \in \mathbb{R}^n,$$

where 0_k is a zero vector of dimension k and $j \in \mathbb{Z}^q$.
In other words, we suppose that the system (1) can be embedded in a manifold $M = \mathbb{R}^k \times \mathbb{S}^q$, where $\mathbb{S}$ is the unit sphere, by a simple projection of the variables θ on $\mathbb{S}^q$.

C. Boundedness of solutions of periodic systems

The so-called cell structure approach was introduced in [7], [24], [26] to establish boundedness of solutions for periodic systems, satisfying Assumption 1 for $q = 1$. This approach can be seen as a fine handling of attractiveness and periodicity, which leads to boundedness of trajectories. That is, the admissible behavior of the trajectories is bounded by a cell covering of the domains of the periodic variables, which is formed by isolated, periodically distributed, attractive, and forward invariant sets.

More precisely, for the system (1), with $q = 1$, the works in [7], [26] show that the existence of a sign-indefinite function $V_0(x)$, whose time derivative satisfies $\dot{V}_0(x) \le -\varrho V_0(x)$ for all $t \ge 0$, where $\varrho > 0$, implies attractiveness and forward invariance of the set $\Omega_0 = \{x \in \mathbb{R}^n : V_0(x) \le 0\}$, i.e., $X(t, x_0) \in \Omega_0$ for all $t \in [0, +\infty]$ provided that $x_0 \in \Omega_0$. Then, by exploiting the periodicity of the system (1), the functions $V_j(x) = V_j((z, \theta)) = V_0((z, \theta + 2\pi j))$ are defined for all $j \in \mathbb{Z}$, such that, the sets $\Omega_j = \{x \in \mathbb{R}^n : V_j(x) \le 0\}$ are also attractive and forward invariant for (1). Finally, it is concluded that the intersections of the sets Ω_j form a kind of cell cover of the domain of the variable θ, where each cell is bounded, isolated and forward invariant.

In [22], a generalization of the cell structure approach is developed for systems whose dynamics are periodic with respect to multiple state variables, i.e., for the case when the system (1) satisfies Assumption 1 for $q > 1$. To this end, [22] introduced the notion of Leonov functions. This concept has been further developed in [23].

Following [23], consider the sets: $\mathcal{U} = \cup_{r\in\mathbb{Z}_+}\mathcal{U}_r$, where $\mathcal{U}_r = \{x = (z, \theta) \in \mathbb{R}^n : |\theta|_\infty = 2r\pi,\ f(x) \equiv 0\}$, and $\mathcal{W} \subset \mathbb{R}^n$, such that for some $\underline{c} \in (0, 2\pi)$ and $\bar{c} \in (\pi, 2\pi)$, with $\underline{c} < \bar{c}$, we have $\{x \in \mathbb{R}^n : \underline{c} \le |\theta|_\infty \le \bar{c}\} \subseteq \mathcal{W}$.

Note that the set $\mathcal{U}$ includes all equilibria of (1) obtained by shifting the one at the origin by using the property that f is $2\pi-$periodic in θ. However, in general, the system (1) may possess other equilibria. Hence, let us define the equilibrium set of (1) as $\mathcal{D} = \{x \in \mathbb{R}^n : f(x) \equiv 0\}$. Clearly, $\mathcal{U} \subseteq \mathcal{D}$. On the other hand, the set $\mathcal{W}$ represents a barrier in $\mathbb{R}^n$ between the origin and the rest of the elements in $\mathcal{U}$.

Definition 1: [23]. A $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ is a strong Leonov function for (1) if there exist a constant $g \ge 0$, functions $\alpha \in \mathcal{K}_\infty$, $\psi \in \mathcal{K}$ and a continuous function $\lambda : \mathbb{R} \to \mathbb{R}$ satisfying $\lambda(0) = 0$, and $\lambda(s)s > 0$ for all $s \ne 0$, such that

$$\begin{gathered}\alpha(|z|) - \psi(|\theta|) - g \le V(x) \quad \forall x = (z, \theta) \in \mathbb{R}^n, \\ \inf_{x\in\mathcal{W}} V(x) > 0, \quad \sup_{x\in\mathcal{U}} V(x) \le 0,\end{gathered} \tag{2}$$

and the following dissipation inequality holds:

$$\frac{\partial V(x)}{\partial x} f(x) \le -\lambda(V(x)) \quad \forall x \in \{x \in \mathbb{R}^n : V(x) \ge 0\}. \tag{3}$$

Theorem 1: [23]. If for the system (1) under Assumption 1, there exists a strong Leonov function as in Definition 1, then for all $x_0 \in \mathbb{R}^n$, the corresponding trajectories $X(t, x_0)$ are bounded for all $t \ge 0$.
By exploiting the periodicity of the system (1), the proof of Theorem 1 in [23] consists of two steps: first, it is shown that the sets $\Omega_j = \{x \in \mathbb{R}^n : V_j(x) \le 0\}$, where $j = [j_1, \ldots, j_q] \in \mathbb{Z}^q$ and $V_j(x) = V(x - \xi_j)$, are globally attractive and forward invariant for (1). Note that $\mathcal{U} \in \Omega_j$ for any j. Then, it is proven that the intersection of the sets Ω_j is composed by compact and isolated "cells". For this, the condition $\inf_{x\in\mathcal{W}} V(x) > 0$, plays a crucial role. So, for any $x_0 \in \mathbb{R}^n$, the solution $X(t, x_0)$ asymptotically enters and remains in one of those cells. Thus, for any $x_0 \in \mathbb{R}^n$, the corresponding solution $X(t, x_0)$ is bounded for all $t \ge 0$.

Corollary 1: [23]. If for the system (1), under Assumption 1, there exists a Leonov function as in Definition 1 with

$$V(0) \le 0, \quad \inf_{x\in\mathcal{U}\setminus\mathcal{U}_0} V(x) \le 0, \quad \inf_{x\in\mathcal{W}} V(x) > 0,$$

and with the following dissipation inequality

$$\frac{\partial V(x)}{\partial x} f(x) < 0 \quad \forall x \in \mathbb{R}^n \setminus \{\mathcal{D} \cup \{x \in \mathbb{R}^n : V(x) \le 0\}\},$$

then for all $x_0 \in \mathbb{R}^n$ the corresponding trajectories $X(t, x_0)$ are bounded for all $t \ge 0$.

III. Main results

Consider a system of the form

$$\dot{\theta} = f_1(x), \qquad \dot{z} = f_2(x) + g(x)u, \tag{4}$$

where $x \in \mathbb{R}^n$ is the state vector, $x = (z, \theta) \in \mathbb{R}^n$ with $z \in \mathbb{R}^k$ and $\theta \in \mathbb{R}^q$, $n = k + q$ for $k, q > 0$, and $u \in \mathcal{L}_\infty^p$ is an admissible control input. Moreover, $f_1 : \mathbb{R}^n \to \mathbb{R}^q$, $f_2 : \mathbb{R}^n \to \mathbb{R}^k$ and $g : \mathbb{R}^n \to \mathbb{R}^{k\times p}$ are supposed to be locally Lipschitz continuous vector functions, with $f_1(0) = 0$ and $f_2(0) = 0$.

Assumption 2: The vector fields f_1, f_2 and g in (4) are $2\pi-$periodic with respect to θ.

Now, the problem is to determine conditions that ensure the existence of a continuous and periodic feedback $u(x)$, such that the trajectories $X(t, x_0)$ of the system (4) are bounded for all $t \geq 0$. Let us define the sets $\mathcal{D}$, $\mathcal{U}$ and $\mathcal{W}$ for the system (4), with $u \equiv 0$, as in Subsection II-C by taking $f(x) = [f_1^\top(x), f_2^\top(x)]^\top$. Then, we present the following definition of CLeF's.

Definition 2: A $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be a CLeF for the system (4) if there exist functions $\varphi, \vartheta, \sigma_1, \sigma_2 \in \mathcal{K}_\infty$, scalars $g_1, g_2 \geq 0$, and a continuous function $\varpi : \mathbb{R} \to \mathbb{R}$ satisfying $\varpi(0) = 0$, and $\varpi(s)s > 0$ for all $s > 0$, such that for all $x = (z, \theta) \in \mathbb{R}^n$:

$$\begin{gathered}\varphi(|z|) - \sigma_1(|\theta|) - g_1 \leq V(x) \leq \vartheta(|x|_\mathcal{U}) - \sigma_2(|\theta|) + g_2,\\ \inf_{x\in\mathcal{W}} V(x) > 0, \quad \sup_{x\in\mathcal{U}} V(x) \leq 0,\end{gathered} \tag{5}$$

and for all $x \in \mathbb{R}^n \setminus \Omega$ with $\Omega = \{x \in \mathbb{R}^n : V(x) < 0\}$:

$$\inf_{u\in\mathbb{R}^p} \{a(x) + b(x)u\} \leq -\varpi(V(x)), \tag{6}$$

where $a(x) = \frac{\partial V(x)}{\partial \theta} f_1(x) + \frac{\partial V(x)}{\partial z} f_2(x)$ and $b(x) = \frac{\partial V(x)}{\partial z} g(x)$, with a function $b(x)$ being 2π-periodic in θ, and satisfying $b(0) \equiv 0$.

Note that the restriction (5) for CLeF's is stronger than the condition (2) for Leonov functions. However, this stronger restriction is needed for the design of a controller u periodic with respect to θ. Hence, we can see from (6) that, under a proper design of the control input u, a CLeF is indeed a Leonov function for the controlled system.

In general, CLeFs are non-periodic, hence the partial derivative of V with respect to θ may contain non-periodic terms of θ, which appears in the function $a(x)$. From (5), we have $\sigma_2(|\theta|) \leq \vartheta(|x|_\mathcal{U}) + g_2$ for all $x \in \mathbb{R}^n \setminus \Omega$, then we can define a function

$$\phi(|x|_\mathcal{U}, z) = \sup_{\sigma_2(|\theta|)\leq\vartheta(|x|_\mathcal{U})+g_2} \{a(x) + \tfrac{1}{2}\varpi(V(x))\}, \tag{7}$$

which is 2π-periodic with respect to θ since $|x|_\mathcal{U} = |x - \xi_j|_\mathcal{U}$. On the other hand, the function $b(x)$ is 2π-periodic with respect to θ by construction.

Now, following [2], consider the so-called "universal formula" given by

$$K(\phi, |b|) = \begin{cases} \frac{\phi + \sqrt{\phi^2 + |b|^4}}{|b|^2} & \text{if} \quad |b| \neq 0, \\ 1 & \text{if} \quad |b| = 0, \end{cases} \tag{8}$$

where the arguments of the functions $\phi(|x|_\mathcal{U}, z)$ and $b(x)$ have been omitted to get a shorter expression. Note that since these functions are supposed to be 2π-periodic with respect to θ, then (8) inherits this property as well. Now, let us check the continuity of the function (8) in the set $\mathbb{R}^n \setminus \Omega$, i.e., wherever $V(x) \geq 0$. By construction the functions $\phi(|x|_\mathcal{U}, z)$ and $b(x)$ are continuous everywhere, then the function (8) is continuous for $|b(x)| \neq 0$. Recall the property $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, for any $a, b \geq 0$. So, we have

$$K(\phi(|x|_\mathcal{U}, z), |b(x)|) \leq \frac{\phi(|x|_\mathcal{U}, z) + |\phi(|x|_\mathcal{U}, z)|}{|b(x)|^2} + 1.$$

Clearly, if $\phi(|x|_\mathcal{U}, z) < 0$ then $K(\phi(|x|_\mathcal{U}, z), |b(x)|) \leq 1$, hence this condition is sufficient to ensure the continuity of (8) for $b(x) \equiv 0$. From (6), it follows that $b(x) \equiv 0$ leads to $a(x) \leq -\varpi(V(x))$ for all $x \in \mathbb{R}^n \setminus \Omega$. Consequently, we obtain that $b(x) \equiv 0$ implies

$$\phi(|x|_\mathcal{U}, z) \leq \sup_{\sigma_2(|\theta|)\leq\vartheta(|x|_\mathcal{U})+g_2} \{-\tfrac{1}{2}\varpi(V(x))\}, \tag{9}$$

for all $x \in \mathbb{R}^n \setminus \Omega$. Hence, for $b(x) \equiv 0$: if $V(x) > 0$ then we have $\phi(|x|_\mathcal{U}, z) < 0$, as it is needed. However, if $V(x) = 0$, we may have $\phi(|x|_\mathcal{U}, z) = 0$. In this case, we need to guarantee that the term $\phi(|x|_\mathcal{U}, z)$ vanishes "faster" than $b(x)$ around the points of the set $\Gamma = \{x \in \mathbb{R}^n : V(x) = 0;\ |b(x)| = 0\}$. For this, let us suppose that for each constant $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sup_{0<|x|_\Gamma\leq\delta} \frac{\phi(|x|_\mathcal{U}, z)}{|b(x)|^2} < \varepsilon. \tag{10}$$

We call this condition a *small control property* in the framework of Leonov functions (SCP-LeF). Hence, if the condition (10) is verified, then the formula (8) satisfies

$$K(\phi(|x|_\mathcal{U}, z), |b(x)|) \leq \frac{2\phi(|x|_\mathcal{U}, z)}{|b(x)|^2} + 1 \leq 2\varepsilon + 1,$$

for all $0 < |x|_\Gamma \leq \delta$. Since we can make ε arbitrarily small then we conclude that the condition (10) ensures the continuity of (8) in all points of Γ. Consequently, we can say that if the SCP-LeF (10) is satisfied, then the function (8) is continuous in $\mathbb{R}^n \setminus \Omega$.

Due to the continuity of the function (8) in the set $\mathbb{R}^n \setminus \Omega$, there exists a continuous periodic function $\bar{K}(x)$, such that

$$K(\phi(|x|_\mathcal{U}, z), |b(x)|) \leq \bar{K}(x) \tag{11}$$

for all $x \in \mathbb{R}^n \setminus \Omega$. For $x \in \Omega$, the latter property cannot be ensured but the function $\bar{K}(x)$ keeps its continuity globally. Note that if we have $\phi(|x|_\mathcal{U}, z) < 0$ wherever $b(x) \equiv 0$ for all $x \in \mathbb{R}^n$ and the SCP (10) is satisfied, then the function (8) is continuous everywhere. Thus, we can set $K(\phi(|x|_\mathcal{U}, z), |b(x)|) = \bar{K}(x)$ in (11), i.e., we can directly apply the Sontag's formula for the design of the controller as in the Lyapunov framework.

Finally, we propose the following feedback control law

$$u(x) = -\bar{K}(x) b^\top(x), \tag{12}$$

which is continuous in $\mathbb{R}^n$, and 2π-periodic in θ. Hence, the closed-loop system (4),(12) inherits these properties.

Moreover, the controller (12) vanishes at every element of the set $\mathcal{U}$ since by definition $b(0) \equiv 0$. Therefore, $\mathcal{U}$ is contained in the equilibrium set of the closed-loop system (4),(12). Finally, we are able to establish the following results.

Theorem 2: Let the system (4) satisfy Assumption 2. If there exists a CLeF as in Definition 2 satisfying the SCP-LeF (10), then the periodic feedback control law (12) is continuous on $\mathbb{R}^n$, and for all $x_0 \in \mathbb{R}^n$ the trajectories $X(t, x_0)$ of the closed-loop system (4),(12) are bounded for all $t \geq 0$.

Corollary 2: Let the system (4) satisfy Assumption 2. If there exists a CLeF as in Definition 2 with

$$V(0) \leq 0, \quad \inf_{x \in \mathcal{U} \setminus \mathcal{U}_0} V(x) \leq 0, \quad \inf_{x \in \mathcal{W}} V(x) > 0, \tag{13}$$

and with the dissipation inequality

$$\inf_{u \in \mathbb{R}^p} \{a(x) + b(x)u\} < 0, \qquad \forall x \in \mathbb{R}^n \setminus \{\mathrm{cl}(\Omega) \cup \mathcal{D}\}, \tag{14}$$

with $a(x)$ and $b(x)$ as in Definition 2, besides the SCP-LeF (10) is satisfied with $\Gamma = \{x \in \mathbb{R}^n : V(x) = 0;\ |b(x)| = 0\} \cup \{x \in \mathcal{D} : V(x) > 0; |b(x)| = 0\}$, then the feedback control law (12) is continuous on $\mathbb{R}^n$, and for all $x_0 \in \mathbb{R}^n$ the trajectories $X(t, x_0)$ of the closed-loop system (4),(12) are bounded for all $t \geq 0$.

Corollary 2 is a counterpart of Corollary 1 that relaxes the conditions of Definition 1. This simplifies the construction of CLeF's. The proofs of these results are omitted due to space limitation.

IV. Application to Controller Design for a Microgrid

Let us follow the modeling of droop-controlled microgrids presented by [31], (see also [18]). So, consider a microgrid of two nodes and one line, where the phase angle and the frequency at the i-th node are represented by θ_i and ω_i, respectively, with $i = 1, 2$. Then, under some standard assumptions like voltage amplitudes are positive real constants for both nodes, and line impedance is purely inductive, the active power flow at the i-th node, with $i = 1, 2$, is given by

$$P_i = G_{ii}V_i^2 + |B_{ik}|V_iV_k \sin(\theta_i - \theta_k), \text{ with } i, k = 1, 2;\ i \neq k,$$

where $V_1, V_2 > 0$ denote the voltage amplitudes, $G_{11}, G_{22} > 0$ denote the active power loads, and $B_{12} = B_{21} < 0$ represents the susceptance of the power line connecting the nodes. Now, by introducing the variable $\eta = \theta_1 - \theta_2$ and the parameter $A = |B_{12}|V_1V_2$, we define the potential function $U(\eta) = -A\cos(\eta)$, and its gradient $\nabla U(\eta) = A\sin(\eta)$. In addition, we denote the active power set-point by $P^d = [P_1^d, P_2^d]^\top \in \mathbb{R}^2$, and the incidence matrix of the network by $\mathcal{B} = [1, -1]^\top$. Therefore, we can write $P - P^d = \mathcal{B}\nabla U(\eta) - P^{\mathrm{net}}$, where $P = [P_1, P_2]^\top \in \mathbb{R}^2$, and $P^{\mathrm{net}} = [P_1^d - G_{11}V_1^2, P_2^d - G_{22}V_2^2]^\top \in \mathbb{R}^2$. Hence, a model for synchronization of the considered microgrid is given by

$$\dot{\eta} = \mathcal{B}\omega, \qquad M\dot{\omega} = -\mathcal{B}\nabla U(\eta) + P^{\mathrm{net}} + u, \tag{15}$$

where $\omega = [\omega_1, \omega_2]^\top \in \mathbb{R}^2$ are frequencies, $M = m\mathbf{I}_2 > 0$ is the inertia matrix ($\mathbf{I}_2$ is an identity matrix of order two), and $u \in \mathbb{R}^2$ is the control input.

Let the left psuedo-inverse matrix of $\mathcal{B}$ be given by $\mathcal{B}^+ = (\mathcal{B}^\top\mathcal{B})^{-1}\mathcal{B}^\top$. For $|A^{-1}\mathcal{B}^+P^{\mathrm{net}}| < 1$, the system (15), with $u = 0$, has equilibrium points at $[\eta^* + 2j\pi, \omega^*, \omega^*]^\top \in \mathbb{R}^3$ and $[(1+2j)\pi - \eta^*, \omega^*, \omega^*]^\top \in \mathbb{R}^3$ for any $j \in \mathbb{Z}$, where ω^* is the nominal frequency of the network, such that the condition $\omega_1 = \omega_2 = \omega^*$ is desired, and η^* is the solution of the equation $\mathcal{B}\nabla U(\eta^*) = P^{\mathrm{net}}$, satisfying $-\frac{\pi}{2} \leq \eta^* \leq \frac{\pi}{2}$.

Then, let us define the error variables $\hat{\eta} = \eta - \eta^* \in \mathbb{R}$ and $\hat{\omega} = \omega - \omega^* \in \mathbb{R}^2$. So, we rewrite the system (15) as

$$\dot{\hat{\eta}} = \mathcal{B}^\top\hat{\omega}, \qquad M\dot{\hat{\omega}} = -\mathcal{B}\chi(\hat{\eta}) + u, \tag{16}$$

where $\chi(\hat{\eta}) = \nabla U(\hat{\eta} + \eta^*) - \nabla U(\eta^*)$. Clearly, the system (16), with $u = 0$, has an equilibrium point at the origin, and in general, at all points $[2j\pi, 0, 0]^\top \in \mathbb{R}^3$ and $[(1+2j)\pi - 2\eta^*, 0, 0]^\top \in \mathbb{R}^3$ for any $j \in \mathbb{Z}$.

Now, consider the function

$$\begin{aligned} V(\hat{\eta}, \hat{\omega}) = \tfrac{1}{2}(m + \varrho^2)\hat{\omega}^\top\hat{\omega} + \varrho\chi(\hat{\eta})\mathcal{B}^\top\hat{\omega} \\ + \chi^2(\hat{\eta}) - \tfrac{1}{2}\kappa A\hat{\eta}^2 - |\mathcal{B}^+P^{\mathrm{net}}|^2 \\ + 2A(U(\hat{\eta} + \eta^*) - U(\eta^*)), \end{aligned} \tag{17}$$

where $\varrho, \kappa > 0$ are constant parameters to be determined.

Since $-\frac{\pi}{2} \leq \eta^* \leq \frac{\pi}{2}$, then $U(\eta^*) > 0$. So, we have

$$-A(1 - \cos(\eta^*)) \leq U(\hat{\eta} + \eta^*) - U(\eta^*) \leq A(1 + \cos(\eta^*)).$$

Moreover, by applying Young's inequality we obtain $2\varrho\|\chi(\hat{\eta})\mathcal{B}\|\|\hat{\omega}\| \leq \varrho^2\|\hat{\omega}\|^2 + 2\chi^2(\hat{\eta})$ (since $\|\mathcal{B}\|^2 = 2$), then

$$\begin{aligned} \tfrac{1}{2}m\|\hat{\omega}\|^2 \leq \tfrac{1}{2}(m + \varrho^2)\hat{\omega}^\top\hat{\omega} + \varrho\mathcal{B}^\top\hat{\omega}\chi(\hat{\eta}) + \chi^2(\hat{\eta}) \\ \leq \tfrac{1}{2}(m + 2\varrho^2)\|\hat{\omega}\|^2 + 2\chi^2(\hat{\eta}). \end{aligned}$$

Thus, we can conclude that for all $\hat{\eta} \in \mathbb{R}$ and $\hat{\omega} \in \mathbb{R}^2$,

$$\begin{aligned} \tfrac{1}{2}m\|\hat{\omega}\|^2 - \tfrac{1}{2}\kappa A\hat{\eta}^2 - \underline{g} \leq V(\hat{\eta}, \hat{\omega}) \\ \leq \tfrac{1}{2}(m + 2\varrho^2)\|\hat{\omega}\|^2 + 2\chi^2(\hat{\eta}) - \tfrac{1}{2}\kappa A\hat{\eta}^2 + \bar{g}, \end{aligned} \tag{18}$$

where $\kappa, m, A > 0$, $\bar{g} = A(1 + \cos(\eta^*)) - |\mathcal{B}^+P^{\mathrm{net}}|^2$, and $\underline{g} = |\mathcal{B}^+P^{\mathrm{net}}|^2 + A(1 - \cos(\eta^*))$.

Now, recall the sets $\mathcal{U}$ and $\mathcal{W}$ introduced in Subsection II-C. Particularly, for the system (16) we have $\mathcal{U} = \{[2j\pi, 0, 0]^\top \in \mathbb{R}^3 : j \in \mathbb{Z}\}$. Note that $\chi(2\pi j) = 0$ and $U(2\pi j + \eta^*) - U(\eta^*) = 0$, for any $j \in \mathbb{Z}$. So, by evaluating the function (17) at the points of the set $\mathcal{U}$, we obtain

$$V(2\pi j, [0, 0]^\top) = -2\kappa A\pi^2 j^2 - |\mathcal{B}^+P^{\mathrm{net}}|^2 < 0, \tag{19}$$

for any $j \in \mathbb{Z}$.

On the other hand, for a sufficiently small $\kappa > 0$ we can establish $\frac{1}{2}\kappa A\hat{\eta}^2 \leq |\mathcal{B}^+P^{\mathrm{net}}|^2$, for all $-2\pi \leq \hat{\eta} \leq 2\pi$. Moreover, let us evaluate the function (17) in the points $[\pi i, 0, 0]^\top$ for $i \in \{-1, 1\}$, this is

$$\begin{aligned} V([\pi i, 0, 0]) = & 4A^2\sin^2(\eta^*) + 4A^2\cos(\eta^*) \\ & - |\mathcal{B}^+P^{\mathrm{net}}|^2 - \tfrac{1}{2}\kappa A(\pi i)^2 \\ \geq & 4A^2\sin^2(\eta^*) + 4A^2\cos(\eta^*) - 2|\mathcal{B}^+P^{\mathrm{net}}|^2 \\ \geq & 2A^2\sin^2(\eta^*) + 4A^2\cos(\eta^*) \geq 0, \end{aligned}$$

since by definition $-\frac{\pi}{2} \leq \eta^* \leq \frac{\pi}{2}$, such that $0 \leq \cos(\eta^*) \leq 1$. Thus, by continuity we can say that the function (17) is positive in a (maybe small) neighborhood of the points where $|\hat{\eta}| = \pi$. Accordingly, we can define a set $\{[\hat{\eta}, \hat{\omega}]^\top \in \mathbb{R}^3 : c_1 \le |\hat{\eta}| \le c_2\} \subseteq \mathcal{W}$, with some $0 < c_1 \le \pi \le c_2 < 2\pi$, and $\mathcal{W}$ as given in Subsection II-C, so we can guarantee that

$$\inf_{[\hat{\eta},\hat{\omega}]^\top \in \mathcal{W}} V(\hat{\eta}, \hat{\omega}) > 0. \tag{20}$$

Now, the derivative of (17) along the trajectories of the system (16) is given by $\dot{V}(\hat{\eta}, \hat{\omega}) = a(\hat{\eta}, \hat{\omega}) + b(\hat{\eta}, \hat{\omega})u$, where

$$\begin{aligned} a(\hat{\eta}, \hat{\omega}) = &-(1 + \varrho^2 m^{-1} - 2A)\hat{\omega}^\top \mathcal{B}\chi(\hat{\eta}) - \kappa A\hat{\eta}\mathcal{B}^\top \hat{\omega} \\ &- 2\varrho m^{-1}\chi^2(\hat{\eta}) + 2A|\mathcal{B}^+ P^{\text{net}}|\mathcal{B}^\top \hat{\omega} \\ &+ (\varrho\hat{\omega}^\top \mathcal{B} + 2\chi(\hat{\eta}))\nabla^2 U(\hat{\eta} + \eta^*)\mathcal{B}^\top \hat{\omega}, \end{aligned}$$

and

$$b(\hat{\eta}, \hat{\omega}) = (1 + \varrho^2 m^{-1})\hat{\omega} + \varrho m^{-1}\mathcal{B}\chi(\hat{\eta}).$$

Note that $\|b(\hat{\eta}, \hat{\omega})\| = 0$ implies $\hat{\omega} = -\varrho(m + \varrho^2)^{-1}\mathcal{B}\chi(\hat{\eta})$. Hence, since $|\nabla^2 U(\hat{\eta} + \eta^*)| \le A$, $2|\hat{\eta}||\chi(\hat{\eta})| \le \hat{\eta}^2 + \chi^2(\hat{\eta})$, and $2|\mathcal{B}^+ P^{\text{net}}||\chi(\hat{\eta})| \le |\mathcal{B}^+ P^{\text{net}}|^2 + \chi^2(\hat{\eta})$, we have

$$\begin{aligned} a\,|_{\|b\|=0} =& -4\varrho(m + \varrho^2)^{-1}(A \\ &+ m(m + \varrho^2)^{-1}\nabla^2 U(\hat{\eta} + \eta^*))\chi^2(\hat{\eta}) \\ &+ 2\kappa\varrho A(m + \varrho^2)^{-1}\hat{\eta}\chi(\hat{\eta}) \\ &- 4\varrho A(m + \varrho^2)^{-1}|\mathcal{B}^+ P^{\text{net}}|\chi(\hat{\eta}) \\ \le& -A\varrho(m + \varrho^2)^{-1}(2 - 4m(m + \varrho^2)^{-1} - \kappa)\chi^2(\hat{\eta}) \\ &+ \varrho A(m + \varrho^2)^{-1}(\kappa\hat{\eta}^2 - 2|\mathcal{B}^+ P^{\text{net}}|^2), \end{aligned}$$

On the other hand, let us evaluate the function (17) in the surface where $\|b(\hat{\eta}, \hat{\omega})\| = 0$, this is

$$\begin{aligned} V\,|_{\|b\|=0} =& m(m + \varrho^2)^{-1}\chi^2(\hat{\eta}) - \tfrac{1}{2}\kappa A\hat{\eta}^2 \\ &- |\mathcal{B}^+ P^{\text{net}}|^2 + 2A(U(\hat{\eta} + \eta^*) - U(\eta^*)). \end{aligned}$$

Thus, $V\,|_{\|b\|=0}\,(\hat{\eta}, \hat{\omega}) \ge 0$ implies $m(m + \varrho^2)^{-1}\chi^2(\hat{\eta}) + 2A(U(\hat{\eta} + \eta^*) - U(\eta^*)) \ge \frac{1}{2}\kappa A\hat{\eta}^2 + |\mathcal{B}^+ P^{\text{net}}|^2$.

Assumption 3: For some $\varrho > 0$, the inequality $m(m + \varrho^2)^{-1}\chi^2(\hat{\eta}) > 2A(U(\hat{\eta} + \eta^*) - U(\eta^*))$ is satisfied wherever $V\,|_{\|b\|=0}\,(\hat{\eta}, \hat{\omega}) \ge 0$.

Therefore, under Assumption 3, we have $2m(m + \varrho^2)^{-1}\chi^2(\hat{\eta}) \ge \frac{1}{2}\kappa A\hat{\eta}^2 + |\mathcal{B}^+ P^{\text{net}}|^2 > 0$, wherever $V\,|_{\|b\|=0}\,(\hat{\eta}, \hat{\omega}) \ge 0$ Thus, for $A > 1$, we obtain

$$a\,|_{\|b\|=0} \le -A\varrho(m + \varrho^2)^{-1}(2 - 8m(m + \varrho^2)^{-1} - \kappa)\chi^2(\hat{\eta}),$$

and, for some sufficiently large $\varrho > 0$ and some sufficiently small $\kappa > 0$, we conclude that

$$a\,|_{\|b\|=0}\,(\hat{\eta}, \hat{\omega}) < 0, \tag{21}$$

wherever $V\,|_{\|b\|=0}\,(\hat{\eta}, \hat{\omega}) \ge 0$.

Finally, recall Definition 2 and Corollary 2, then note that the properties (18), (19) and (20) are equivalent to the conditions (5) and (13), while the property (21) is equivalent to the condition (14). Thus, if Assumption 3 is satisfied with some sufficiently large $\varrho > 0$, then we can say that the function (17) is a CLeF for the system (16) in the sense of Corollary 2, provided that $\kappa > 0$ is sufficiently small.

Now, we can follow the procedure developed in Section III to design a controller for the system (16). First, from (17) we can easily see that $\frac{1}{2}(m + \varrho^2)\hat{\omega}^\top\hat{\omega} + \varrho\chi(\hat{\eta})\mathcal{B}^\top\hat{\omega} + \chi^2(\hat{\eta}) + 2A(U(\hat{\eta} + \eta^*) - U(\eta^*)) > \frac{1}{2}\kappa A\hat{\eta}^2 + |\mathcal{B}^+ P^{\text{net}}|^2$, wherever $V(\hat{\eta}, \hat{\omega}) > 0$. Then, similarly to (7) we define

$$\begin{aligned} \phi(|(\hat{\eta}, \hat{\omega})|_{\tilde{\mathcal{U}}}, \hat{\omega}) =& \sup_{V(\hat{\eta},\hat{\omega})>0} \{a(\hat{\eta}, \hat{\omega})\} \\ \le& \sup_{V(\hat{\eta},\hat{\omega})>0} \big\{-(1 + \varrho^2 m^{-1} - 2A)\hat{\omega}^\top \mathcal{B}\chi(\hat{\eta}) \\ &+ (1 + \tfrac{1}{2}\kappa)A\hat{\omega}^\top\hat{\omega} - 2\varrho m^{-1}\chi^2(\hat{\eta}) \\ &+ (\varrho\hat{\omega}^\top \mathcal{B} + 2\chi(\hat{\eta}))\nabla^2 U(\hat{\eta} + \eta^*)\mathcal{B}^\top \hat{\omega} \\ &+ A(2|\mathcal{B}^+ P^{\text{net}}|^2 + \kappa\hat{\eta}^2)\big\} \\ \le& -(1 + \varrho^2 m^{-1} - 2A)\hat{\omega}^\top \mathcal{B}\chi(\hat{\eta}) + \chi^2(\hat{\eta}) \\ &+ (1 + \tfrac{1}{2}\kappa)A\hat{\omega}^\top\hat{\omega} - 2\varrho m^{-1}\chi^2(\hat{\eta}) \\ &+ (\varrho\hat{\omega}^\top \mathcal{B} + 2\chi(\hat{\eta}))\nabla^2 U(\hat{\eta} + \eta^*)\mathcal{B}^\top \hat{\omega} \\ &+ 2A(\tfrac{1}{2}(m + \varrho^2)\hat{\omega}^\top\hat{\omega} + \varrho\chi(\hat{\eta})\mathcal{B}^\top\hat{\omega} \\ &+ 2A(U(\hat{\eta} + \eta^*) - U(\eta^*))) \\ =& \bar{\phi}(|(\hat{\eta}, \hat{\omega})|_{\tilde{\mathcal{U}}}, \hat{\omega}). \end{aligned}$$

where Young's inequality was applied to split the terms $2\||\mathcal{B}^+ P^{\text{net}}|\mathcal{B}\|\|\hat{\omega}\| \le 2|\mathcal{B}^+ P^{\text{net}}|^2 + \hat{\omega}^\top\hat{\omega}$, and $2\|\mathcal{B}\hat{\eta}\|\|\hat{\omega}\| \le 2\hat{\eta}^2 + \hat{\omega}^\top\hat{\omega}$.

Clearly, $b(\hat{\eta}, \hat{\omega})$ and $\bar{\phi}(|(\hat{\eta}, \hat{\omega})|_{\tilde{\mathcal{U}}}, \hat{\omega})$ are 2π-periodic with respect to $\hat{\eta}$. Then, from the formula (8) we obtain

$$K(\phi, b) \le \frac{\bar{\phi} + \sqrt{\bar{\phi}^2 + |b|^4}}{|b|^2}, \tag{22}$$

where we have omitted the arguments of the respective function due to the space limitation. Next, let us check the continuity of the function (22) for $V(\hat{\eta}, \hat{\omega}) \ge 0$. Clearly, this property is obtained wherever $\|b(\hat{\eta}, \hat{\omega})\| \ne 0$, by construction. Therefore, consider Assumption 3 and the constraint $\|b(\hat{\eta}, \hat{\omega})\| = 0$, which implies $\hat{\omega} = -\varrho(m + \varrho^2)^{-1}\mathcal{B}\chi(\hat{\eta})$:

$$\begin{aligned} \bar{\phi}\,|_{\|b\|=0} \le& -4\varrho A(m + \varrho^2)^{-1}(1 - m(m + \varrho^2)^{-1} \\ &- \tfrac{1}{2}\varrho(\tfrac{1}{2}\kappa + 1)(m + \varrho^2)^{-1} - m\rho^{-1})\chi^2(\hat{\eta}). \end{aligned}$$

Thus, we can see that for some sufficiently large $\varrho > 0$ and some sufficiently small $\kappa > 0$,

$$\bar{\phi}\,|_{\|b\|=0}\,(|(\hat{\eta}, \hat{\omega})|_{\tilde{\mathcal{U}}}, \hat{\omega}) \le 0, \tag{23}$$

wherever $V\,|_{\|b\|=0}\,(\hat{\eta}, \hat{\omega}) \ge 0$. Therefore, we can ensure that the function (22) is continuous, wherever (23) is satisfied with strict inequality. So, it remain us to check the points where the equality occurs, this is at every element of the set $\{[2j\pi, 0, 0]^\top, [(1 + 2j)\pi - 2\eta^*, 0, 0]^\top : j \in \mathbb{Z}\}$. Recall that we have already proven that $V([2j\pi, 0, 0]) < 0$ for any $j \in \mathbb{Z}$. On the other hand, since $-\frac{1}{2}\pi < \eta^* < \frac{1}{2}\pi$, then $2A(U((1 + 2j)\pi - \eta^*) - U(\eta^*)) = 4A^2\cos(\eta^*) > 0$. Hence, by Assumption 3 we can establish that $V([(1 + 2j)\pi - 2\eta^*, 0, 0]) < 0$ for any $j \in \mathbb{Z}$, as well. Thus, we can ensure that the function (22) is continuous, wherever $V(\hat{\eta}, \hat{\omega}) \ge 0$, as it is required. Note that the SCP-LeF (10) is straightforwardly satisfied for the function (17), since the function $\bar{\phi}\,|_{\|b\|=0}\,(|(\hat{\eta}, \hat{\omega})|_{\tilde{\mathcal{U}}}, \hat{\omega})$ is strictly negative at every element of the set Γ given in Corollary 2.

Thus, since the function $\chi(\hat{\eta})$ is bounded, then we can verify that there exists a constant $K_1 > 0$ such that

$$\lim_{|b(\hat{\eta},\hat{\omega})|\to 0} K(\bar{\phi}(|(\hat{\eta},\hat{\omega})|_{\tilde{\mathcal{U}}},\hat{\omega}), b(\hat{\eta},\hat{\omega})) = K_1,$$

On the other hand, note that the function $\bar{\phi}(|(\hat{\eta},\hat{\omega})|_{\tilde{\mathcal{U}}},\hat{\omega})$ is quadratic in the variables $\hat{\omega}$, hence we can also claim that there exist a constant $K_2 > 0$, such that

$$\lim_{\|\hat{\omega}\|\to \infty} K(\bar{\phi}(|(\hat{\eta},\hat{\omega})|_{\tilde{\mathcal{U}}},\hat{\omega}), b(\hat{\eta},\hat{\omega})) = K_2,$$

Hence, we have $K(\bar{\phi}(|(\hat{\eta},\hat{\omega})|_{\tilde{\mathcal{U}}},\hat{\omega}), b(\hat{\eta},\hat{\omega})) \leq \max\{K_1, K_2\} = \bar{K}$, wherever $V(\hat{\eta},\hat{\omega}) \geq 0$.

Finally, we propose the controller

$$u(\hat{\eta},\hat{\omega}) = -\bar{K}((1+\varrho^2 m^{-1})\hat{\omega} + \varrho m^{-1}\mathcal{B}\chi(\hat{\eta})), \quad (24)$$

which is continuous in $\mathbb{R}^3$, 2π-periodic with respect to $\hat{\eta}$, and vanishes at every equilibrium point of the system (16), since all of them lay in th surface $\|b(\hat{\eta},\hat{\omega})\| = 0$. Hence, the closed-loop system

$$\begin{aligned} \dot{\hat{\eta}} &= \mathcal{B}^\top \hat{\omega}, \\ M\dot{\hat{\omega}} &= -(1+\varrho^2 m^{-1})\bar{K}\hat{\omega} - (1+\varrho m^{-1}\bar{K})\mathcal{B}\chi(\hat{\eta}), \end{aligned}$$

is continuous in $\mathbb{R}^3$, 2π-periodic with respect to $\hat{\eta}$, and has equilibria at all points $[2j\pi, 0, 0]^\top \in \mathbb{R}^3$ and $[(1+2j)\pi - 2\eta^*, 0, 0]^\top \in \mathbb{R}^3$ for any $j \in \mathbb{Z}$. Moreover, boundedness of solutions of this closed-loop system is supported by the results of [31] through Corollary 1 and a Leonov function slightly different to (17).

V. Conclusions

In this paper, we addressed the design of continuous and periodic in θ feedback laws for boundedness of trajectories of the system (4). To this end, the concept of CLeF was introduced, with the corresponding SCP-LeF condition. Two variants of control synthesis were proposed based on Sontag's formula. Finally, the advantage of this methodology was illustrated by an application in a small microgrid.

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