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Adaptive Estimation for Environmental Monitoring using an Autonomous Underwater Vehicle

Ziwen Yang, Joana Fonseca, Shanying Zhu, Cailian Chen, and Karl H. Johansson

***Abstract*—This paper considers the problem of monitoring and adaptively estimating an environmental field, such as temperature or salinity, using an autonomous underwater vehicle (AUV). The AUV moves in the field and persistently measures environmental scalars and its position in its local coordinate frame. The environmental scalars are approximately linearly distributed over the region of interest, and an adaptive estimator is designed to estimate the gradient. By orthogonal decomposition of the velocity of the AUV, a linear time-varying system is equivalently constructed, and the sufficient conditions on the motion of the AUV are established, under which the global exponential stability of the estimation error system is rigorously proved. Furthermore, an estimate of the exponential convergence rate is given, and a reference trajectory that maximizes the estimate of the convergence rate is obtained for the AUV to track. Numerical examples verify the stability and efficiency of the system.**

***Index Terms*—Adaptive Estimation, Environmental Monitoring, Exponential Stability, Autonomous Underwater Vehicle.**

1. INTRODUCTION

Studying environmental fields such as temperature, chemical concentration, or radiation intensity has attracted the attention of researchers due to its wide applications. For example, in oceanography, for monitoring large oil spills using autonomous underwater vehicles (AUVs) [1]. AUVs are a tool about two-decade-old [2] with diverse oceanographic capabilities, including for environmental monitoring. One of their most significant drawbacks is battery time which is influenced by inertial characteristics and propulsion efficiency [3]. Efficient battery-saving algorithms are needed, especially for demanding scenarios such as off-shore monitoring [4]. This paper addresses efficiency by improving the estimation performance under the AUV's energy constraints, deriving the exponential convergence of the estimation error, and finding the motion characteristics that guarantee convergence.

Adaptive estimation for environmental monitoring has been a thoroughly studied topic over the past decades [5], yet, it is still fairly novel regarding its deployment in underwater surveys as its open-loop counterpart is generally adopted [6]. Source localization using pseudo gradient-based (Chemotaxis) method for a single agent is proposed in [7]. By maximiz ing the determinant of Fisher information matrix, a proper choice of the marine vehicle's motion is given for source localization in [8]. Multi-agent systems are sometimes essential for collecting measurements in larger or more complex environments. The work in [9] provides an effective estimation scheme for multi-agents systems, which only uses the on-board sensors to solve the distributed estimation problem of unknown field or sensor networks. Still, most of these multi-agent systems also rely on adaptively estimating the unknown field. In [10], AUVs use local interpolation rules to estimate the scalar field in their regions. The distributed Kriged Kalman filter of [11] uses average consensus estimators and distributed Jacobi over-relaxation to estimate the scalar field. In [12], the environmental field is modeled as a Gaussian process (GP), which is learned by maximization of the log marginal likelihood from the accumulated measurements gathered by the multi-AUV system. However, the absence of GPS signals and acoustic communication underwater poses great challenges for the cooperative scheme of multiple AUVs. Therefore, a limited number of mobile sensors are considered in [13] to reduce memory and communication requirements when environmental monitoring over large areas.

We design a systematic framework for the AUV to realize the environmental monitoring of a region while considering efficiency. An adaptive estimator is proposed to estimate the uniform gradient using the scalar measurement and position in the local coordinate frame. Compared with the online parameter estimation problem in adaptive estimation [14], [15] where the persistent excitation condition is often required on the regressor function in the real space, the adaptive estimator proposed in this paper is defined in the complex plane, and thus the persistent excitation condition is not directly established on the measuring position, but on one of the features of the measuring trajectory, the angular velocity. By decomposing the velocity of the AUV into a time-varying orthogonal basis, a second-order linear time-varying system in real space is equivalently constructed from the dynamics of the

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Ziwen Yang, Shanying Zhu, Cailian Chen, and Xinping Guan are with the Department of Automation, Shanghai Jiao Tong University; Key Laboratory of System Control and Information Processing, Ministry of Education of China, and also Shanghai Engineering Research Center of Intelligent Control and Management, Shanghai 200240, China. E-mail:{1106385445,shyzhu,cailianchen,xpguan}@sjtu.edu.cn

Ziwen Yang, Joana Fonseca, and Karl H. Johansson are with Division of Decision and Control Systems, School of Electrical Engineering and Computer Science, KTH Royal Institute of Technology, Stockholm, Sweden, SE-100 44 Stockholm, Sweden. They are also affiliated with Digital Futures. E-mail: {ziwyan,jfgf,kallej}@kth.se

estimation error in the complex plane. Sufficient conditions are established on the distance and angular velocity with respect to the origin of the local frame. It is rigorously proved that a persistently exciting angular velocity guarantees the global exponential stability of the estimation error system. Furthermore, an explicit expression is given to estimate the convergence rate of the estimation error. By analyzing its relationship with the features of the measuring motion, estimation efficiency is considered under the constrained energy of the mobility. It is proved that the maximum convergence rate estimate can be reached, given its excitation level, when the angular velocity is constant. Accordingly, a reference trajectory is designed to guarantee exponential stability of the estimation error system and enable efficient environment monitoring by the AUV as it tracks the reference trajectory.

The rest of the paper is organized as follows. In Section 2, the environmental monitoring problem is formulated. A framework for the AUV to solve the previously stated problem is given in Section 3. Section 4 demonstrates the stability and efficiency of the system. Numerical examples follow in Section 5, and conclusions in Section 6.

Notations: ι is the unit imaginary number with $\iota^2 = -1$. The principal argument angle and magnitude of c are denoted as $\angle c$ and $|c|$, respectively.

2. Problem Formulation

A. Kinematic and Dynamic Model of the Single Agent

Define the local coordinate frame $\{n\}$ as shown in Fig. 1, whereas its origin is defined as the initial position of the AUV, the x_n-axis points towards Due North, and the y_n-axis points towards Positive East. Note that the directions of axes are the same as the familiar North-East coordinate system. Denote the position in the local frame $\{n\}$ as $p \in \mathbb{C}$. The surge-yaw kinematic model of the AUV can be described as

$$\begin{cases} \dot{p} = \nu \mathrm{e}^{\iota\psi}, \\ \dot{\psi} = w, \end{cases} \tag{1}$$

where $\psi \in \mathbb{R}$ is the yaw angle, $\nu \in \mathbb{R}$ is the surge velocity and $w \in \mathbb{R}$ is the yaw velocity. The dynamic model is described as follows, which is based on the classic model of INFANTE AUV [16]

$$\begin{cases} m_\nu \dot{\nu} + d_\nu = F, \\ m_w \dot{w} + d_w = \Gamma, \end{cases} \tag{2}$$

where $m_\nu = m - X_{\dot{\nu}}$, $d_\nu = -X_\nu \nu^2 w$, $m_w = I_z - N_{\dot{w}}$, $d_w = -N_w \nu w$. The variables m, I_z, $X_{\{.\}}$, and $N_{\{.\}}$ denote the mass, moment of inertia, and the classical hydrodynamic derivatives of the AUV. (F, Γ) defines the control input vector of force and torque to be designed.

B. Environmental Model and Sensor Model

The environment model is a scalar field defined in the local frame $\{n\}$. The scalar field is a mapping function denoted as $f(p) : \mathbb{C} \to \mathbb{R}$, which is given by

$$f(p) = \langle p, h \rangle + \Sigma, \tag{3}$$

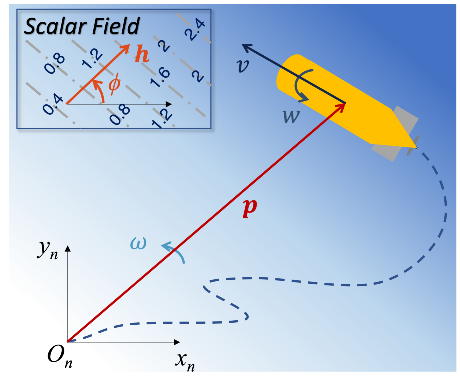

Fig. 1. Illustration of the scalar field monitored and estimated by an AUV.

where $h \in \mathbb{C}$ is the gradient, and $\Sigma \in \mathbb{R}$ is the offset at the origin of $\{n\}$. For the environmental scalars, which are linearly distributed over the region of interest as shown in Fig. 1, the gradient h has the form as

$$h = \nabla f(p) = \hbar \mathrm{e}^{\iota\phi}, \tag{4}$$

where $\hbar \in \mathbb{R}^+$ and $\phi \in \mathbb{R}$ are the magnitude and the argument of the gradient h, respectively. h is constant and independent of the position p. Denote $\varsigma \in \mathbb{R}$ as the measurement at time t, then the measuring model of the AUV is

$$\varsigma = f(p(t)). \tag{5}$$

Since the initial position in the local frame is known to the AUV as $p(0) = 0$, denote $\varsigma_0 = f(p(0))$ as the scalar measurement at time $t = 0$, then it follows from (3) and (5) that $\Sigma = \varsigma_0$, which is thus known to the AUV.

C. Environmental Monitoring and Estimation Convergence Rate

The environmental monitoring problem for an AUV is to estimate h and Σ. Since Σ is available by recording the scalar measurement at the time $t = 0$, the problem is thus reduced to merely estimating the gradient h.

Problem 1 (Environmental monitoring Problem)**.** *For an AUV with kinematic model* (1)*, dynamic model* (2)*, and measuring model* (5)*, design a gradient estimator $\hat{h}$ and a control law (F, Γ), satisfying that there exist positive constants $\mu, \lambda_T \in \mathbb{R}^+$ such that for any $t_0 \geq 0$, $t \geq t_0$,*

$$|\epsilon(t)| \leq \mu |\epsilon(t_0)| \mathrm{e}^{-\lambda_T (t - t_0)}, \tag{6}$$

where $\epsilon = \hat{h} - h$ is the estimation error. That is, ϵ converges to the origin globally exponentially fast.

The convergence rate, λ_T, is largely determined by the properties of the AUV's motion. Increased mobility can result in improved observability and estimation performance, but it also incurs high consumption costs and battery capacity requirements that may limit available mobility resources. Consequently, we aim to investigate the efficiency of environmental monitoring by addressing the question: *With a given amount*

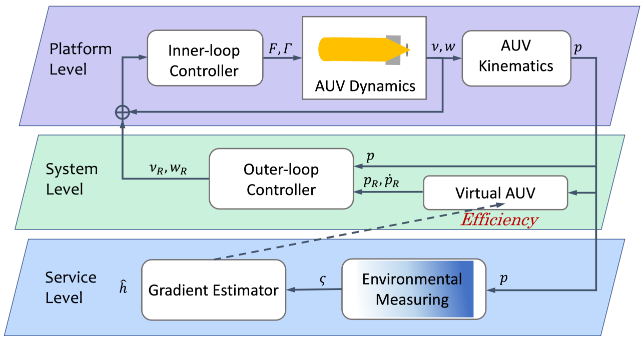

Fig. 2. Proposed framework with platform, system, and service levels.

of resources, what motion strategy for the AUV will enable the highest estimation convergence rate?

3. Efficient Environmental Monitoring

The AUV has different roles when solving the environmental monitoring problem. It is both a data collector and an observer for estimating the scalar field, as well as a controlled object for motion control. Additionally, the motion of the AUV not only determines the performance of gradient estimation but also is driven by the controller. To divide the roles of the AUV, a multi-level systematic framework is proposed, as illustrated in Fig. 2. This framework defines different tasks for the AUV at different levels. Specifically, it consists of three layers, i.e., the platform level, the system level, and the service level.

At the *service level*, the gradient estimator is proposed. The stability of the estimation error system is analyzed. Estimation performance is determined by the trajectory p of the AUV, along which the environmental scalar ς is collected.

At the *system level*, since the controller also drives p, we introduce the reference trajectory denoted as p_R to decouple the desired measuring motion, enabling efficient estimation from the actual motion. A virtual AUV with the same kinematic model as (1) is introduced, for which a reference trajectory is designed in the frame $\{n\}$. The design criteria for p_R takes into account the AUV's feasibility to track and estimate efficiently. The outer-loop controller of the actual AUV generates the desired input (ν_R, w_R) to track the reference trajectory.

At the *platform level*, the inner-loop controller is designed for the dynamics of the AUV, and the desired input (ν_R, w_R) is tracked by designing the control law $(F, \varGamma)$. An AUV's reference trajectory tracking control problem is well investigated, e.g., [17], and the following proposition is given.

Proposition 3.1. *Suppose that the reference trajectory p_R is sufficiently smooth. Then there exists trajectory tracking control law $(F, \varGamma)$, such that*

$$\begin{aligned} \lim_{t\to\infty} p(t) &= p_R(t), \\ \lim_{t\to\infty} \dot{p}(t) &= v_R(t), \end{aligned} \tag{7}$$

where p_R is generated by a system $\dot{p}_R = v_R$, where v_R is the designed reference velocity.

Proof. The proof is similar to [17], and thus is omitted. □

Based on the proposed framework and Proposition 3.1, efficient estimation can be achieved as time goes to infinity, by characterizing the reference trajectory p_R with v_R, then designing the control law for the AUV to realize (7). In this paper, we mainly focus on the service and system levels, where an online gradient estimator is designed, and the reference trajectory is characterized to realize efficient estimation.

4. Main Results

We first propose an adaptive estimator in the complex plane to solve the environmental monitoring problem. Subsequently, efficiency is considered based on the stability analysis of the estimation error system.

A. Gradient Estimator

At the service level, we first propose the online estimator for the AUV to accomplish its environmental monitoring task. Using the scalar measurement ς, the gradient estimator $\hat{h}$ is proposed as follows:

$$\dot{\hat{h}} = -k_0(\langle p, \hat{h}\rangle - \varsigma + \varsigma_0)p, \tag{8}$$

where the positive constant $k_0 \in \mathbb{R}^+$ is the estimator gain. Since h is constant, combining (3), (5) with (8) yields the dynamics of the estimation error ϵ as

$$\dot{\epsilon} = \dot{\hat{h}} - \dot{h} = -k_0\langle p, \hat{h} - h\rangle p = -k_0\langle p, \epsilon\rangle p. \tag{9}$$

Note that (9) has a similar form with $n-$dimensional linear regression model in adaptive estimation [18], where the regressor function is defined as $p : \mathbb{R}^+ \to \mathbb{R}^n$. In this case, it is well known that the estimation error dynamics are globally exponentially stable under the persistently exciting (PE) condition on the matrix $pp^\top \in \mathbb{R}^{n\times n}$. However, p is defined in the complex plane in (9), and it is difficult to find the PE condition directly on $p \in \mathbb{C}$. In addition, p is governed by the kinematic model (1) and the dynamic model (2), which is quite different from the regressor function in [18]. It is necessary to analyze the physical meaning of p as a measuring trajectory, to design a reference trajectory that satisfies the PE condition.

To facilitate the analysis of mobility and limited resources, we give the physical meaning of variables in the motion. Denote the distance with respect to the origin of the frame $\{n\}$ as $\rho(t) = |p(t)|$, and the unit-length direction as $\varrho(t) = \frac{p(t)}{\rho(t)}$. Then the estimation error ϵ can be equivalently given by $\epsilon = \eta\varrho + \zeta\iota\varrho$, where $\eta = \langle \epsilon, \varrho\rangle$ and $\zeta = \langle \epsilon, \iota\varrho\rangle$. By definition, $\dot{\varrho} = \frac{v\rho - \dot{\rho}p}{\rho^2}$, where the velocity of the AUV in the frame $\{n\}$ is denoted as $v = \dot{p} \in \mathbb{C}$. Since $\rho = \sqrt{\langle p, p\rangle}$, one has $\dot{\rho} = \frac{2\langle v, p\rangle}{2\sqrt{\langle p, p\rangle}} = \frac{\langle v, p\rangle}{\rho}$. Substituting it into $\dot{\varrho}$ and combining the orthogonal decomposition[1] $v = \langle v, \varrho\rangle\varrho + \iota\langle v, \iota\varrho\rangle\varrho$ yields

$$\dot{\varrho} = \frac{v\rho - \langle v, p\rangle\varrho}{\rho^2} = \frac{(\langle v, \varrho\rangle + \iota\langle v, \iota\varrho\rangle)\varrho\rho - \langle v, \rho\varrho\rangle\varrho}{\rho^2} = \iota\omega\varrho,$$

[1] $\langle a, b\rangle\, b + \iota\,\langle a, \iota b\rangle\, b = (\mathrm{Re}(a/b) + \iota\mathrm{Re}(-\iota a/b))b = (\mathrm{Re}(a/b) + \iota\mathrm{Im}(a/b))b = (a/b)b = a.$

where $\omega(t) = \frac{\langle v(t), \iota\varrho(t)\rangle}{\rho(t)}$ denotes the angular velocity of the AUV with respect to the origin of the frame $\{n\}$. Then the dynamics of η and ξ can be expressed by (9) as

$$\begin{aligned}\dot{\eta} &= -k_0\rho^2\langle\varrho,\epsilon\rangle\langle\varrho,\varrho\rangle + \langle\epsilon,\iota\omega\varrho\rangle = -k_0\rho^2\eta + \omega\zeta,\\ \dot{\zeta} &= \langle\epsilon,\iota^2\omega\varrho\rangle = -\omega\eta.\end{aligned} \tag{10}$$

Stack η, ζ into a vector $X = [\eta\ \zeta]^\top \in \mathbb{R}^2$. (10) can be reorganized in the Linear Time Varying (LTV) system, denoted by (C, A), as

$$\begin{aligned}\dot{X}(t) &= A(t)X(t),\\ z(t) &= C^\top X(t),\end{aligned} \tag{11}$$

where $z = \eta = \langle\epsilon, \frac{p}{\rho}\rangle = \frac{\langle\hat{h}-h,p\rangle}{\rho} = \frac{\langle p,\hat{h}\rangle-\varsigma+\varsigma_0}{\rho}$ is the output of the system since it is measurable, and

$$A(t) = \begin{bmatrix}-k_0\rho^2(t) & \omega(t)\\ -\omega(t) & 0\end{bmatrix}, \quad C = [1\ 0]^\top.$$

Before analyzing the stability of the LTV system (11), we introduce the following lemma.

Lemma 4.1 ([19], Lemma 4.8.1)**.** *Assume that there exist positive constants $T, w_T \in \mathbb{R}^+$ such that $W(t) \in \mathbb{R}^2$ satisfies the inequality*

$$\int_t^{t+T} |W(\tau)|^2 d\tau \le w_T, \quad \forall t \ge 0. \tag{12}$$

If there exist positive constants $\beta_1, \beta_2 \in \mathbb{R}^+$ such that the observability gramian $\tilde{N}(t, t+T)$ of the system $(C, \tilde{A})$ where $\tilde{A} \in \mathbb{R}^{2\times 2}, C \in \mathbb{R}^2$ satisfies $\beta_1 I \le \tilde{N}(t, t+T) \le \beta_2 I$, then there exist positive constants $\beta_1', \beta_2' \in \mathbb{R}^+$ where

$$\beta_1' = \frac{\beta_1}{2(1+\beta_2 w_T)}, \tag{13}$$

such that the observability gramian $N(t, t+T)$ of (C, A) where $A = \tilde{A} + WC^\top$ satisfies $\beta_1' I \le N(t, t+T) \le \beta_2' I$.

Denote the time derivative of ω as $\dot{\omega} = a_\omega$, then we have the following stability results on the gradient estimation error.

Theorem 4.1. *Suppose that ρ is time invariant, and there exist positive constants $d, \bar{d} \in \mathbb{R}^+$ such that*

$$d \le \rho \le \bar{d}, \quad \forall t \ge 0. \tag{14}$$

Besides, the following conditions on ω hold:

C1) *ω is sufficiently smooth, and there exist constants $\bar{a}_\omega, \bar{\omega} \in \mathbb{R}$ where $\bar{a}_\omega \ge 0, \bar{\omega} > 0$, such that $|a_\omega(t)| \le \bar{a}_\omega$, $|\omega(t)| \le \bar{\omega}$, $\forall t \ge 0$;*

C2) *and ω is persistently exciting (PE), i.e., there exist positive constants $\nu, \beta \in \mathbb{R}^+$ such that*

$$\frac{1}{\nu}\int_t^{t+\nu} |\omega(\tau)|^2 d\tau \ge \beta, \forall t \ge 0. \tag{15}$$

Let $k_0 > \frac{4\bar{a}_\omega \bar{d}^2}{d^4\sqrt{\beta}}$ for the estimator (8)*. Then the estimation error ϵ converges to the origin globally exponentially fast.*

Proof. The stability of the system (9) is equivalent to that of the LTV system (11), where $A(t)$ is always a stable matrix for any ω since $k_0 \in \mathbb{R}^+$ and C1) holds. For simplicity, we drop t for the constant ρ. Then there is $A(t) + A^\top(t) = -2k_0\rho^2 CC^\top$. Defining a Lyapunov function as $V(t) = X^\top X = \|X\|^2$, then its time derivative is $\dot{V}(t) = -2k_0\rho^2 X(t)^\top CC^\top X(t)$. Note that $V(t+T) - V(t) = \int_t^{t+T}\dot{V}(\tau)d\tau$, and one can derive by (14) that for any $t \ge 0$ and $T \in \mathbb{R}^+$,

$$V(t+T) - V(t) \le -2aX^\top(t)N(t, t+T)X(t), \tag{16}$$

where $a = k_0 d^2$, and $N(t, t+T)$ is the observability gramian of the system (C, A).

For (16), we next prove the existence of the positive constant β_1' such that $N(t, t+T) \ge \beta_1' I$. Motivated by Lemma 4.1, we consider an auxiliary system as follows:

$$\begin{aligned}\dot{\tilde{\eta}} &= -k_0\rho^2\tilde{\eta} + \omega\tilde{\zeta},\\ \dot{\tilde{\zeta}} &= 0,\end{aligned} \tag{17}$$

with the state matrix $\tilde{A}(t) = \begin{bmatrix}-k_0\rho^2 & \omega\\ 0 & 0\end{bmatrix}$. By letting $W = [0\ -\omega]^\top$, one has $A = \tilde{A} + WC^\top$. Since C1) holds, (12) is satisfied for W with $w_T = \bar{\omega}^2 T$. Then by (13) in Lemma 4.1, β_1' for $N(t, t+T)$ of the system (C, A) can be obtained by finding β_1 and β_2 for $\tilde{N}(t, t+T)$ of the system $(C, \tilde{A})$.

Considering (17), $\tilde{\zeta}^2$ is constant since $\dot{\tilde{\zeta}}$ equals to zero. Then finding β_1 and β_2 for $\tilde{N}(t, t+T)$ is equivalent to establishing the following inequality

$$\beta_1(\tilde{\eta}^2(t) + \tilde{\zeta}^2) \le \int_t^{t+T}\tilde{\eta}^2(\tau)d\tau \le \beta_2(\tilde{\eta}^2(t) + \tilde{\zeta}^2), \tag{18}$$

where $\tilde{\eta}(\tau) = \Phi(\tau, t)\tilde{\eta}(t) + \int_t^\tau \Phi(\tau,\sigma)\omega(\sigma)d\sigma\tilde{\zeta}$, with $\Phi(\tau, t) = \mathrm{e}^{-k_0\rho^2(t-\tau)}$ being the state-transition function. Denote $x(\tau) = \Phi(\tau, t)\tilde{\eta}(t)$ and $y(\tau) = \int_t^\tau \Phi(\tau,\sigma)\omega(\sigma)d\sigma\tilde{\zeta}$. Then $\int_t^{t+T}\tilde{\eta}^2(\tau)d\tau \ge \int_t^{t+T'}(\frac{x^2(\tau)}{2} - y^2(\tau))d\tau + \int_{t+T'}^{t+T}(\frac{y^2(\tau)}{2} - x^2(\tau))d\tau$ holds[2] for $T' \in \mathbb{R}^+$ where $0 < T' < T$. In this way, we can separately analyze x and y.

For x, we have $\mathrm{e}^{-k_0\bar{d}^2(\tau-t)} \le \Phi(\tau, t) \le \mathrm{e}^{-a(\tau-t)}$for any $\tau \ge t$. Since $\bar{d} \ge d$, it yields that $\int_t^{t+T'}\frac{x^2(\tau)}{2}d\tau - \int_{t+T'}^{t+T}x^2(\tau)d\tau \ge (1 - \mathrm{e}^{-2k_0\bar{d}^2T'} - \frac{2\bar{d}^2}{d^2}(\mathrm{e}^{-2aT'} - \mathrm{e}^{-2aT}))\frac{\tilde{\eta}^2(t)}{4k_0\bar{d}^2} \ge (1 - (1 + \frac{2\bar{d}^2}{d^2})\mathrm{e}^{-2aT'})\frac{\tilde{\eta}^2(t)}{4k_0\bar{d}^2}$. Choose

$$T' = \ln(2(1 + 2\bar{d}^2/d^2))/2a, \tag{19}$$

then for $\beta_1 = \frac{1}{8k_0\bar{d}^2}$, there is

$$\int_t^{t+T'}\frac{x^2(\tau)}{2}d\tau - \int_{t+T'}^{t+T}x^2(\tau)d\tau \ge \beta_1\tilde{\eta}^2(t). \tag{20}$$

For y, we first prove the upper bound of $\int_t^{t+T'}y(\tau)d\tau$. Taking Laplace transform of $y(\tau) = \int_t^\tau \Phi(\tau,\sigma)\omega(\sigma)d\sigma\tilde{\zeta}$ yields $y(\tau) = \frac{\omega(s)}{s+k_0\rho^2}\tilde{\zeta}$, then $y(\tau) = y_{[0,t)}(\tau) + y_{[t,t+T']}(\tau)$, for any $0 \le \tau \le t + T'$. Thus

$$\int_t^{t+T'}y^2(\tau)d\tau \le 2\int_t^{t+T'}y_1^2(\tau)d\tau + 2\int_t^{t+T'}y_2^2(\tau)d\tau, \tag{21}$$

[2] $(x+y)^2 + y^2 - \frac{x^2}{2} = \frac{1}{2}(x+2y)^2 \ge 0.$

where $y_1 := \frac{\omega_{[0,t)}}{s+a}\tilde{\zeta}$ and $y_2 := \frac{\omega_{[t,t+T')}}{s+a}\tilde{\zeta}$. Following the same procedure in the proof of Lemma 4.8.2 in [19], one can derive that $\int_t^{t+T'} y_1^2(\tau)d\tau \le \frac{\bar{\omega}^2}{2a^3}\tilde{\zeta}^2$, and for the second term on the right-hand side of (21), one can also yield by that $\int_t^{t+T'} y_2^2(\tau)d\tau \le \frac{\int_t^{t+T'}\bar{\omega}^2 d\tau}{\|s+a\|_\infty^2}\tilde{\zeta}^2 \le \frac{4\bar{\omega}^2}{a^2}T'\tilde{\zeta}^2$, since $\|\frac{1}{s+a}\|_\infty^2 \le \frac{4}{a^2}$ as proved in Lemma A.2 in [19] and C1) holds for ω. Put these into (21), then

$$\int_t^{t+T'} y^2(\tau)d\tau \le \frac{\bar{\omega}^2}{a^2}(\frac{1}{a}+8T')\tilde{\zeta}^2. \quad (22)$$

Next we prove the lower bound of $\int_{t+T'}^{t+T} y^2(\tau)\rho\tau$. Let $\hat{y}(\tau)$ as $\hat{y}(\tau) = \frac{\omega}{k_0\rho^2}(1-\frac{s}{s+k_0\rho^2})\tilde{\zeta} = \frac{\omega}{k_0\rho^2}\tilde{\zeta} - \frac{\dot{\omega}}{k_0\rho^2(s+k_0\rho^2)}\tilde{\zeta}$. Following the same procedure to derive (22), we have $\int_{t+T'}^{t+T}\frac{\dot{\omega}^2}{k_0^2 d^4(s+k_0\rho^2)^2}d\tau \le \frac{\bar{a}_\omega^2}{(a^2)^2}(\frac{1}{a}+8(T-T'))$, since C1) holds for a_ω. By (15), one obtains by (15) that

$$\begin{aligned}\int_{t+T'}^{t+T} y^2(\tau)d\tau &\ge \frac{\int_{t+T'}^{t+T}\omega^2 d\tau}{2k_0^2\rho^4}\tilde{\zeta}^2 - \int_{t+T'}^{t+T}\frac{\dot{\omega}^2\tilde{\zeta}^2}{k_0^2\rho^4(s+k_0\rho^2)^2}d\tau,\\ &\ge \frac{n\beta\nu}{2k_0^2\bar{d}^4}\tilde{\zeta}^2 - \frac{\bar{a}_\omega^2}{a^4}(\frac{1}{a}+8(T-T'))\tilde{\zeta}^2,\end{aligned}$$

where n is the integer satisfying $\frac{T-T'}{\nu}-1 \le n \le \frac{T-T'}{\nu}$. Thus $\int_{t+T'}^{t+T}\frac{y^2(\tau)}{2}d\tau - \int_t^{t+T'} y^2(\tau)d\tau \ge (b-\frac{4\bar{a}_\omega^2}{a^4})(T-T')\tilde{\zeta}^2 - \frac{8\bar{\omega}^2}{a^2}T'\tilde{\zeta}^2 - (b\nu+\frac{1}{a^3}(\frac{\bar{a}_\omega^2}{2a^2}+\bar{\omega}^2))\tilde{\zeta}^2$, where $b = \frac{\beta}{4k_0^2\bar{d}^4}$. Denote $\alpha_1 = b-\frac{4\bar{a}_\omega^2}{a^4}$, $\alpha_2 = b\nu+\frac{1}{a^3}(\frac{\bar{a}_\omega^2}{2a^2}+\bar{\omega}^2)$, and since $k_0 > \frac{4\bar{a}_\omega\bar{d}^2}{d^4\sqrt{\beta}}$, one has $\alpha_1,\alpha_2 > 0$. Let T be

$$T = \frac{\beta_1}{\alpha_1} + (1+\frac{8\bar{\omega}^2}{a^2\alpha_1})T' + \frac{\alpha_2}{\alpha_1}, \quad (23)$$

and it yields that

$$\int_{t+T'}^{t+T}\frac{y^2(\tau)}{2}d\tau - \int_t^{t+T'} y^2(\tau)d\tau \ge \beta_1\tilde{\zeta}^2. \quad (24)$$

For the upper bound β_2 on $N(t,t+T)$, following the same procedure of deriving (22) obtains that

$$\int_t^{t+T}\tilde{\eta}^2(\tau)d\tau \le \frac{1-\mathrm{e}^{-2aT}}{a}\tilde{\eta}^2(t) + \frac{2\bar{\omega}^2}{a^2}(\frac{1}{a}+8T)\tilde{\zeta}^2. \quad (25)$$

Combining (20), (24), and (25), yields (18), where $\beta_1 = \frac{1}{8k_0\bar{d}^2}$ and $\beta_2 = \max\{\frac{1-\mathrm{e}^{-2aT}}{a}, \frac{2\bar{\omega}^2}{a^2}(\frac{1}{a}+8T)\}$. Equivalently, we have $\beta_1 I \le \tilde{N}(t,t+T) \le \beta_2 I$ for the system $(C,\tilde{A})$, and by Lemma 4.1, $N(t,t+T)$ of the system (C,A) satisfies $N(t,t+T) \ge \beta_1' I$, where β_1' is given by (13).

With the derived β_1', it follows from (16) that for any $t \ge 0$ and $T \in \mathbb{R}^+$,

$$V(t+T) - V(t) \le -\frac{a\beta_1}{(1+\beta_2\bar{\omega}^2 T)}X^\top(t)X(t),$$

and $0 < \frac{a\beta_1}{1+\beta_2\bar{\omega}^2 T} = \frac{d^2}{8\bar{d}^2(1+\beta_2\bar{\omega}^2 T)} < \frac{1}{8} < 1$. Denote

$$\lambda = \frac{d^2}{8\bar{d}^2(1+\beta_2\bar{\omega}^2 T)}, \quad (26)$$

and following the proof of Theorem 8.5 in [20] there exists $\lambda_T \in \mathbb{R}^+$ satisfying

$$\lambda_T = \frac{1}{2T}\ln\frac{1}{1-\lambda}, \quad (27)$$

such that $V(X(t)) \le \frac{1}{1-\lambda}\mathrm{e}^{-2\lambda_T(t-t_0)}V(t_0,X(t_0))$ for any $t \ge t_0$. Thus the estimation error ϵ converges to the origin globally exponentially fast □

Note that λ_T given by (27) is an estimate of the convergence rate. Besides, characteristics of the measuring motion are the distance ρ and the rotating angular velocity ω, on which the sufficient conditions (14), C1) and C2) guarantee the exponential convergence of the estimation error ϵ. The trajectory of the AUV satisfying (14), C1) and C2) can be circular and quasi-circular (e.g., elliptical, wavy), etc. A natural question arises: *what kind of trajectory p satisfying* (14)*, C1), and C2), is efficient for the AUV to estimate the gradient h?*

B. Efficient Estimation with Outer-Loop Controller

At the system level, considering the implementation constraints of the AUV, it is desired that good services including efficiency can be provided. That is, the best estimation performance, in terms of the maximum convergence rate of the estimation error, can be achieved under the constraint of the AUV motion energy. By Proposition 3.1, it is necessary to design a reference trajectory that is sufficiently smooth to guarantee the existence of the outer-loop controller. Moreover, the reference trajectory should enable the maximum of λ_T under the energy constraint. By (27), λ_T relates to the motion features: the distance ρ and the angular velocity ω. Since ρ and ω can be independently controlled to satisfy (14) and C1)-C2) respectively, the constrained resources can be merely imposed on one of them. Out of this observation, we consider the constraint on ω, since (15) in C2) puts forward a lower bound requirement, which means that at least a certain amount of energy needs to be provided.

An example is given to show the limited energy of mobility by posing a constraint on ω as below. A measuring motion has the dynamics as $\dot{p} = \iota\omega_0 p$, with $\omega_0 : \mathbb{R}^+ \to \mathbb{R}$ being designed, and let $|p(0)| > 0$. In this case, $\dot{\rho} = \langle\dot{p},\varrho\rangle = \omega_0\langle\iota p,\varrho\rangle = 0$, such that $\rho = \rho(0)$ holds for all $t \ge 0$ and thus (14) holds by letting $d = \bar{d} = |p(0)|$. $\omega = \frac{\langle\iota\omega_0 p,\iota p\rangle}{\rho^2} = \omega_0$ holds for all $t \ge 0$. Then p is a variable with the fixed magnitude and the rotating angular velocity ω_0. By Theorem 4.1, ω_0 is designed to satisfy C1) and C2) to essentially guarantee the exponential convergence of the estimation error. On the other hand, to maintain such a motion of the AUV, it follows from (1) that $\nu\mathrm{e}^{\iota\psi} = \dot{p} = \iota\omega_0 p$, such that $\nu = \omega_0\rho(0)$, $\dot{\nu} = \dot{\omega}_0\rho(0)$ and $w = \dot{\psi} = \omega = \omega_0$. Substituting them into (2) and letting $\bar{F}(t) := \int_t^{t+T}|F(\tau)|d\tau$, $\bar{\Gamma}(t) := \int_t^{t+T}|\Gamma(\tau)|d\tau$, the corresponding force and torque satisfy that

$$\begin{cases}\bar{F}(t) \ge X_1\int_t^{t+T}|\omega_0(\tau)|^3 d\tau - X_2\int_t^{t+T}|\dot{\omega}_0(\tau)|d\tau,\\ \bar{\Gamma}(t) \ge N_1\int_t^{t+T}|\omega_0(\tau)|^2 d\tau - N_2\int_t^{t+T}|\dot{\omega}_0(\tau)|d\tau,\end{cases}$$

where $X_1 = |X_\nu|\rho^2(0)$, $X_2 = |m_\nu|\rho(0)$, $N_1 = |N_w|\rho(0)$ and $N_2 = |m_w|$ are constants. It implies that for a given $\bar{a}_\omega = \sup(|\dot{\omega}_0|)$, the lower bound on energy for the control input over the period T is decided by the lower bound β in the inequality (15) for ω_0. When β is larger, a larger amount of energy is required by the mobility of the AUV, which is indicated by $\bar{F}(t)$ and $\bar{\Gamma}(t)$.

Therefore, more than basically satisfying (15) in C2), we consider the energy constraint as the following equation of ω:

$$\frac{1}{\nu}\int_t^{t+\nu} |\omega(\tau)|^2 d\tau = \beta, \forall t \geq 0. \tag{28}$$

for some $\nu \in \mathbb{R}^+$, where $\beta \in \mathbb{R}^+$ is the so-called excitation level. (28) is the boundary of the feasible region of ω defined by (15). Note that β is given as a predefined positive value indicating a fixed amount of the energy afforded to the AUV to achieve successful estimation. Thus the efficiency of estimation can be derived by maximizing λ_T under the constraint (28), then we have the following result.

Theorem 4.2. *For the proposed estimator* (8)*, suppose that there exists a control law* (F, Γ) *such that* p *satisfies the following conditions* $\forall t \geq 0$*:*

1) *p is sufficiently smooth;*
2) (14) *holds for a constant ρ;*
3) *C1) in Theorem 4.1 holds for ω;*
4) *and* (28) *holds for ω with a given $\beta \in \mathbb{R}^+$.*

Given a fixed $k_0 > \frac{4\bar{a}_\omega \bar{d}^2}{d^4\sqrt{\beta}}$. *Then* λ_T *in* (27) *is maximized if and only if ω is constant.*

Proof. Since the sufficient conditions in Theorem 4.1 are also assumed to hold, the estimation error converges to the origin exponentially fast, and (27) holds. Moreover, the maximum of λ_T is found by minimizing T given by (23) while maximizing λ given by (26) simultaneously. Since d and $\bar{d}$ in (14) are given and β, k_0 is fixed, a, b, β_1 and T' by (19) are also fixed. The rest of the parameters in (23) and (26) are all related to ω, i.e., the maximum $\bar{a}_\omega$ and $\bar{\omega}$ where $|a_\omega(t)| \leq \bar{a}_\omega$, $|\omega(t)| \leq \bar{\omega}$, $\forall t \geq 0$. To see the relationship between λ_T and $\bar{a}_\omega$ as well as $\bar{\omega}$, recall that $0 < \lambda < 1$, then it follows from (27) that

$$\begin{cases} \partial\lambda_T/\partial T = \ln(1-\lambda)/(2T^2) < 0, \\ \partial\lambda_T/\partial\lambda = 1/(2T(1-\lambda)) > 0, \end{cases} \tag{29}$$

which implies that λ_T is negative and positive correlated with respect to T and λ, respectively. It then yields by (23) that

$$\begin{cases} \dfrac{\partial T}{\partial \bar{a}_\omega} = \dfrac{\partial T}{\partial \alpha_1}\dfrac{\partial \alpha_1}{\partial \bar{a}_\omega} + \dfrac{\partial T}{\partial \alpha_2}\dfrac{\partial \alpha_2}{\partial \bar{a}_\omega} = -\dfrac{\gamma}{\alpha_1^2 a^2}\dfrac{\partial \alpha_1}{\partial \bar{a}_\omega} + \dfrac{1}{\alpha_1}\dfrac{\partial \alpha_2}{\partial \bar{a}_\omega}, \\ \dfrac{\partial T}{\partial \bar{\omega}} = \dfrac{16T'\bar{\omega}}{a^2\alpha_1} + \dfrac{\partial T}{\partial \alpha_2}\dfrac{\partial \alpha_2}{\partial \bar{\omega}} = \dfrac{16T'\bar{\omega}}{a^2\alpha_1} + \dfrac{1}{\alpha_1}\dfrac{\partial \alpha_2}{\partial \bar{\omega}}, \end{cases}$$

where $\gamma = (\beta_1 + \alpha_2)a^2 + 8\bar{\omega}^2 T' > 0$. By the expressions of α_1 and α_2, we have $\frac{\partial \alpha_1}{\partial \bar{a}_\omega} = -\frac{8\bar{a}_\omega}{a^4} \leq 0$, $\frac{\partial \alpha_2}{\partial \bar{a}_\omega} = \frac{\bar{a}_\omega}{a^5} \geq 0$, and $\frac{\partial \alpha_2}{\partial \bar{\omega}} = \frac{2\bar{\omega}}{a^3} > 0$. Substitute the inequalities into the above equations, one has $\frac{\partial T}{\partial \bar{a}_\omega} \geq 0$ and $\frac{\partial T}{\partial \bar{\omega}} > 0$.

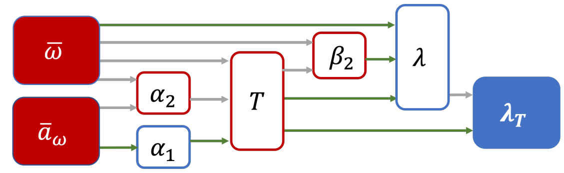

Fig. 3. λ_T is monotonically negative correlated to $\bar{\omega}$ and $\bar{a}_\omega$.

On the other hand, one obtains from (26) that

$$\begin{cases} \dfrac{\partial \lambda}{\partial \bar{a}_\omega} = \left(\dfrac{\partial \lambda}{\partial \beta_2}\dfrac{\partial \beta_2}{\partial T} + \dfrac{\partial \lambda}{\partial T}\right)\dfrac{\partial T}{\partial \bar{a}_\omega}, \\ \dfrac{\partial \lambda}{\partial \bar{\omega}} = \dfrac{\partial \lambda}{\partial \beta_2}\dfrac{\partial \beta_2}{\partial \bar{\omega}} + \dfrac{\partial \lambda}{\partial T}\dfrac{\partial T}{\partial \bar{\omega}} - \dfrac{d^2\beta_2\bar{\omega}T}{4\bar{d}^2(1+\beta_2\bar{\omega}^2 T)^2}, \end{cases}$$

where $\frac{\partial \lambda}{\partial \beta_2} = -\frac{d^2\bar{\omega}^2 T}{8\bar{d}^2(1+\beta_2\bar{\omega}^2T)^2} < 0$, $\frac{\partial \lambda}{\partial T} = -\frac{d^2\beta_2\bar{\omega}^2}{8\bar{d}^2(1+\beta_2\bar{\omega}^2T)^2} < 0$. When $\frac{1-\mathrm{e}^{-2aT}}{a} \geq \frac{2\bar{\omega}^2}{a^2}(\frac{1}{a}+8T)$, one has $\frac{\partial \beta_2}{\partial T} = 2\mathrm{e}^{-2aT} > 0$, and thus $\frac{\partial \beta_2}{\partial \bar{\omega}} = \frac{\partial \beta_2}{\partial T}\frac{\partial T}{\partial \bar{\omega}} > 0$. When $\frac{1-\mathrm{e}^{-2aT}}{a} < \frac{2\bar{\omega}^2}{a^2}(\frac{1}{a}+8T)$, one has $\frac{\partial \beta_2}{\partial T} = \frac{16\bar{\omega}^2}{a^2} > 0$ and $\frac{\partial \beta_2}{\partial \bar{\omega}} = \frac{4\bar{\omega}}{a^2}(\frac{1}{a}+8T) + \frac{\partial \beta_2}{\partial T}\frac{\partial T}{\partial \bar{\omega}} > 0$. To sum up, $\frac{\partial \beta_2}{\partial T} > 0$ and $\frac{\partial \beta_2}{\partial \bar{\omega}} > 0$ always hold. Substituting these results into the above equations yields $\frac{\partial \lambda}{\partial \bar{a}_\omega} < 0$ and $\frac{\partial \lambda}{\partial \bar{\omega}} < 0$. Combing the above results with (29) obtains that

$$\begin{cases} \dfrac{\partial \lambda_T}{\partial \bar{a}_\omega} = \dfrac{\partial \lambda_T}{\partial T}\dfrac{\partial T}{\partial \bar{a}_\omega} + \dfrac{\partial \lambda_T}{\partial \lambda}\dfrac{\partial \lambda}{\partial \bar{a}_\omega} < 0, \\ \dfrac{\partial \lambda_T}{\partial \bar{\omega}} = \dfrac{\partial \lambda_T}{\partial T}\dfrac{\partial T}{\partial \bar{\omega}} + \dfrac{\partial \lambda_T}{\partial \lambda}\dfrac{\partial \lambda}{\partial \bar{\omega}} < 0, \end{cases} \tag{30}$$

such that the maximum of λ_T is reached when $\bar{a}_\omega$ and $\bar{\omega}$ are at their minimum simultaneously under the constraint that (28) holds with a given β. The correlation is depicted in Fig. 3, where the grey and green arrows represent the positive and negative correlations between the two ends, respectively.

Next we show that for an ω satisfying (28), the minimum of $\bar{\omega} = \sup(|\omega|)$ equals to $\sqrt{\beta}$. For any $\Delta \in (0, \sqrt{\beta})$ where $|\omega(\tau)| \leq \Delta$ holds for any time τ, $\frac{1}{\nu}\int_t^{t+\nu}\omega^2(\tau)d\tau \leq \Delta^2 < \beta$ which contradicts with (28). Then $\bar{\omega} \geq \sqrt{\beta}$ must hold.

Secondly, we prove that $\bar{\omega} = \sqrt{\beta}$ holds if and only if ω is constant. *Sufficiency:* For a constant ω satisfying (28), we have $|\omega(\tau)| = \sqrt{\beta}$ for any $\tau \in [t, t+\nu]$, and $\bar{\omega} = \sqrt{\beta}$ holds. *Necessity:* since $|\omega(\tau)| \leq \sqrt{\beta}$ holds $\forall \tau \in [t, t+\nu]$, we prove that $|\omega(\tau)| = \sqrt{\beta}$ must hold $\forall \tau \in [t, t+\nu]$ using the contradiction argument. Suppose that there exists a t_2 in the interval $[t, t+\nu]$, $|\omega(t_2)| = \Delta < \sqrt{\beta}$. For the sufficiently smooth ω, there must exist a σ-neighborhood of t_2, such that $|\omega(\tau) - \Delta| < \frac{\sqrt{\beta}-\Delta}{2}$ and thus $\omega(\tau) < \frac{\sqrt{\beta}+\Delta}{2}$ strictly holds $\forall \tau \in [t_2 - \sigma, t_2 + \sigma]$. Let $t_a = \max\{t_2 - \sigma, t\}$, $t_b = \min\{t_2 + \sigma, t + \nu\}$, and due to $|\omega(\tau)| \leq \sqrt{\beta}$ one has $\int_t^{t+\nu}\omega^2(\tau)d\tau = \int_t^{t_a}\omega^2(\tau)d\tau + \int_{t_b}^{t+\nu}\omega^2(\tau)d\tau + \int_{t_a}^{t_b}\omega^2(\tau)d\tau < (t_a - t)\beta + (t_b - t_a)\frac{\Delta^2+\beta}{2} + (t+\nu-t_b)\beta < \nu\beta$, which contradicts with (28).

Note that $\bar{a}_\omega = \sup(|\dot{\omega}|)$ equals to 0 only when ω is constant. Since $\bar{a}_\omega \geq |a_\omega(t)| \geq 0$ holds for any $t \geq 0$, it means that the minimum of $\bar{a}_\omega$ and $\bar{\omega}$ are simultaneously reached when ω is constant, such that by (30) the maximum of λ_T is reached, which completes the proof. $\square$

Based on the above results, we can design the reference trajectory p_R, which allows efficient estimation. Basically, p_R is required to be sufficiently smooth to guarantee the control law (F, Γ). Moreover, denote the reference distance as $\rho_R = |p_R|$, the unit-length reference direction as $\varrho_R = \frac{p_R}{\rho_R}$, and the reference angular velocity as $\omega_R = \frac{\langle \dot{p}_R, \iota\varrho_R\rangle}{\rho_R}$. Then it implies by Theorem 4.2 that conditions 1)-4) should be satisfied by ρ_R and ω_R, and especially, ω_R should be constant for enabling efficient estimation.

Next we testify the extension case of time-varying ρ which satisfies (14), to see the efficiency of estimation influenced by moving trajectory of the AUV. Select a time $t_0 \in \mathbb{R}^+$ where $p_R(t_0)$ satisfies $\rho_R(t_0) = |p_R(t_0)| > 0$. Design the reference trajectory p_R as generated by the following dynamic:

$$\dot{p}_R = \iota\kappa p_R + r\varrho_R, \tag{31}$$

where $\kappa, r : \mathbb{R}^+ \to \mathbb{R}$ are the functions to be designed. It follows from (31) that

$$\omega_R = \frac{\langle \dot{p}_R, \iota\varrho_R\rangle}{\rho_R} = \frac{\langle \iota\kappa p_R + r\varrho_R, \iota\varrho_R\rangle}{\rho_R} = \kappa. \tag{32}$$

Then we design $\kappa(t) = \varpi$, $\forall t \geq t_0$, where $\varpi \in \mathbb{R}^+$ is a positive constant, and thus ω_R is constant. On the other hand, by (31) we have

$$\dot{\rho}_R = \langle \dot{p}_R, \varrho_R\rangle = \langle \iota\kappa\varrho_R + rp_R, \varrho_R\rangle = r. \tag{33}$$

Then we can design r as a sufficiently smooth function which enables $d \leq \rho_R \leq \bar{d}$ for some $d, \bar{d} \in \mathbb{R}^+$. For example, for the circular reference trajectory centered at the origin, one can design $r = 0$, such that $d \leq \rho_R \leq \bar{d}$ hold with $d = \bar{d} = |p_R(t_0)|$. For the elliptical reference trajectory centered at the origin, where $q_1, q_2 \in \mathbb{R}^+$ denote semi-major and semi-minor axes respectively, choose $p_R(t_0)$ as on the ellipse and take the derivatives of both sides of $\rho_R = \frac{q_1 q_2}{\sqrt{q_1^2 \sin^2(\theta_R) + q_2^2 \cos^2(\theta_R)}}$, where $\theta_R = \angle p_R$. Then design $r(t) = r_1(t)$ for $t \geq t_0$, where

$$r_1 = \frac{\rho_R^3 (q_2 - q_1)}{q_1^2 q_2^2} \cos(\theta_R)\sin(\theta_R)\kappa, \tag{34}$$

such that $d \leq \rho_R \leq \bar{d}$ hold with $\bar{d} = q_1, d = q_2$. Sufficient smoothness of ρ_R can thus also be guaranteed.

5. Numerical Examples

In this simulation, we provide two numerical examples to illustrate the effectiveness and efficiency of the proposed algorithm. Consider the reference trajectories generated by (31). Let the initial measurement at the origin of the frame $\{n\}$ be $\varsigma_0 = 0.6$. The parameters of the gradient in (4) are $\phi = 3.3\pi/4$, $\hbar = 6.5$. In this section, $\mathsf{t} = t - t_0$ denotes the time shift starting from t_0. For both cases, let the initial estimate $\hat{h}(t_0)$ be randomly selected but common, and the estimator gain $k = 1.2$.

Case I). Fixed ρ_R. Let $r(t) = 0$ and thus $\dot{\rho}_R = 0$ hold for all $t \geq t_0$. Let the initial position $p_R(t_0) = 1$ hold for all the cases, which is shown by the yellow triangle. Then $\rho_R(t) = 1$ holds for all $t \geq t_0$, and $d, \bar{d}$ are both

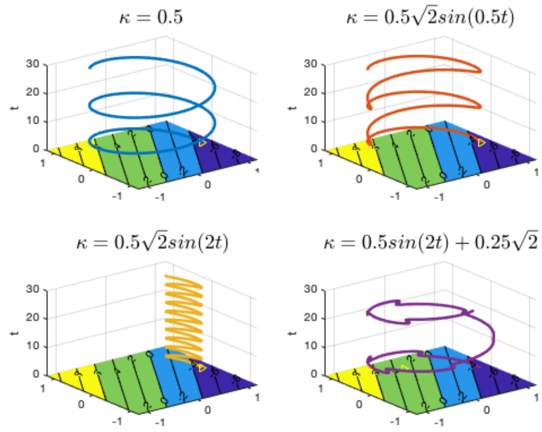

Fig. 4. Reference trajectories with fixed ρ_R along which the measurements are collected from the scalar field.

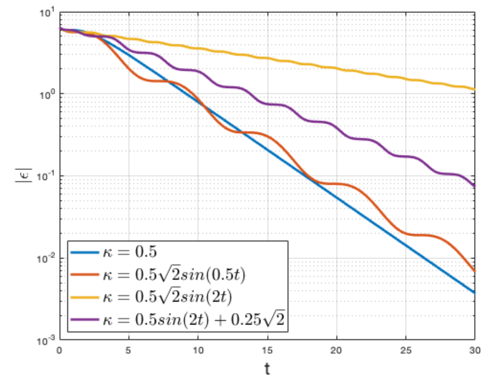

Fig. 5. Evolution of estimation error under different reference trajectories at the common excitation level.

1. We consider 4 different periodic functions for κ. Let $\kappa = 0.5, \frac{\sqrt{2}}{2}\sin(0.5t), \frac{\sqrt{2}}{2}\sin(2t), \frac{1}{2}\sin(2t) + \frac{\sqrt{2}}{4}$ respectively, such that (28) holds with the common period $\nu = \pi$ and the common excitation level $\beta = 0.25$. The corresponding reference trajectories are depicted in Fig. 4. The corresponding evolution of the magnitude of the estimation error $|\epsilon|$ is shown in Fig. 5. It can be observed that, although λ_T in (27) is just an estimate of the convergence rate of ϵ, the actual convergence rate is the largest when κ and thus ω_R is constant, which is shown by the blue line. It can be further inferred that a larger $\bar{a}_\omega$ may result in a slower convergence. This problem needs further investigation.

Case II). Time-varying ρ_R. Apart from the case of constant ρ_R where only ω_R influences the convergence rate of the estimation error, we further analyze the effectiveness and efficiency of the estimator for the case where ρ_R is time-varying. In the simulation, we let κ be $0.5, 0.5, 0.5\sqrt{2}\sin(0.5)t, 0.5$, which are at the common expectation level. Let r be $0, r_1, r_1, 0$ where r_1 is given by (34) with $q_1 = 1$ and $q_2 = 0.7$. The first and last reference trajectories are circular, and the other two are elliptical. Let $\rho_R(t_0) = 1$ for both the elliptical reference trajectories, such that $d = 0.7$ and $\bar{d} = 1$ hold. Let $p_R(t_0)$ be 1 and 0.7ι for the circular reference trajectories, such that ρ_R

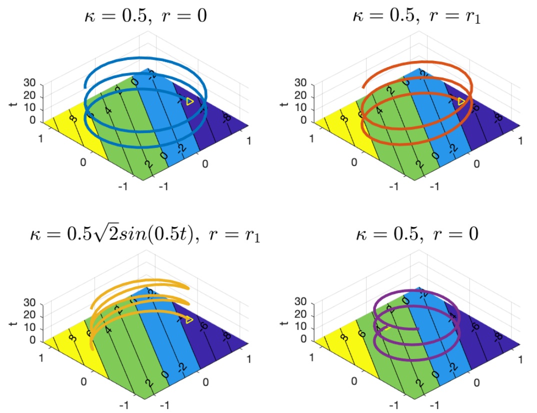

Fig. 6. Reference trajectories with fixed/time-varying ρ_R and ω_R.

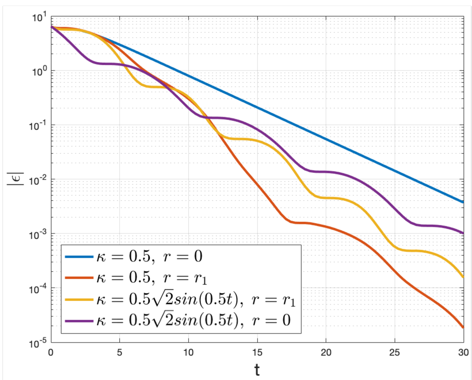

Fig. 7. Evolution of estimation error under reference trajectories with fixed/time-varying ρ_R and ω_R.

are always 1 and 0.7, respectively. The reference trajectories are shown in Fig. 6 with $p_R(t_0)$ shown by the yellow triangles. The evolution of the estimation error $|\epsilon|$ is shown in Fig. 7.

It is shown that with the same d and $\bar{d}$, a constant ω_R allows a faster convergence rate. See the comparison between the red and yellow curves in Fig. 6 and Fig. 7. However, the measuring distance also influences the efficiency. When ω_R are all constant, the time-varying ρ_R allows the fastest convergence rate, compared with that fixed at its lower and upper bound, respectively. See the comparison among the figures' blue, red, and purple curves. The influence of measuring distance ρ_R on the efficiency of estimation needs further investigation, and the reference distance needs to be characterized.

6. CONCLUSION

The problem of a single AUV monitoring a scalar field in a local coordinate system is considered. Global exponential stability of the estimation error system is rigorously proved, under the sufficient conditions on the distance and angular velocity. An estimate of the exponential convergence rate is given. Efficiency is achieved by maximizing the estimated convergence rate under the constraint of a given excitation level of the angular velocity. The effect of measuring distance on estimation efficiency will be studied in the future. Besides, a proper measure of energy consumption regarding the angular velocity and the distance needs further investigation.

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