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Design of nonlinear coupling for efficient synchronization in networks of nonlinear systems

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Abstract— **This paper proposes a design methodology of nonlinear coupling functions for guaranteed network synchronization. Compared to commonly used linear coupling, the proposed nonlinear coupling allows for a significant reduction of coupling energy cost and output noise sensitivity. This is achieved by activating the coupling only where necessary. Using the novel concept of strict incremental feedback passivity with a nonlinear gain, we estimate the magnitude and state-space location of potential incremental instabilities present in the systems' intrinsic dynamics, which could drive systems apart in the absence of coupling. Then we introduce a nonlinear coupling design that provides a gain only in the part of the coupled systems' state-space where the estimated incremental instabilities need to be suppressed. We provide constructive methods to design the nonlinear couplings for guaranteed synchronization over any connected, undirected, weighted network. By means of a numerical example, we demonstrate that our nonlinear coupling design, compared to linear couplings, results in significant performance improvements in terms of noise sensitivity and required coupling energy.**

I. Introduction

Synchronization, referring to any persistent state of coherent behavior of systems (agents, entities, etc.) in time, is omnipresent in both the natural and man-made world [1]–[3]. In this paper, we study synchronization in networks in its strongest form, which is that all systems in the network converge to a common trajectory (which may be oscillatory and is not required to be unique). For this type of synchronization, the literature mostly covers either *systems with "simple" intrinsic dynamics with nonlinear coupling*, cf. [4]–[6], or *systems with "complex" (even chaotic) intrinsic dynamics with linear coupling*, cf. [7]–[11].

We consider networks of systems with *nonlinear, possibly complex, intrinsic dynamics*, which mutually interact via *nonlinear coupling functions*. The considered nonlinear coupling functions take the form of a definite integral of a non-negative nonlinear coupling density function over the outputs of pairs of interacting systems. As shown in [12], this type of nonlinear coupling has performance advantages over conventional linear coupling. This is achieved by activating couplings only in those parts of the coupled systems' state-space where instabilities that could drive the systems apart need to be suppressed. These instabilities can be estimated using the system property: *incremental strict feedback passivity* (iSFP) with a *nonlinear gain function*. In [12], it was shown that, on so-called *sequentially decolorable* (SD) networks, synchronization is guaranteed by using a nonlinear coupling density function that is larger or equal to the nonlinear iSFP gain function, which is multiplied by global coupling strength that exceeds a computable threshold value. However, as the class of SD networks is restrictive (for instance, any ring network with more than four systems is not SD), the problem of designing these nonlinear coupling functions that synchronize systems on *any* (given) network is still open.

The main contribution of this paper is in providing a constructive design of nonlinear coupling functions for synchronization of systems on *any undirected weighted network*. Compared to [13], in which the existence of a class of synchronizing nonlinear coupling functions for unweighted networks was proved, this paper presents a *constructive design* for guaranteed synchronization in weighted networks. We also show that the extra assumptions on the coupling density function imposed in [13] (compared to [12], [14]) are redundant. Therefore, our new results enable systematic design of, among others, saturated couplings [15] and coupling functions with deadzones [16] for synchronization in any undirected weighted network. However, a price to pay for allowing for more general coupling density functions is that we need extra conditions for boundedness of solutions of the coupled systems (which is at least a technical requirement). A second contribution of this paper is to prove that the solutions of the coupled systems are bounded if the (isolated) systems satisfy a semipassivity (SP) property [17], which is a generalization of the well-known concept of passivity.

We demonstrate our results with a numerical study in a ring network with five FitzHugh-Nagumo oscillators [18]. We show that this oscillator is iSFP and SP, and we compute nonlinear coupling functions that guarantee synchrony in the network. The obtained results are constructive and close to optimal in the sense that synchronization would not be possible for lower coupling strengths. We present simulation results in which we corrupted the signals in the coupling with noise. In these simulations, we demonstrate that, compared to linear couplings, the nonlinear coupling functions are superior in terms of coupling energy cost and measurement noise sensitivity.

This paper is organized as follows. After introducing the

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adopted notation, we discuss the problem setting and we introduce the notion of synchronization. Next, in Section III we introduce iSFP. In Section IV, we present our coupling design philosophy and we state our main synchronization results. Boundedness of solutions is discussed in Section V. (As the order of these latter two sections suggests, our synchronization conditions do not rely on a-priori bounds on the solutions of the coupled systems). The numerical example is presented in Section VI. Concluding remarks are given in Section VII.

Notation. $\mathbb{R}$ denotes the real numbers, and $\mathbb{R}_+ := \{s \in \mathbb{R} | \, s \geq 0\}$. We let $|\cdot|$ be the vector 2-norm, i.e. for any $x \in \mathbb{R}^n$, $|x|^2 = x^T x$, where ^{T} denotes transposition. $\mathbf{1} \in \mathbb{R}^n$ and $\mathbf{0} \in \mathbb{R}^n$ denote the vectors with all entries identical to 1, respectively, 0. We also use $\mathbf{0}$ to denote the zero matrix of appropriate dimension. I denotes the identity matrix (of appropriate dimension). $\mathcal{C}^r(X, Y)$ is the space of (at least) r-times continuously differentiable functions that assign elements of X to elements of Y. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called positive semidefinite if $f(x) \geq 0$ for all $x \in \mathbb{R}^n$. If, in addition, and $f(x) = 0 \Leftrightarrow x = \mathbf{0}$, then f is positive definite. A square matrix $A \in \mathbb{R}^{n \times n}$ is positive (semi)definite if $f(x) = x^T A x$ is positive (semi)definite. The function f is called radially unbounded if $f(x) \to \infty$ as $\|x\| \to \infty$. $\mathcal{L}_\infty$ is the space of (essentially) bounded functions. $\mathcal{L}_1$ is the space of measurable functions over $\mathbb{R}$ with finite 1-norm, i.e., for any $f \in \mathcal{L}_1$, $\int_{\mathbb{R}} |f(s)| ds < \infty$.

II. Problem setting

We represent a network by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{1, 2, \ldots, N\}$ being the set of nodes and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ being the unordered set of edges. An element $(i, j) \in \mathcal{E}$ defines the edge between nodes i and j (and thus $\mathcal{G}$ is an undirected graph). We adopt the following assumption.

Assumption 1.
The graph $\mathcal{G}$ is *simple* and *connected*.

In a *simple* graph, there is at most one edge between nodes i and j, and edges of the form (i, i) (self-loops) are absent. A graph is *connected* if, for every two distinct nodes i and j in $\mathcal{G}$, there exists a path in $\mathcal{G}$ (which is a sequence of edges that join a sequence of disjoint nodes) connecting i and j (which are the start and end nodes of the path). $\mathcal{N}_i := \{j \in \mathcal{V} \,|\, (i, j) \in \mathcal{E}\}$ denotes the set of neighbors of node i, i.e. the set of nodes that connect to node i.

We consider systems on $\mathcal{G}$ of the form

$$\begin{cases} \dot{x}_i = f(t, x_i) + B u_i, \\ y_i = C x_i, \end{cases} \tag{1}$$

defined for all $t \geq t_0$, where, for any $i \in \{1, 2, \ldots, N\} =: \mathcal{V}$, $x_i \in \mathbb{R}^n$ is the state, $y_i \in \mathbb{R}$ is the output, and $u_i \in \mathbb{R}$ is the input. The function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is piece-wise continuous in the first argument, and at least once continuously differentiable in its second argument, and matrices B and C are of appropriate dimension. The systems (1) on $\mathcal{G}$ interact via the following coupling functions:

$$u_i = \sigma \sum_{j \in \mathcal{N}_i} w_{ij} \int_{y_i}^{y_j} \lambda(s) \mathrm{d}s, \tag{2}$$

where the real constant $\sigma > 0$ is the global coupling strength, real constants $w_{ij} \in (0, \infty)$ denote the interaction weights, and $\lambda : \mathbb{R} \to \mathbb{R}_+$ is the *coupling density function*. Note that $u_i \equiv 0$ if $y_i = y_j$ for all $j \in \mathcal{N}_i$, i.e. the couplings vanish if the outputs of all systems are identical.

We define *synchronization* as follows:

Definition 1. The network of systems (1), which interact via (2), *synchronizes* if

- the solutions of the coupled systems (1) and (2) are bounded in forward time, and
- for any initial condition, the states asymptotically match, i.e., for every $i \in \mathcal{V}$, it holds that for every $j \in \mathcal{V} \backslash \{i\}$,

$$|x_i(t) - x_j(t)| \to 0 \quad \text{as} \quad t \to \infty.$$

Remark 1. In the definition of synchronization, we explicitly state that the solutions of the coupled systems must be bounded. This is obviously a condition of practical interest. Besides, many theoretical results rely (often implicitly) on solutions to be bounded. An example illustrating the necessity of bounded solutions is found in [19].

The main objective that we pursue in this paper is to determine conditions on the systems (1) and the coupling (2) – in particular, the design of the coupling density function $\lambda(\cdot)$ and the coupling strength σ – leading to synchronization of all systems on a given graph $\mathcal{G}$ with given weights w_{ij} (which can be thought of as edge weights).

III. Incremental Strict Feedback Passivity

In this section, we present a definition and characterization of the system property *incremental Strictly Feedback Passivity* (iSFP) with nonlinear gain, which are taken from [12]. We sometimes write iSFP systems, or systems are iSFP, in which the P then stands for *passive*.

Definition 2. The system (1) is called iSFP with *nonlinear gain function* $\gamma : \mathbb{R} \to \mathbb{R}$ if there exists an *incremental storage function* $\tilde{S} \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_+)$ such that for any two solutions $x_a(\cdot)$ and $x_b(\cdot)$ of (1) corresponding to inputs $u_a(\cdot)$ and $u_b(\cdot)$, respectively, which are defined on $[t_0, \infty)$, the following dissipation inequality holds true for all $t \geq t_0$:

$$\frac{d}{dt} \tilde{S}(x_a(t) - x_b(t)) \leq -\rho(|x_a(t) - x_b(t)|) + (y_a(t) - y_b(t)) \left[(u_a(t) - u_b(t)) + \int_{y_b(t)}^{y_a(t)} \gamma(s) ds \right],$$

where $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is a positive definite function, and $y_a(t) = C x_a(t)$, $y_b(t) = C x_b(t)$.

In case $\gamma(s) > 0$ for all s in a closed interval with endpoints y_a and y_b, the nonlinear gain function γ specifies the amount of potential *shortage of incremental passivity* at every point in that interval; If $\gamma(s) < 0$ for some s in the closed interval with endpoints y_a and y_b, there is a guaranteed *excess of incremental passivity* in that interval.

Note that for constant $\gamma(s) \equiv \text{constant} > 0$, i.e., when the potential shortage of incremental passivity is uniform over all $s \in \mathbb{R}$, our iSFP definition becomes identical to the definition of iSFP from [11]. As will be demonstrated in the next section, compensating $\gamma(s)$ by means of feedback in the areas where the system is potentially short of incremental passivity renders the closed-loop system incrementally globally asymptotically stable. This targeted compensation has a number of benefits for synchronization, compared to uniform compensation, as demonstrated in Section VI.

The challenge in characterizing the iSFP property is in finding the incremental storage function $\tilde{S}$ and the nonlinear gain function $\gamma(\cdot)$. If $\gamma(\cdot)$ is chosen (or known), then the iSFP property can be verified using the following result [12]:

Theorem 1. *If there exists positive definite matrices $P = P^T > 0$ and $R = R^T > 0$ such that, for all $x_i \in \mathbb{R}^n$ and for all $t \geq t_0$,*

$$\frac{\partial f^T}{\partial x_i}(t, x_i)P + P\frac{\partial f}{\partial x_i}(t, x_i) - 2C^T C\gamma(Cx_i) \leq -R, \quad (3)$$

$$PB = C^T, \quad (4)$$

then the system (1) *is iSFP with nonlinear gain function $\gamma(\cdot)$, incremental storage function $\tilde{S}(x_a - x_b) = \frac{1}{2}(x_a - x_b)^T P(x_a - x_b)$ and dissipation rate $\rho(|x_a - x_b|) = \lambda_{\min}(R)|x_a - x_b|^2$, where $\lambda_{\min}(R)$ is the smallest eigenvalue of R (which is positive).*

In [12], one can find an even more constructive result, which gives (an estimate of) $\gamma(\cdot)$ for a class of systems with relative degree one.

IV. Coupling design to synchronize iSFP systems

In this section, we show how to use the iSFP property in designing the coupling density function in (2) to synchronize two iSFP systems. Next we show that essentially the same coupling density functions can be used to achieve synchronization in any connected network, provided that the coupling strength σ is chosen appropriately. We present *constructive* results for determining a threshold value (lower bound) for σ. We remark that the use of the iSFP property, which is a property of a *single system*, to establish conditions for synchronization of *multiple systems* is justified as the considered systems (1) are identical.

A. Two coupled systems

Consider two iSFP systems (1) with $i \in \{1, 2\}$, which interact via

$$u_1 = -u_2 = \sigma \int_{y_1}^{y_2} \lambda(s)ds. \quad (5)$$

The iSFP property implies that, along solutions of the two coupled systems, it holds that

$$\frac{d}{dt}\tilde{S}(x_1 - x_2) \leq -\rho(|x_1 - x_2|) + (y_1 - y_2)\int_{y_2}^{y_1} (\gamma(s) - 2\sigma\lambda(s))\, ds.$$

Take any $\lambda \in \mathcal{L}_\infty$ satisfying $\lambda(s) \geq \max\{\gamma(s), 0\}, \quad \forall s \in \mathbb{R}$. With this choice of $\lambda(\cdot)$, we have, for any $\sigma \geq \frac{1}{2}$,

$$(y_1 - y_2)\int_{y_2}^{y_1} (\gamma(s) - 2\sigma\lambda(s))\, ds \leq 0,$$

which implies $\frac{d}{dt}\tilde{S}(x_1 - x_2) \leq -\rho(|x_1 - x_2|)$. This, in turn, under some additional mild conditions, implies that two the systems synchronize [12].

The key observation is that any properly designed coupling (that is, the coupling density function satisfying $\lambda(\cdot) \geq \max\{\gamma(\cdot), 0\}$ and sufficiently high coupling strength σ) compensates for the potential shortage of incremental passivity in the systems, which results in synchronization.

B. Network synchronization

Below we show that the same philosophy used to synchronize two coupled systems – the design of coupling density functions using the knowledge of the nonlinear gain function $\gamma(\cdot)$ – leads to synchronization in any network of coupled systems (1), (2), provided the coupling strength σ exceeds some computable threshold value. We impose the following assumptions.

Assumption 2. The solutions of the coupled systems (1), (2) are bounded in forward time.

Assumption 3. The systems (1) are iSFP with a radially unbounded, positive definite incremental storage function $\tilde{S} \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}_+)$, and $\gamma(\cdot)$ satisfies $\gamma^* := \sup_{s\in\mathbb{R}} \gamma(s) < \infty$. In addition, there exists a positive definite function $\tilde{\rho} : \mathbb{R}_+ \to \mathbb{R}_+$ and a positive real constant ϵ_1 such that, for all $x_a, x_b \in \mathbb{R}^n$ (as $y_i = Cx_i$),

$$\rho(|x_a - x_b|) \geq \tilde{\rho}(|x_a - x_b|) + \epsilon_1|y_a - y_b|^2.$$

Assumption 4. The coupling density function $\lambda \in \mathcal{L}_\infty$ satisfies $\lambda(s) \geq \max\{\gamma(s), 0\}$ for all $s \in \mathbb{R}$.

Remark 2. Assumption 3 is satisfied if the iSFP property is established using Theorem 1. Indeed, in that case one can take $\epsilon_1 > 0$ such that $R - \epsilon_1 C^T C > 0$ (which restricts the value of ϵ_1 not to be too large). Note that the constant ϵ_1 can also be introduced by using $\gamma(\cdot) + \epsilon_1$ instead of $\gamma(\cdot)$ in the iSFP definition (as $-\epsilon_1|y_a - y_b|^2 + (y_a - y_b)\int_{y_b}^{y_a}(\gamma(s) + \epsilon_1)ds = (y_a - y_b)\int_{y_b}^{y_a}\gamma(s)ds$). Using this construction, the value of ϵ_1 can be chosen freely (at the cost of possibly increasing the support of $\lambda(\cdot)$ that satisfies Assumption 4).

We let $L = \left(L_{ij}\right) \in \mathbb{R}^{N\times N}$ be the *Laplacian matrix* of $\mathcal{G}$, whose entries are $L_{ii} = \sum_{j\in\mathcal{N}_i} w_{ij}$ and

$$L_{ij} = \begin{cases} -w_{ij} & \text{if } (i,j) \in \mathcal{E}, \\ 0 & \text{otherwise}, \end{cases}$$

where w_{ij} are the (edge) weights specified in (2). Furthermore, we let $L_K = NI - \mathbf{1}\mathbf{1}^T$ be the Laplacian matrix of the *complete graph*, which is the graph in which every pair of nodes is joined by an edge (of weight 1). The $N \times M$ matrix E, where $M := \frac{N(N-1)}{2}$, is the *oriented incidence matrix* of this complete graph. (See Appendix A for its definition.)

Our main result is summarized in the following theorems, the proofs of which are presented in Appendix B.

Theorem 2. *Consider N systems* (1) *on any graph $\mathcal{G}$ that satisfies Assumption 1. Suppose that these systems satisfy Assumption 3. Let the systems interact via* (2) *with a coupling density function $\lambda(\cdot)$ satisfying Assumption 4, and suppose that the coupled systems have bounded solutions (Assumption 2 holds true). Then the coupled systems synchronize for any $\sigma > 0$ for which there exist an $M \times M$ diagonal real matrix Φ such that*

$$\Phi \geq 0,$$
$$\begin{pmatrix} \epsilon_1 L_K & \frac{1}{2}(\sigma\lambda^* NL - \gamma^* L_K - E\Phi E^T) \\ \star & E\Phi E^T \end{pmatrix} \geq 0, \tag{6}$$

where $\star$ denotes the symmetric part, $\lambda^* = \sup_{s\in\mathbb{R}} \lambda(s)$ and $\gamma^* = \sup_{s\in\mathbb{R}} \gamma(s)$.

Theorem 3. *There exists a constant $\sigma^* \in (0, \bar{\sigma}]$, where*

$$\bar{\sigma} := \frac{(\gamma^*)^2}{4\epsilon_1 \lambda^* \lambda_2(L)}, \tag{7}$$

such that for any $\sigma \geq \sigma^*$ there is a diagonal $M \times M$ matrix Φ such that (6) *is satisfied. Here $\lambda_2(L)$ is the smallest non-zero eigenvalue of L, i.e., the* algebraic connectivity *of the weighted graph $\mathcal{G}$.*

As shown in the proof of Theorem 2 in Appendix B, an incremental storage function is defined for the network of coupled systems. The LMIs (6) guarantee that the total potential shortage of incremental passivity is compensated for by the coupling design.

Remark 3. In case $L = L_K$, i.e. complete graphs with weights $w_{ij} = 1$, one can easily show that (6) holds true for $\Phi = \mathbf{0}$ and any $\sigma \geq \sigma^* = \frac{\gamma^*}{N\lambda^*}$. This result is independent of the value of $\epsilon_1 > 0$. In fact, with $\epsilon_1 = 0$ and noticing that $\gamma^* \leq \lambda^*$ by Assumption 4, one recovers the results presented in [14] and [12].

V. Bounded solutions

Our synchronization conditions require the solutions of the coupled systems to be bounded (i.e., Assumption 2 is to be satisfied). In this section, we provide conditions under which this assumption is valid. In [12], it is proved that iSFP systems, for which the iSFP property is established using Theorem 1, have bounded solutions when coupled via (2) with $\lambda \in \mathcal{L}_1$. Therefore, we conclude the following.

Corollary 1. *Theorem 2 is true without Assumption 2 if the iSFP property is established using Theorem 1 and $\lambda \in \mathcal{L}_1$.*

In case $\lambda \notin \mathcal{L}_1$, e.g. the linear coupling case for which $\lambda(s) \equiv \text{constant}$, we need alternative methods to verify Assumption 2.

The system property *semi-passivity* (SP) was introduced in [17] (in a strict form) to establish ultimately bounded solutions of linearly coupled systems.

Definition 3. The system (1) is called *semi-passive* (SP) if there exists a *storage function* $S \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_+)$ such that

$$\dot{S}(t, x_i, y_i, u_i) \leq y_i u_i, \quad \forall t \geq t_0, \quad \forall |x_i| > R,$$

where $\dot{S}$ is the time derivative of S along the solutions of (1), and R is a positive constant.

It follows from this definition that SP is a relaxation of the well-known notion of passivity, for which the dissipation inequality holds with $R = 0$ (and thus every passive system is SP.) Roughly speaking, an SP system behaves as a passive system outside the ball of radius R in $\mathbb{R}^n$ (the system's state-space). Using the fact that our nonlinear couplings (2) satisfy a passivity property, namely $\sum_{i=1}^N y_i u_i \leq 0$ as shown in Appendix B, we derive the following result:

Theorem 4. *Suppose that the systems* (1) *are SP with a positive definite and radially unbounded storage function. Then for any graph $\mathcal{G}$, the solutions of the coupled systems* (1), (2) *are bounded for any non-negative coupling density function $\lambda(s)$.*

VI. Example and numerical simulations

We consider networks of FitzHugh-Nagumo (FHN) [18] systems, whose dynamics are given by

$$\underbrace{\begin{pmatrix} \dot{z}_i \\ \dot{y}_i \end{pmatrix}}_{=:x_i} = \underbrace{\begin{pmatrix} \mu(y_i - \kappa z_i) \\ y_i - \frac{1}{3}y_i^3 - z_i + \xi \end{pmatrix}}_{=:f(t,x_i)} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{=:B} u_i \tag{8}$$

with positive, real constants μ, κ, ξ, and, obviously, $y_i = Cx_i = B^T x_i$. The FHN system is iSFP as Theorem 1 is satisfied with

$$P = \begin{pmatrix} 1/\mu & 0 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 2\kappa & 0 \\ 0 & 2\epsilon \end{pmatrix}.$$

(thus $\rho(s) = 2\min\{\kappa, \epsilon\} s^2$) and

$$\gamma(s) = \epsilon + 1 - s^2, \tag{9}$$

for some arbitrary, positive, real constant ϵ. As $\gamma(s) > 0$ for $s \in (-\sqrt{1+\epsilon}, \sqrt{1+\epsilon})$ we conclude that the FHN system is potentially short of incremental passivity in that interval.

In addition, the FHN system is SP with the radially unbounded storage function $S = \frac{1}{2\mu} z_i^2 + \frac{1}{2} y_i^2$. Indeed, $\dot{S} = y_i u_i - \kappa z_i^2 - \frac{1}{3} y_i^4 + y_i^2 + \xi y_i$, and it is straightforward to show that there exists a positive constant R_1 such that $-\frac{1}{3} y_i^4 + y_i^2 + \xi y_i \leq -\kappa y_i^2 + R_1^2$. Therefore, $\dot{S} \leq y_i u_i$ for all $| \begin{pmatrix} z_i & y_i \end{pmatrix}^T | > R = \frac{R_1}{\sqrt{\kappa}}$. Thus by Theorem 4, the FHN systems coupled via (2) have bounded solutions.

We consider these FHN systems on the network depicted in Figure 1, which satisfies Assumption 1. Our motivation to include this network is that we can demonstrate (see section VI-A) that the estimate of the coupling strength provided by Theorem 2 is the best one can obtain. We emphasize that our theory covers weighted networks as well, yet we did

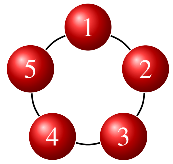

Fig. 1. The network of FHN systems. All edge weights w_{ij} equal 1.

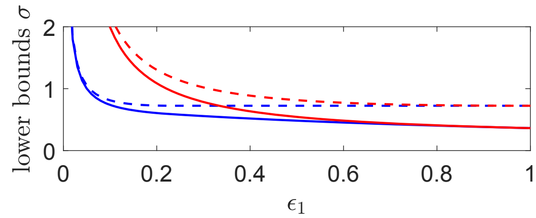

Fig. 2. Lower bounds on σ as function of ϵ_1 computed using (6) (σ^* in solid blue) and (7) ($\bar{\sigma}$ in solid red). Corresponding normalized (multiplication by $\lambda^*/(1+\epsilon)$) lower bounds are represented by the dashed lines.

not include an example with a weighted network because of limited space.

We take as coupling density function

$$\lambda(s) = \max\{\epsilon+1-s^2+\epsilon_1, 0\} > \gamma(s)+\epsilon_1, \quad \forall s \in \mathbb{R}, \tag{10}$$

with constant $\epsilon_1 > 0$. Clearly, Assumption 4 is satisfied. As we wish ϵ_1 to be a design variable (that can be chosen freely), we redefine $\gamma(s) \mapsto \gamma(s) + \epsilon_1$ such that $\gamma^* = 1 + \epsilon + \epsilon_1$ and $\lambda^* = \gamma^*$. (See Remark 2.) For this coupling density function $\lambda(\cdot)$, Theorem 2 guarantees synchronization of the FHN systems for any σ and ϵ_1 such that LMIs (6) are satisfied.

A. Computation of the lower bounds on σ

The Laplacian matrix for this example is

$$L = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

We remark that the methods from [12] are not applicable, as this ring network is not *sequentially decolorable*.

We fix $\epsilon = 0.001$ and, for various values of $\epsilon_1 > 0$, we solve the LMIs (6) using YalMip/SeDuMi [20], [21] for σ. This allows us to determine σ^* and compare it with the guaranteed lower bound $\bar{\sigma}$ (Theorem 3). The results of this numerical exercise are shown in Figure 2, where the *blue solid line* corresponds to the bound computed using (6), and the *red solid line* corresponds to (7). In both cases, we observe that the lower bound on σ decreases monotonically with ϵ_1. Moreover, for the larger ϵ_1, the more identical σ^* and $\bar{\sigma}$ become.

We compare these results with the linear coupling case. As shown in [22], for this network with the linear coupling (2) with $\lambda(s) \equiv 1+\epsilon$, a threshold value for σ that guarantees synchronization is

$$\bar{\sigma}_{\text{lin}} = \tfrac{1}{2(1-\cos(2\pi/5))} \approx 0.7236.$$

For a fair comparison of the obtained lower bounds on σ for the nonlinear and the linear coupling cases, we multiplied the computed nonlinear coupling strength lower bounds by $\frac{\lambda^*}{1+\epsilon} = \frac{\epsilon+1+\epsilon_1}{1+\epsilon}$. (Note that for our nonlinear coupling, $\lambda^* = 1 + \epsilon + \epsilon_1$, which increases with ϵ_1, whereas in the considered linear coupling case, $\lambda^* = 1 + \epsilon$ is constant.) These "normalized" coupling strength lower bounds are represented by the dashed lines in Figure 2. We observe that both dashed lines converge to $\bar{\sigma}_{\text{lin}}$ with increasing ϵ_1.

Interestingly, when taking the FHN system parameter $\xi = 0$, one can show the existence of a Hopf bifurcation at $\sigma_H = \frac{1-\mu\kappa}{2(1-\cos(2\pi/5))}$ in the network with linear coupling, which corresponds to the birth of a non-synchronized (unstable) periodic solution that exists for any $\sigma \in [0, \sigma_H]$ [22]. Note that $\sigma_H \to \bar{\sigma}_{\text{lin}}$ for $\kappa\mu \to 0$. For the FHN system, the obtained $\gamma(\cdot)$ in (9) is independent of the FHN parameters ξ, μ and κ, and therefore, the computed threshold values shown in Figure 2 hold true for all (positive) values of ξ, μ and κ. Given that $\sigma^*\lambda^*/(1+\epsilon) \approx \sigma_{\text{lin}}$ for $\epsilon_1 > 0.2$ (see Figure 2), and $\sigma_{\text{lin}} \approx \sigma_H$ for the given FHN parameters, one cannot expect a tighter estimate of σ^* for values of $\epsilon_1 > 0.2$. (A similar statement can be made for $\bar{\sigma}$.) Note that by the synchronization Definition 1, for which we developed our theory, synchrony is not possible if a non-synchronized (unstable periodic) solution exists.

B. Numerical simulations

We simulated the coupled FHN systems with parameters $\mu = 0.01$, $\kappa = 0.8$ and $\xi = -0.4$ for various σ and ϵ_1. In our simulations, we numerically integrated the differential equations in Matlab 2022b using the `ODE23s` solver (with default solver settings). For every simulation, we selected the initial conditions for the FHN systems uniformly at random.

Figure 3 shows the outputs of the FHN systems. We remark that synchronization of the outputs of the FHN systems implies that the states of the FHN systems synchronize, which is easily concluded given that the internal z_i-states have stable, linear and time-invariant dynamics. In the top panel of that figure, which shows the uncoupled case, we observe that the outputs of the FHN systems do not synchronize, which is attributed to the potential shortage of incremental passivity. In the bottom panel of Figure 3, for which the coupling parameters σ and ϵ_1 are chosen just above the blue line in Figure 2 such that synchrony is guaranteed, we see that the outputs of the FHN systems synchronize.

Figure 4 shows the evolution of the *coupling gains*

$$g_{ij} := \sigma w_{ij} \left| \int_{y_i}^{y_j} \lambda(s) ds \right| / |y_i - y_j|$$

for two pairs of coupling parameters σ and ϵ_1, which are chosen again just above the blue line in Figure 2 such that synchrony is guaranteed. Looking at these coupling gains, we observe that, after the transient to synchrony

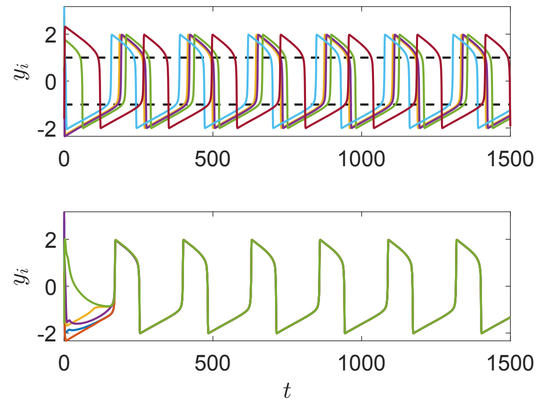

Fig. 3. Simulation results of the FHN systems coupled via (2) with (10). Top panel: $\sigma = 0$, i.e. the FHN systems are uncoupled. Within the two dashed black lines, the FHN system is potentially short on incremental passivity. Bottom panel: Synchronization for $(\sigma, \epsilon_1) = (0.72, 0.1)$

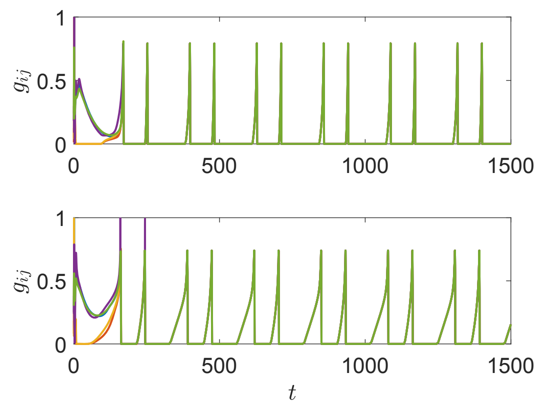

Fig. 4. Evolution of the realized coupling gains. Top panel: $(\sigma, \epsilon_1) = (0.72, 0.1)$, Bottom panel: $(\sigma, \epsilon_1) = (0.37, 1)$

in the interval $0 \leq t \leq 300$, the maximal value of the coupling gains decreases slightly with increasing ϵ_1, which is in correspondence with the normalized results displayed in Figure 2. However, as also seen in the evolution of the maximal coupling gains, increasing ϵ_1 comes at the cost of increasing the size of the support of $\lambda(\cdot)$, which results in the coupling being active on longer intervals of time.

To illustrate the benefits of our proposed coupling over linear coupling, we simulated the coupled FHN neurons with noise (normally distributed with variance 0.25) added to the outputs that are used to establish the interactions (i.e. a example of measurement noise). Our results, as well as results for linear coupling with $\sigma = 0.7236$ (for which synchronization occurs as discussed above), are displayed in Figure 5. In this figure, $\tilde{y}$ are the synchronization output errors, which is the vector with entries $y_i - y_j$ for all pairs (i, j) (with $i \neq j$). We observe from the bottom panel, which displays the maximum of the absolute values of the inputs (i.e. couplings), that the nonlinear coupling transmits (on average) significantly less output noise to the systems, resulting in smaller synchronization errors as seen in the top panel. Comparing the low $\epsilon_1 = 0.1$ case with

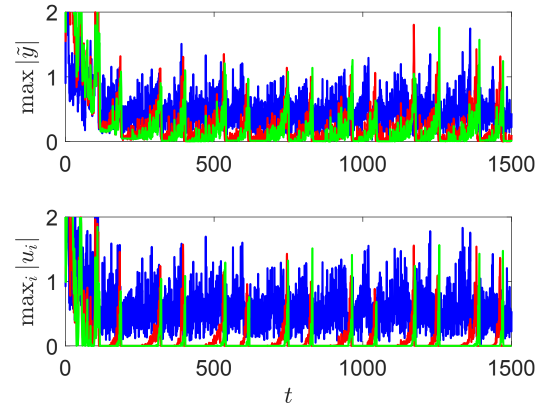

Fig. 5. Noise simulation results of the four FHN systems on the undirected graph with nonlinear coupling with $(\sigma, \epsilon_1) = (0.72, 0.1)$ in green, $(\sigma, \epsilon_1) = (0.37, 1)$ in red, and linear coupling with $\sigma = 0.7236$ in blue.

the high $\epsilon_1 = 1$ case, we observe that the higher ϵ_1, the larger is the amount of noise entering the systems. This is attributed to the increase of the support of $\lambda(\cdot)$ with increasing ϵ_1. We also computed the average energy of $\max|u_i|$ as $E_{\text{avg}} := \sum_{\#\text{samples}} \max|u_i|^2$, where we used linear interpolation along the time axis (with fixed step size of 0.5) to ensure the same number of samples for every case. The results are: $E_{\text{avg}} = 0.1349$ for nonlinear coupling with $(\sigma, \epsilon_1) = (0.72, 0.1)$, $E_{\text{avg}} = 0.1559$ for nonlinear coupling with $(\sigma, \epsilon_1) = (0.37, 1)$, and $E_{\text{avg}} = 0.5034$ for linear coupling with $\sigma = 0.7236$. This demonstrates that our nonlinear coupling design outperforms the linear coupling in terms of required energy.

VII. Concluding Remarks

We considered networks of nonlinearly coupled incrementally strictly feedback passive (iSFP) systems. We proved that, by designing a coupling density function $\lambda(\cdot)$ that compensates for the potential shortage of incremental passivity characterized by $\gamma(\cdot)$, synchronization can be achieved on any connected simple network, provided that the coupling strength σ is sufficiently large. The coupling strength σ that guarantees synchrony can be easily determined by solving a system of linear matrix inequalities (LMIs). Alternatively, an explicit equation that specifies a (possibly conservative) lower bound on σ is provided. We demonstrated our results in a numerical study with a ring of five coupled FitzHugh-Nagumo (FHN) systems. Interestingly, in this setting, for larger values of the parameter ϵ_1 in the nonlinear coupling, one cannot get better estimates of the threshold value of σ. We demonstrated the benefits of the proposed nonlinear coupling design over the conventional linear coupling in a setting in which the outputs used to establish coupling are corrupted with noise.

Appendix A: Algebraic Graph Theory

Clearly, any Laplacian matrix L is singular by construction. It is well-known [23] that, under Assumption 1, L has a simple zero eigenvalue with (left and right) eigenvector in $\operatorname{span}\{\mathbf{1}\}$. Let M be the cardinality of $\mathcal{E}$. For any edge $e_\ell = (i,j) \in \mathcal{E}$, $\ell \in \{1,2,\ldots,M\}$, we construct an N-dimensional vector E_ℓ such that its ith is equal to 1, its jth is identical to -1, and all other entries are 0. (One may swap the -1 and 1 elements, as $(i,j)=(j,i)$.) The matrix $E = \begin{pmatrix} E_1 & E_2 & \cdots & E_M \end{pmatrix}$ is the *oriented incidence matrix* of $\mathcal{G}$. For any graph with symmetric weights $w_{ij} = w_{ji}$, its *symmetric* Laplacian matrix L can be written as $L = EWE^T$, where W is diagonal matrix whose (M positive) diagonal entries the weights of the associated edges, i.e., for the edge $e_\ell = (i,j)$, $W_\ell = w_{ij}$. Furthermore, any symmetric Laplacian matrix L is positive semi-definite (which follows, e.g., from Gershgorin's Disc theorem [24]). We denote the eigenvalues of L by $\lambda_i(L)$, and we adopt the convention that $\lambda_1(L) = 0$ and all other eigenvalues are ranked such that $\lambda_2(L) \leq \lambda_3(L) \leq \ldots \leq \lambda_N(L)$. Under Assumption 1, $\lambda_2(L)$, which is known is the *algebraic connectivity*, is positive.

Appendix B: Proofs of Theorems 2, 3 and 4

The proof of Theorem 2 is based on the construction of the *network incremental storage function* $\tilde{\mathcal{S}}$:

$$\tilde{\mathcal{S}} = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \tilde{S}_{ij}, \tag{11}$$

where $\tilde{S}_{ij} := \tilde{S}(x_i - x_j)$ is the *system incremental storage function* from Assumption 3. Thus this network storage function is the sum of the *system incremental storage functions* taken along all pairs of systems in $\mathcal{G}$. Note that, by construction, $\tilde{\mathcal{S}} \geq 0$.

The derivative of $\tilde{\mathcal{S}}$ along solutions of the coupled systems (1), (2), given Assumptions 3, satisfies

$$\begin{aligned}\dot{\tilde{\mathcal{S}}} \leq& -\left(\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \tilde{\rho}(|x_i - x_j|)\right) \\ &- \epsilon_1 y^T E E^T y + y^T E \left(E^T u + \gamma^* E^T w(y)\right) \\ =& -\left(\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \tilde{\rho}(|x_i - x_j|)\right) \\ &- \epsilon_1 y^T L_K y + y^T L_K u + \gamma^* y^T L_K w(y),\end{aligned} \tag{12}$$

where $y := \begin{pmatrix} y_1 & \cdots & y_N \end{pmatrix}^T$, $u := \begin{pmatrix} u_1 & \cdots & u_N \end{pmatrix}^T$, $w = w(y) := \begin{pmatrix} w_1 & \cdots & w_N \end{pmatrix}^T$ with $w_i = w_i(y_i) := \int_0^{y_i} \gamma(s)/\gamma^* ds$. Furthermore, with $v = v(y) := \begin{pmatrix} v_1 & \cdots & v_N \end{pmatrix}^T$, where $v_i = v_i(y_i) := \int_0^{y_i} \lambda(s)/\lambda^* ds$, we can write (2) as

$$u = -\sigma\lambda^* L v(y). \tag{13}$$

In addition, by Assumption 4, for all $y_i, y_j \in \mathbb{R}$,

$$(y_i - y_j)(v_i - v_j) \geq (y_i - y_j)(w_i - w_j),$$

which leads to

$$y^T L_K v \geq y^T L_K w. \tag{14}$$

Combining (12), (14) and (13) yields

$$\begin{aligned}\dot{\tilde{\mathcal{S}}} \leq& -\left(\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \tilde{\rho}(x_i - x_j)\right) \\ &- \epsilon_1 y^T L_K y - y^T \left(\sigma\lambda^* N L - \gamma^* L_K\right) v(y).\end{aligned} \tag{15}$$

Let us now first present a supporting lemma.

Lemma 1. *Consider N systems* (1) *on a graph $\mathcal{G}$ that satisfies Assumption 1. Suppose that these systems satisfy Assumptions 3. Let the systems interact via* (2) *with a coupling density function $\lambda(\cdot)$ satisfying Assumption 4. Then if Assumption 2 holds true, and if*

$$-\epsilon_1 y^T L_K y - y^T \left(\sigma\lambda^* N L - \gamma^* L_K\right) v(y) \leq 0 \quad \forall y \in \mathbb{R}^N, \tag{16}$$

then the coupled systems on $\mathcal{G}$ synchronize.

Proof of Lemma 1. Clearly, if (16) holds true, then we obtain from (15) that $\dot{\tilde{\mathcal{S}}} \leq -\left(\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \tilde{\rho}(|x_i - x_j|)\right) \leq 0$ with $\dot{\tilde{\mathcal{S}}} = 0$ if and only if $x_1 = x_2 = \cdots = x_N$, because $\tilde{\rho}$ is positive definite. This implies that $\tilde{\mathcal{S}} \geq 0$ has a finite limit for $t \to \infty$. Because $\tilde{\mathcal{S}} \in \mathcal{C}^2$ by Assumption 3, and the solutions of the coupled systems are bounded (Assumption 2), $\dot{\tilde{\mathcal{S}}}$ is uniformly continuous in t. Then Barbalat's Lemma [25] implies that any solution of the coupled systems converge to the set at which $\dot{\tilde{\mathcal{S}}} = 0$, i.e. the coupled systems synchronize. $\square$

The difficulty in Lemma 1 is the verification of (16) due to the presence of the nonlinearity $v(y)$. The following property is key in deriving a verifiable alternative for (16) .

Property 1. *For all $y_i, y_j \in \mathbb{R}$,*

$$(v_i - v_j)\left((y_i - y_j) - (v_i - v_j)\right) \geq 0. \tag{17}$$

Proof of Property 1. For less cluttered notation, we use $v_i = v_i(y_i)$. Since $\lambda(\cdot) \geq 0$ and thus $(v_i - v_j)(y_i - y_j) \geq 0$, $|v_i - v_j| \leq |y_i - y_j|$ and $y_i = y_j$ implies $v_i = v_j$ by the definition of v_i, v_j, we have $0 \leq (v_i - v_j)^2 \leq (v_i - v_j)(y_i - y_j)$. $\square$

From this property, using the S-Lemma [26], we derive the that LMIs (6) imply (16).

Lemma 2. *The LMIs* (6) *imply* (16).

Proof of Lemma 2. For less cluttered notation, we use $v = v(y)$. Inequality (16) reads in matrix form as

$$\begin{pmatrix} y \\ v \end{pmatrix}^T \underbrace{\begin{pmatrix} \epsilon_1 L_K & \frac{1}{2}(\sigma\lambda^* N L - \gamma^* L_K) \\ \star & 0 \end{pmatrix}}_{=:H} \begin{pmatrix} y \\ v \end{pmatrix} \geq 0. \tag{18}$$

Using Property 1, we observe that for any pair (i,j) with $i \neq j$, there exists $\ell \in \{1,2,\ldots,M\}$ such that (17) can be written as

$$\begin{aligned}&\begin{pmatrix} E_\ell^T y \\ E_\ell^T v \end{pmatrix}^T \begin{pmatrix} 0 & \frac{1}{2} \\ \star & -1 \end{pmatrix} \begin{pmatrix} E_\ell^T y \\ E_\ell^T v \end{pmatrix} \\ &= \begin{pmatrix} y \\ v \end{pmatrix}^T \underbrace{\begin{pmatrix} 0 & \frac{1}{2} E_\ell E_\ell^T \\ \star & -E_\ell E_\ell^T \end{pmatrix}}_{=:G_\ell} \begin{pmatrix} y \\ v \end{pmatrix} \geq 0,\end{aligned}$$

where E_ℓ denotes the ℓ^{th} column of E. Invoking the S-Lemma, if there exist real constants $\phi_\ell \geq 0$, $\ell \in$

$\{1, 2, \ldots, M\}$ such that

$$\begin{pmatrix} y \\ v \end{pmatrix}^T H \begin{pmatrix} y \\ v \end{pmatrix} \geq \sum_{\ell=1}^{\frac{N(N-1)}{2}} \phi_\ell \begin{pmatrix} y \\ v \end{pmatrix}^T G_\ell \begin{pmatrix} y \\ v \end{pmatrix},$$

then (18) holds true for all $y \in \mathbb{R}^N$ and $v \in \mathbb{R}^N$, whose components are relate via (17). Denoting $\Phi = \mathrm{diag}(\phi_1, \phi_2, \ldots, \phi_M)$, we obtain (6). □

Combining Lemma 1 and Lemma 2 proves Theorem 2.

Proof of Theorem 3. Because L is symmetric, there exist a unitary matrix U such that

$$U^T L U = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \Lambda \end{pmatrix}, \quad U^T L_K U = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & NI \end{pmatrix},$$

where Λ is a diagonal matrix with the positive eigenvalues of L as entries. (Recall that L has a simple zero eigenvalue under Assumption 1, with an eigenvector (first column of U) in $\mathrm{span}\{\mathbf{1}\}$.) Next, for any $\ell \in \{1, 2, \ldots, M\}$, set $\phi_\ell = \sigma\lambda^* N w_{ij}$ if E_ℓ corresponds to the edge (i, j) in the graph $\mathcal{G}$, or set $\phi_\ell = 0$ otherwise. Then $\Phi \geq 0$ and $\sigma\lambda^* NL = E\Phi E^T$. For this choice of Φ, (6) becomes

$$\begin{pmatrix} \epsilon_1 L_K & -\frac{\gamma^*}{2} L_K \\ \star & \sigma\lambda^* NL \end{pmatrix} \geq 0 \quad \Leftrightarrow \quad \begin{pmatrix} \epsilon_1 I & -\frac{\gamma^*}{2} I \\ \star & \sigma\lambda^* \Lambda \end{pmatrix} \geq 0,$$

which is true if and only if $\sigma\lambda^*\Lambda - \frac{(\gamma^*)^2}{4\epsilon_1} I \geq 0$, which is true if and only if $\sigma \geq \bar{\sigma}$ with $\bar{\sigma}$ in (7). □

Proof of Theorem 4. Consider the network storage function $\mathcal{S} = S(x_1) + S(x_2) + \ldots + S(x_N)$. Clearly, $\mathcal{S}$ is positive definite and radially unbounded. By the SP property, the derivative of $\mathcal{S}$ along solutions of (1), (2) satisfies $\dot{\mathcal{S}} \leq \sum_{i=1}^N y_i u_i = -\sigma\lambda^* y^T L v(y)$, with $v(y)$ as in the proof of Theorem 2. Writing $L = E_\mathcal{G} W E_\mathcal{G}^T$, with $E_\mathcal{G}$ the oriented incidence matrix of the graph (network) $\mathcal{G}$ corresponding to L, we conclude that

$$\begin{aligned} \sum_{i=1}^N y_i u_i &= -\sigma\lambda^* y^T L v(y) = -\sigma\lambda^* y^T E_\mathcal{G} W E_\mathcal{G}^T v(y) \\ &= -\sigma \sum_{(i,j)\in\mathcal{E}} w_{ij}(y_i - y_j) \int_{y_j}^{y_i} \lambda(s) ds \leq 0. \end{aligned}$$

Thus $\dot{\mathcal{S}} \leq 0$ outside the ball of radius $R^* = R\sqrt{N}$ in the coupled systems' state-space. Standard results, cf. Theorem 4.18 of [25], then imply that solutions of the coupled systems are *(uniformly) bounded.* □

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