2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Beyond Common Randomness: Quantum Resources in Decentralized Control

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***Abstract*— Ananthram and Borkar [1] showed that there exist strategies that are consistent with the requirements of a decentralized information structure but are unattainable through the use of common randomness. This opens the question of discovering physically realisable mechanisms that provide access to this region of the strategic space. In our previous work we introduced a class of quantum strategies that allow such access in a two-agent setting. In this paper, we consider the problem of optimal allocation of a k-partite quantum resource amongst n agents, $k < n$. We study the problem of decentralized estimation of a binary source by agents that are informed through independent binary symmetric channels, and face a cost that is homogeneous in their actions. We show a k-partite quantum resource produces the maximum advantage over classical strategies when allocated to the agents with the k most reliable channels.**

I. Introduction

Decentralised control asks collaborative agents to choose their actions as a function of noisy and asymmetric observations to optimize some collective objective of their actions and some unknown underlying state. The precise structure of the observation channels is described by the *information structure* of the problem; the information structure imposes constraints on the set of strategies available to these agents whereby the decentralized control problem is to optimize an expected cost over all strategies that respect these information constraints.

The strategic spaces investigated in decentralized control include the set of deterministic strategies, and stochastic strategies implemented through either behavioural or local randomness. These spaces can be enriched further using externally supplied common randomness. But it is well known that both local and common randomizations do not improve upon the cost of the optimal deterministic strategy; see e.g., [2].

However, there are decision strategies that do not violate the constraints imposed by the information structure of the problem but still lie outside the scope of the above strategies. Ananthram and Borkar [1] demonstrate that there are decision strategies, or equivalently correlations, that respect the information structure but remain inaccessible even with common randomness. They view this as a limitation of common randomness in decentralised control problems of both cooperative and competitive nature.

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It is then natural to ask if there exist other, physically realizable mechanisms that allow an access to the strategic space beyond what common randomness avails, without violating the constraints of the information structure. We affirmatively answered this question in [3] through a novel class of entanglement assisted *quantum* strategies. In particular, we showed that the set of quantum strategies is a strict convex *superset* of the space of stochastic strategies availed by classical common randomness. For a numerical example we showed that such strategies achieve lower costs than any achievable through classical common randomness, thereby demonstrating the existence of a *quantum advantage* in decentralised control.

The key feature of quantum mechanics that makes this advantage possible is *entanglement*. In quantum strategies we introduced in [3] agents choose their actions based on measurements they perform on a multipartite composite quantum system that is entangled across its components. Entanglement has many stunning consequences, but perhaps the most relevant for decentralized control is the existence correlations beyond those possible classically, without any additional communication requirements. Thus, our quantum strategies *do not* violate the information structure, and yet outperform all classical strategies. This is a manifestation of the famed spooky action-at-a-distance produced by entanglement. That entanglement is a real physical phenomenon was demonstrated in ingenious experiments that were awarded this year's Nobel prize in physics.

Though entanglement lends itself naturally to decentralized control as we have shown in [3], practically speaking entanglement remains an expensive resource. Motivated by this we ask the following question in this paper. Consider a problem with n agents that are asymmetrically and noisily informed through independent binary symmetric channels about an underlying state, but are required to collectively estimate this state via a cost function that is homogeneous across the agents. Agents do not signal to each other and thus obey a static information structure [4]. If these agents were given access to a k-partite quantum resource, $k < n$, which k of these agents should it be given to produce the largest improvement over the classical cost? Our first main contribution is in showing that the answer to this question is rather clean – the k-partite resource produces the maximum advantage when assigned to the top k best informed agents.

This implies that in this setting, the channel reliability is, in some sense, a sufficient statistic for optimal allocation of quantum resources. For more general problems, it would be illuminating to discover a more general 'measure' or rubric by which quantum resources should be allocated. Though our setting is specialized, we believe it opens a fascinating new line of inquiry of optimal quantum resource allocation in decentralized control.

The present paper also follows up on our other previous works [5] and [6]. Here we show that the quantum advantage in a two-agent decentralized setting is essentially due to an underlying decision-theoretic difficulty called the 'coordination dilemma'. We study a problem class superstructure and show that the only classes within it that admit a quantum advantage are those with this dilemma. Our present paper is an attempt at furthering the understanding of use of quantum resources in n-agent settings for $n > 2$.

II. Problem Class, Decision Strategies and the Non-Local Advantages

A. Notation

We denote the negation of $a \in \{0,1\}$ by $\overline{a}$. For $a, b \in \{0,1\}$, we denote the binary addition or 'XOR' of a and b by $a \oplus b$. For $\{a_i\}_{i=1}^k \in \{0,1\}$, $\oplus_i a_i := a_1 \oplus a_2 \oplus ... \oplus a_k$. For a set $\mathcal{V}$, we use the notation $\mathcal{V}_{-j}$ to denote $\mathcal{V} \setminus \{j\}$ for a $j \in \mathcal{V}$. Similarly for a tuple $x := (x_1, ..., x_k)$, x_{-j} denotes $(x_1, ..., x_{j-1}, x_{j+1}, ..., x_k)$. Further if $\mathcal{V} \supset K := \{i^{(1)}, \cdots, i^{(m)}\}$, then x_K denotes the tuple $(x_{i^{(1)}}, \cdots, x_{i^{(m)}})$. $\mathcal{P}(\Xi)$ denotes the set of probability distributions on Ξ, and $\mathcal{P}(\mathcal{U}|\Xi)$ denotes the set of conditional probability distributions on $\mathcal{U}$ given an element in Ξ. $\delta(x,y) := \delta_{xy}$ denotes the standard Kronecker Delta with $\delta_{xy} = 1$ if $x = y$ and 0 otherwise.

B. The class of static team problems.

Let $\mathcal{N} := \{1, 2, \ldots, n\}$ be a set of agents. For each agent $i \in \mathcal{N}$, we have an observation $\xi_i \in \Xi_i$ and an action $u_i \in \mathcal{U}_i$. Let $\xi := (\xi_i)_i$, $u = (u_i)_i$, $\Xi := \prod_i \Xi_i$ and $\mathcal{U} := \prod_i \mathcal{U}_i$. Every ξ_i records a noisy observation about an underlying state of nature $\xi_W \in \Xi_W$ which distributed according to $\mathbb{P}(\xi_W) \in \mathcal{P}(\Xi_W)$ and their observations are jointly correlated with this natural state according to a distribution $\mathbb{P}(\xi, \xi_W) \in \mathcal{P}(\Xi \times \Xi_W)$. We denote $u_{-k} := (u_i)_{i \neq k}$, and likewise for $K \subseteq \mathcal{N}$, $u_K = (u_i)_{i \in K}$, $u_{-K} = (u_i)_{i \notin K}$, and similarly for other variables. The agents face a static team decision problem $D := (\mathbb{P}, \mathcal{U}, \Xi \times \Xi_W, \ell)$ where they jointly attempt to minimize a joint cost $\ell(u, \xi_W)$ dependent on their actions u and the natural state ξ_W. We will specialize later in the next section to a specific cost structure ℓ.

C. Decision strategies for static teams.

We work in the space of occupation measures following [7] so that a decision strategy for the above problem class is given by a distribution $Q \in \mathcal{P}(\mathcal{U}|\Xi)$. In the language of stochastic control, any such strategy can be thought of as *occupation measure* as done in Markov decision processes [8]. Under a strategy Q, the expected cost of a problem $D = (\mathbb{P}, \mathcal{U}, \Xi \times \Xi_W, \ell)$ is given by a linear objective in Q:

$$J(Q; D) = \sum_{u, \xi, \xi_W} \mathbb{P}(\xi, \xi_W) Q(u|\xi) \ell(u, \xi_W). \qquad (1)$$

We classify the space $\mathcal{P}(\mathcal{U}|\xi)$ of decision strategies based upon specification of further restrictions on the strategies. Our strategies will be distinguished based on the resources available to the agents.

1) Space of local strategies $\mathcal{L}$:: We first introduce the space of strategies, Π in which agents choose actions as a function of only their (local) information. Toward specifying $\mathcal{L}$, first denote a locally randomized policy by the tuple $\{Q_i\}_{i \in \mathcal{N}}$ where $Q_i \in \mathcal{P}(\mathcal{U}_i|\Xi_i)$. We let Π be the set of all occupation measures that correspond generated from such policies, defined as follows.

$$\Pi := \{Q | \exists\ Q_i \in \mathcal{P}(\mathcal{U}_i|\Xi_i) : Q(u|\xi) \equiv \prod_i Q(u_i|\xi_i)\}. \qquad (2)$$

a) Common randomness: Suppose now that in addition to their local information, the agents have access to a passively generated random variable $w \in \mathcal{W} : \ w \sim \Phi \in \mathcal{P}(\mathcal{W})$ in order to classically correlate their actions beyond the inherent correlation provided by that among their observations. We define $\mathcal{L}$ as the set of strategies so achievable with arbitrary w, Φ

$$\mathcal{L} = \{Q | \exists \mathcal{W}, \Phi \in \mathcal{P}(\mathcal{W}), Q_i \in \mathcal{P}(\mathcal{U}_i|\Xi_i) : \\ Q(u|\xi) \equiv \sum_{w \in \mathcal{W}} \Phi(w) \prod_i Q_i(u_i|\xi_i, w)\}. \qquad (3)$$

We refer to $\mathcal{L}$ as the space of local or *classical* strategies.

2) Space of quantum strategies $\mathcal{Q}$: We briefly specify this space of strategies here; a more detailed description can be found in [3]. $\mathcal{Q}$ is the space of decision strategies allowed by quantum physical reality. A strategy $Q \in \mathcal{Q}$ is specified by a tuple $(\{\mathcal{H}_i\}_i, \rho, \{P^{(i)}_{u_i}(\xi_i)\}_{i, \xi_i, u_i})$ where $\mathcal{H}_i$ is a finite dimensional Hilbert space for each $i \in \mathcal{N}$, $\rho \in \bigotimes_i \mathcal{H}_i =: \mathcal{H}$ is such that $\rho \succeq 0, \mathrm{Tr}\, \rho = 1$ and $P^{(i)}_{u_i}(\xi_i)$ denotes a projection operator in $\mathcal{H}_i$ for each $i, \xi_i \in \Xi_i, u_i \in \mathcal{U}_i$. ρ is a composite quantum system whose subsystems are accessible to each decision maker i for measurement. The projection operators are required to satisfy for each i, ξ_i:

$$\sum_{u_i} P^{(i)}_{u_i}(\xi_i) = \mathbf{I};$$
$$P^{(i)}_{u_i}(\xi_i) P^{(i)}_{u_i'}(\xi_i) = \delta(u_i, u_i') P^{(i)}_{u_i}(\xi_i).$$

The occupation measure corresponding to the above-mentioned strategy Q is then computed as

$$Q(u|\xi) = \mathrm{Tr}\left(\rho \bigotimes_i P^{(i)}_{u_i}(\xi_i)\right). \qquad (4)$$

A quantum strategy is executed as follows. An order of measurements is decided (say 1 to n). Upon observing ξ_1, player 1 performs a local measurement on the shared quantum resource ρ in the POVM $\{P_{u_1}(\xi_1)\}_{u_1 \in \mathcal{U}_1}$, then player 2 performs his measurement, and so on. The action is then determined by the measurement outcome $(u_1, \ldots, u_n)$. We refer the reader to [9] for details on POVMs and composite systems.

3) Space of no-signalling strategies $\mathcal{NS}$: This is the set of all decision strategies $Q \in \mathcal{P}(\mathcal{U}|\Xi)$ that respect the absence of signalling between the agents. This is provided by imposing that u_i is conditionally independent of ξ_{-i} given ξ_i for each $i \in \mathcal{N}$. Thus for each $i \in \mathcal{N}$,

$$\mathcal{NS} = \{Q | Q(u_i|\xi_i, \xi_{-i}) \equiv Q(u_i|\xi_i) \; \forall i \in \mathcal{N}\}.$$

These restrictions enforce the condition $Q(u_K|\xi_K, \xi_{-K}) = Q(u_K|\xi_K)$ for all $K \subset \mathcal{N}$ [10]. It is easy to see that $Q \in \mathcal{NS}$ is equivalent to,

$$\sum_{u_i} Q(u|\xi_i, \xi_{-i}) = \sum_{u_i} Q(u|\xi_i', \xi_{-i}) \; \forall \; \xi_i, \xi_i', \xi_{-i}, u_{-i}, \quad (5)$$

and $Q \in \mathcal{P}(\mathcal{U}|\Xi)$. Notice that these are finite a set of linear equality constraints on $\mathcal{P}(\mathcal{U}|\Xi)$, and hence $\mathcal{NS}$ is a polytope.

4) Centralised polytope $\mathcal{P}(\mathcal{U}|\Xi)$: We call the set of all distributions $\mathcal{P}(\mathcal{U}|\Xi)$ the centralised polytope. This is indeed the set of all strategies when the information in the problem is centralised.

Later we will introduce additional strategic classes. For any class S, say $S = \Pi, \mathcal{L}, \mathcal{Q}, \mathcal{NS}$ and $\mathcal{P}(\mathcal{U}|\Xi)$, denote the infimum of the expected cost of a problem D over set S as

$$J_S^*(D) = \inf_{Q \in S} J(Q; D). \quad (6)$$

Denote the centralised optimum $J^*_{\mathcal{P}(\mathcal{U}|\Xi)}(D)$ by $J^{**}(D)$. It can be shown that our strategic spaces obey the inclusion [6]

$$\mathcal{L} \subseteq \mathcal{Q} \subseteq \mathcal{NS} \subseteq \mathcal{P}(\mathcal{U}|\Xi), \quad (7)$$

from which, it then follows that

$$J^*_{\mathcal{L}}(D) \geq J^*_{\mathcal{Q}}(D) \geq J^*_{\mathcal{NS}}(D) \geq J^{**}(D). \quad (8)$$

Further, since $J(Q; D)$ is a linear objective in Q and $\mathcal{L} = \mathrm{conv}(\Pi)$, it follows that $J^*_\Pi(D) = J^*_{\mathcal{L}}(D)$. We say that at instance D admits a quantum no-signalling, or a centralisation advantage if the following equations respectively hold:

$$J^*_{\mathcal{Q}}(D) < J^*_{\mathcal{L}}(D),$$
$$J^*_{\mathcal{NS}}(D) < J^*_{\mathcal{L}}(D),$$
$$J^{**}(D) < J^*_{\mathcal{L}}(D).$$

D. Limitations of common randomness

We now come to a key matter in decentralized control, which is also our main motivation for this paper. Prima-facie, it appears plausible that decentralized players tied by a static information structure can access strategies in $\mathcal{NS}$; after all the only requirement of $\mathcal{NS}$ is that players cannot communicate, which is the essence of the information structure of the problem. But the natural question then is, what is the mechanism by which they can access this region? Ananthram and Borkar [1] have highlighted the inherent limitation of common randomness in achieving this access. In [1] they demonstrate a strategy that lies in $\mathcal{NS}$, but cannot be expressed in the form (3), i.e., as arising by providing common randomness to the decision makers. Thus the inclusion $\mathcal{L} \subseteq \mathcal{NS}$ is, in general, strict.

One can then ask, what physically realizable mechanism is available that does not violate the no-signalling requirement, and yet provides access to the $\mathcal{NS}$ region? In [3], we construct a class of physically-implementable quantum strategies $\mathcal{Q}$ that escape the limitations of common randomness and produce strategies in $\mathcal{NS} \backslash \mathcal{L}$. We showed that there exist decentralized control problems with the property that there exists a quantum strategy that strictly outperforms all strategies in $\mathcal{L}$, thereby demonstrating the existence of a *quantum advantage* in decentralized control.

E. Partitioned quantum strategies

Let $\mathsf{P} = \{K_1, \cdots, K_p\}$ be a partition on $\mathcal{N}$. Denote $u_K = (u_i)_{i \in K}$ and similarly $\xi_K = (\xi_i)_{i \in K}$ for $K \in \mathsf{P}$. Consider a strategy Q that allows agents within each K_j access to a shared, possibly entangled, quantum resource. Hence, if $Q = (\{\mathcal{H}_i\}_i, \rho, \{P^{(i)}_{u_i}(\xi_i)\}_{i, \xi_i, u_i})$, then ρ decomposes as $\rho = \bigotimes_i \rho_{K_i}$ where $\rho_K \in \bigotimes_{i \in K} \mathcal{H}_i$. It follows that

$$Q(u|\xi) = \prod_{j=1}^{p} Q_{K_j}(u_{K_j}|\xi_{K_j}) = \prod_j \mathrm{Tr}(\rho_{K_j} \bigotimes_{i \in K_j} P^{(i)}_{u_i}(\xi_i)). \quad (9)$$

Denote $Q_{-K_i} = \prod_{j \neq i} Q_{K_j}(u_{K_j}|\xi_{K_j})$. We call such a strategy Q P-quantum, and by $\mathcal{Q}_{\mathsf{P}} \subseteq \mathcal{Q}$ we denote the set of such strategies. If an element of the partition P is singleton, i.e. say $K_j = \{k_j\}$, then we call the agent k_j a *classical* agent. $\mathcal{Q}_{\mathsf{P}}$ is thus the set of decision strategies where agents across different elements in the partition P do not share a resource entangled across partitions. Notice that

$$J^*_{\mathcal{L}}(D) \geq J^*_{\mathcal{Q}_{\mathsf{P}}}(D) := \inf_{Q \in \mathcal{Q}_{\mathsf{P}}} J(Q; D) \geq J^*_{\mathcal{Q}}(D).$$

In the following proposition, we quickly show that it is optimal for each classical agent to play deterministically.

Proposition 2.1: Consider a singleton element of the partition P, say, $K = \{k\} \in \mathsf{P}$. For each $Q \in \mathcal{Q}_{\mathsf{P}}$, there exists a $\gamma^*_k : \Xi_k \to \mathcal{U}_k$ such that $J(Q; D) \geq J(Q_{\gamma^*_k}; D)$ where $Q_{\gamma^*_k}(u|\xi) = \delta(u_k, \gamma^*_k(\xi_k)) Q(u_{-k}|\xi_{-k})$.

Proof: Let $Q \in \mathcal{Q}_{\mathsf{P}}$. The result follows from linearity of $J(Q; D)$ in Q and structure (9). Fixing $Q(u_{-k}|\xi_{-k})$, the cost is linear in Q_k. It follows that a deterministic strategy for the classical player k as required exists. ■

Let k-$\mathcal{Q}$ be the set of all quantum strategies where a subset of k agents $K \subset \mathcal{N}$ share a quantum resource, and other $n - k$ agents are classical. Then k-$\mathcal{Q} = \bigcup_{\mathsf{P}} \mathcal{Q}_{\mathsf{P}}$ where the union is over all partitions P with $n - k$ singleton elements.

III. Decentralized Estimation with Independent Observations

Consider a team of n agents where the agents ought to collectively produce an estimate of a state ξ_W that is partially observed by each of them. The cost of the problem is given by the following general expression that penalizes an incorrect estimate depending upon ξ_W

$$\ell(u, \xi_W) := -l(\xi_W)\delta(\xi_W, f(u)); \; l(\xi_W) \geq 0. \quad (10)$$

where $f : \mathcal{U} \to \Xi_W$ and $l : \Xi_W \to \mathbb{R}$. We assume that $\mathcal{U}_i \equiv \mathcal{U}_j$ for all i, j and that f admits a permutation symmetry with respect to the agents so that if the tuple u' is a permutation of u, then $f(u') = f(u)$. It follows that $\ell(u', \xi_W) = \ell(u, \xi_W)$. We also assume that irrespective of other agents' actions, each agent has a complete sway on their collective estimate so that $\mathrm{range}(f(\cdot, u_{-i})) = \Xi_W$ for all $u_{-i} \in \mathcal{U}_{-i}$. In this article, we work with $\mathcal{U}_i = \Xi_W = \{0, 1\}$. It is easy to show that it is sufficient to consider $f(u) = \bigoplus_i u_i$ upto relabelling of agents' actions. Each agent i records its observations ξ_i of ξ_W through independent binary symmetric channels so that the following hold for their joint prior:

$$\mathbb{P}(\xi_i|\xi_W) = \begin{cases} \lambda_i & \xi_i = \xi_W \\ (1-\lambda_i) & \xi_i \neq \xi_W \end{cases},$$
$$\mathbb{P}(\xi, \xi_W) = \mathbb{P}(\xi_W) \prod_{i \in \mathcal{N}} \mathbb{P}(\xi_i|\xi_W), \tag{11}$$

where $1/2 \leq \lambda_i \leq 1$ for each $i \in \mathcal{N}$. Without loss of generality, we assume $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ so that the agent 1 is the most 'well-informed' agent. We now work with the instance $D = (\mathcal{N}, \mathbb{P}, \mathcal{U}, \Xi \times \Xi_W, \ell)$ with elements of the tuple as described above.

A. Optimal strategies for classical agents.

The following lemma is a key ingredient of the rest of our analysis. We find that *over all no-signalling strategies* it is optimal for every classical agent except $i = 1$ to play the constant, deterministic strategy: $\gamma_i(\xi_i) \equiv 0$.

Lemma 3.1: Let $j \in \mathcal{N}$ be a classical agent and consider a joint strategy $Q(u|\xi) = \delta(u_j, \gamma_j(\xi_j))Q_{-j}(u_{-j}|\xi_{-j})$ where γ_j is a deterministic non-constant strategy,i.e., $\gamma_j(0) \neq \gamma_j(1)$. Then the cost of Q is bounded below as follows: $J(Q; D) \geq -\lambda_j \sum l(\xi_W)\mathbb{P}(\xi_W)$.

Proof: Denote $Q_{-j}(u_{-j}|\xi_{-j})\mathbb{P}(\xi_{-j}|\xi_W) =: R(u_{-j}, \xi_{-j}|\xi_W)$. Clearly, $\sum_{u_{-j}, \xi_{-j}} R(u_{-j}, \xi_{-j}|\xi_W) = 1$. Now notice that

$$J(Q; D) = \sum_{\xi_W, \xi, u} \mathbb{P}(\xi_W)\ell(u, \xi_W)\mathbb{P}(\xi_j|\xi_W)\delta(u_j, \gamma_j(\xi_j))R(u_{-j}, \xi_{-j}|\xi_W).$$

Now summing over ξ_j and using $\mathbb{P}(\xi_j = \xi_W|\xi_W) = \lambda_j$ gives

$$\begin{aligned} &J(Q; D) \\ &= \sum_{\xi_W, u, \xi_{-j}} \mathbb{P}(\xi_W)(\lambda_j \delta(u_j, \gamma_j(\xi_W))\ell(u, \xi_W) \\ &+(1-\lambda_j)\delta(u_j, \gamma_j(\overline{\xi}_W))\ell(u, \xi_W)\big) R(u_{-j}, \xi_{-j}|\xi_W) \\ &= \sum_{\xi_W, u_{-j}, \xi_{-j}} \mathbb{P}(\xi_W)(\lambda_j \ell(\gamma_j(\xi_W), u_{-j}, \xi_W) \\ &+(1-\lambda_j)\ell(\gamma_j(\overline{\xi}_W), u_{-j}, \xi_W)\big) R(u_{-j}, \xi_{-j}|\xi_W) \end{aligned}$$

where we have used the definition of $\delta(\cdot, \cdot)$ to sum over u_j. Now observe that

$$\min_{\gamma_j} \lambda_j \ell(\gamma_j(\xi_W), u_{-j}, \xi_W) + (1-\lambda_j)\ell(\gamma_j(\overline{\xi}_W), u_{-j}, \xi_W) = -\lambda_j l(\xi_W) \ \forall u_{-j}$$

since $\ell(u, \xi_W) \in \{0, -l(\xi_W)\}$, $\gamma(\xi_W) \neq \gamma(\overline{\xi}_W)$ and $\ell(u_j, u_{-j}, \xi_W) \neq \ell(\overline{u}_j, u_{-j}, \xi_W)$. Hence,

$$\begin{aligned} J(Q; D) &\geq -\sum_{\xi_W, u, \xi_{-j}} \mathbb{P}(\xi_W)R(u_{-j}, \xi_{-j}, \xi_W)\lambda_j l(\xi_W) \\ &= -\lambda_j \sum_{\xi_W} l(\xi_W)\mathbb{P}(\xi_W). \end{aligned}$$

This completes the proof. ∎

The first corollary of this lemma provides a classical optimum for our problem.

Proposition 3.2: Let

$$\widehat{J} = \min(-\chi(0)\mathbb{P}(0), -\lambda_1 \sum_{\xi_W} \mathbb{P}(\xi_W)\chi(\xi_W), -\chi(1)\mathbb{P}(1)), \tag{12}$$

Then an optimal classical strategy for D is given by:

$$\gamma_i^*(\xi_i) \equiv 0 \ \forall \ i \neq 1;$$
$$\gamma_1^*(\xi_1) = \begin{cases} 0 & \text{if } \widehat{J} = -\mathbb{P}(0)\chi(0) \\ 1 & \text{if } \widehat{J} = -\mathbb{P}(1)\chi(1) \\ \xi_1 & \text{if } \widehat{J} = -\lambda_1 \sum_{\xi_W} \mathbb{P}(\xi_W)\chi(\xi_W). \end{cases} \tag{13}$$

Moreover $J_{\mathcal{L}}^*(D) = \widehat{J}$.

Proof: To begin, one can verify that the specified strategy γ^* gives a cost exactly equal to $\widehat{J}$. Let Q be a deterministic strategy $Q(u|\xi) = \prod_{i \in \mathcal{N}} \delta(u_i, \gamma_i(\xi_i))$ where $\gamma_i : \Xi_i \to \mathcal{U}_i$ for each i. From Lemma 3.1, we know that for any j, if $\gamma_j(0) \neq \gamma_j(1)$ then

$$\begin{aligned} J(Q; D) &\geq -\lambda_j \sum_{\xi_W} l(\xi_W)\mathbb{P}(\xi_W) \\ &\geq -\lambda_1 \sum_{\xi_W} l(\xi_W)\mathbb{P}(\xi_W). \end{aligned}$$

$\geq \widehat{J}$. On the other hand, if $\gamma_j(0) = \gamma_j(1)$ for all j, then $J(Q; D) \in \{-l(0)\mathbb{P}(0), -l(1)\mathbb{P}(1)\}$, thus $J(Q; D) \geq \widehat{J}$. Thus, the cost of all deterministic strategies is bounded below by $\widehat{J}$, which is achieved by γ^*. ∎

The following proposition eases our subsequent analysis when dealing with optimality in quantum settings. We quickly state and prove it.

Proposition 3.3: Consider a partition P of $\mathcal{N}$ with $\{j\}$ as its singleton element and suppose that $J_{\mathcal{Q}_{\mathsf{P}}}^*(D) < J_{\mathcal{L}}^*(D)$. Then for each $Q \in \mathcal{Q}_{\mathsf{P}}$ of the form $Q(u|\xi) = \delta(u_j, \gamma_j(\xi_j))Q(u_{-j}|\xi_{-j})$ that obeys $J(Q; D) < J_{\mathcal{L}}^*(D)$, there is a $Q' \in \mathcal{Q}_{\mathsf{P}}$ of the form $Q'(u|\xi) = \delta(u_j, 0)Q'(u_{-j}|\xi_{-j})$ such that $J(Q'; D) \leq J(Q; D)$.

Proof: Since $J(Q; D) < J_{\mathcal{L}}^*(D)$, from Proposition 3.2 we also have $J(Q; D) < -\lambda_1 \sum \mathbb{P}(\xi_W)l(\xi_W) \leq -\lambda_j \sum \mathbb{P}(\xi_W)l(\xi_W)$ (recall $\lambda_j \leq \lambda_1$ for all j). We can conclude from Lemma 3.1 that $\gamma_j(0) = \gamma_j(1)$.

If $\gamma_j \equiv 0$, then we are done. Otherwise if $\gamma_j(0) = 1$, pick a $K \in \mathsf{P}$ such that $|K| > 1$. Then Q decomposes as $Q(u|\xi) = \delta(u_j, 1)Q_{-K\cup\{j\}}(u_{-K\cup\{j\}}|\xi_{-K\cup\{j\}})Q_K(u_K|\xi_K)$ where for some state $\rho_K \in \bigotimes_{r\in K} \mathcal{H}_r$ and projectors $\{P_{u_r}^r(\xi_r)\}_{r, u_r, \xi_r}$,

$$Q_K(u_K|\xi_K) = \mathrm{Tr}\left(\rho_K \bigotimes_r P_{u_r}^r(\xi_r)\right).$$

Now, pick a $k \in K$ and define $P^{k\prime}_{u_k}(\xi_k) = P^k_{\overline{u}_k}(\xi_k)$ for all $u_k, \xi_k \in \{0,1\}$ (recall $\overline{u}_k$ is the negation of u_k). Further define $Q'_K(u_K|\xi_K) := \mathrm{Tr}(\rho_K P^{k\prime}_{u_k}(\xi_k) \bigotimes_{r\neq k} P^r_{u_r}(\xi_r))$, and $Q'(u|\xi) := \delta(u_j,0) Q_{-K\cup\{j\}}(u_{-K\cup\{j\}}|\xi_{-K\cup\{j\}}) Q'_K(u_K|\xi_K)$.

It follows that $\ell(1,u_k,u_{-\{j,k\}})Q(u_j=1,u_k,u_{-\{j,k\}}|\xi) = \ell(0,\overline{u}_k,u_{-\{j,k\}})Q'(u_j = 0,\overline{u}_k,u_{-\{j,k\}}|\xi)$ and $Q(u_j=1,u_k,u_{-\{j,k\}}|\xi) = Q'(u_j=1,\overline{u}_k,u_{-\{j,k\}}|\xi) = 0$. for all $u_k \in \{0,1\}$.
Hence $\sum_{u_j,u_k} \ell(u,\xi_W)\delta(u_j,0)Q'_{-j}(u_{-j}|\xi_{-j}) = \sum_{u_j,u_k} \ell(u,\xi_W)\delta(u_j,1)Q_{-j}(u_{-j}|\xi_{-j})$.
It follows that,

$$\begin{aligned} &J(Q';D)\\ &= \sum_{\xi_W,\xi,u_{-\{j,k\}}} \mathbb{P}(\xi,\xi_W) \sum_{u_j,u_k} \ell(u,\xi_W)\delta(u_j,0)Q'_{-j}(u_{-j}|\xi_{-j})\\ &= \sum_{\xi_W,\xi,u_{-\{j,k\}}} \mathbb{P}(\xi,\xi_W) \sum_{u_j,u_k} \ell(u,\xi_W)\delta(u_j,1)Q_{-j}(u_{-j}|\xi_{-j})\\ &= J(Q;D). \end{aligned}$$

This completes the proof. ■

IV. Non-Classical Resources and Agent Information.

A. *Allocation of a k-partite non-classical resource.*

We now come to main theoretical contributions of the paper, namely the optimal allocation of quantum resources. Suppose that the agents have the assistance of a k-partite quantum resource. Thus their strategy is chosen from the set $k\text{-}\mathcal{Q} := \bigcup_{\mathsf{P}} \mathcal{Q}_{\mathsf{P}}$ where the union is over all partitions P with $n-k$ classical agents and a k-partite quantum resource shared amongst the remaining k agents. Denote, $J^*_{k\text{-}\mathcal{Q}}(D) := \inf_{Q\in k\text{-}\mathcal{Q}} J(Q;D)$.

We show that an optimal allocation of this resource corresponds to the k agents with *most information* (agents 1 to k) sharing the resource and the other $n-k$ agents playing classically. In other words if one has access to a k-partite quantum resource it can be best exploited by agents that are most informative. This illustrates a simple role of the agents' information in the allocation problem for the assumed correlation structure of observations (11). Although our result is limited to a particular setting, we find it nontrivial that the *value of this resource* in a team is largest when given to best informed.

The following lemma is a precursor to the proposition that follows. It shows that the cost improves when a less informed agent with access to a shared quantum resource is replaced by a more informed agent.

Lemma 4.1: Consider a partition P where a set K of k agents share a k-partite quantum resource and all other agents are classical. Consider an arbitrary strategy $Q \in \mathcal{Q}_{\mathsf{P}}$ and let agent $j \in K$ If $i \notin K$ is such that $\lambda_i \geq \lambda_j$, then define $K' = (K\setminus\{j\})\cup\{i\}$. Let P' be another partition such that such that the k agents in K' share the quantum resource. We claim that there is a $Q'' \in \mathrm{conv}(\mathcal{Q}_{\mathsf{P}'})$ such that $J(Q;D) \geq J(Q'';D)$ and hence it holds that $J^*_{\mathcal{Q}_{\mathsf{P}'}}(D) \leq J^*_{\mathcal{Q}_{\mathsf{P}}}(D)$.

Proof: Let Q be as in the statement of the lemma. First consider the case where $J(Q;D) = J^*_{\mathcal{L}}(D)$. It suffices to take Q'' as the optimal classical strategy specified by Proposition 3.2. Clearly $Q'' \in \mathcal{Q}_{\mathsf{P}'}$ and $J(Q;D) \geq J(Q'';D)$ by hypothesis. This shows the existence of the $Q'' \in \mathcal{Q}_{\mathsf{P}'}$ as required by the statement of the lemma.

Now consider $J(Q;D) < J^*_{\mathcal{L}}(D)$. Following Proposition 3.3 Q is in the form $Q(u|\xi) = \delta(u_{-K},0)\,\mathrm{Tr}\left(\rho_K \bigotimes_{r\in K} P^r_{u_r}(\xi_r)\right)$ for some state $\rho_K \in \bigotimes_r \mathcal{H}_r$ and respective projectors $\{P^r_{u_r}(\xi_r)\}_{r,u_r,\xi_r}$. This assumption is without loss of generality because if Q is not in this form, we can leverage Proposition 3.3 and find another $Q^o \in \mathcal{Q}_{\mathsf{P}}$ in the desired form.

We define two strategies $\widehat{Q}$ and $\widetilde{Q} \in \mathrm{conv}(\mathcal{Q}_{\mathsf{P}'})$ using Q as follows. In defining $\widehat{Q}$, we simply exchange the agents i and j in Q. So

$$\widehat{Q}(u|\xi) = \delta(u_{-K'},0)\,\mathrm{Tr}\left(\widehat{\rho}_{K'} \bigotimes_{r\in K'} \widehat{P}^r_{u_r}(\xi_r)\right)$$

where $\widehat{\rho}_{K'} = \rho_K \in \bigotimes_{r\in K'} \widehat{\mathcal{H}}_r$ with $\widehat{\mathcal{H}}_r \equiv \mathcal{H}_r, \widehat{P}^r_{u_r}(\xi_r) \equiv P^r_{u_r}(\xi_r)$ for all $r \in K'\setminus\{i\}$ and $\widehat{\mathcal{H}}_i = \mathcal{H}_j, \widehat{P}^i_{u_i}(\xi_i) = P^j_{u_j}(\xi_j)$. By construction, $\widehat{Q} \in \mathcal{Q}_{\mathsf{P}'}$.

In defining $\widetilde{Q}$, we exchange the agents i and j with a small modification. So $\widetilde{Q}(u|\xi) = \delta(u_{-K'},0)\,\mathrm{Tr}\left(\widetilde{\rho}_{K'} \bigotimes_{r\in K'} \widetilde{P}^r_{u_r}(\xi_r)\right)$ where $\widetilde{\rho}_{K'} = \rho_K \in \bigotimes_{r\in K'} \widetilde{\mathcal{H}}_r$ with $\widetilde{\mathcal{H}}_r \equiv \mathcal{H}_r, \widetilde{P}^r_{u_r}(\xi_r) \equiv P^r_{u_r}(\xi_r)$ for all $r \in K'\setminus\{i\}$ and $\widetilde{\mathcal{H}}_i = \mathcal{H}_j, \widetilde{P}^i_{u_i}(\xi_i) = P^j_{u_j}(\overline{\xi}_j)$. Again, by construction, $\widetilde{Q} \in \mathcal{Q}_{\mathsf{P}'}$. Now we expand the costs of each of these two (We denote the tuple $w_i = \xi_i, u_i$ for compact presentation). Summing $J(\widehat{Q};D)$ over ξ_i,

$$\begin{aligned} J(\widehat{Q};D) = \sum_{\xi_W,w_{-i},u_i} &\mathbb{P}(\xi_W)\ell(u,\xi_W)(\lambda_i\widehat{Q}(u|\xi_W,\xi_{-i})\\ &+ (1-\lambda_i)\widehat{Q}(u|\overline{\xi}_W,\xi_{-i})) \end{aligned} \tag{14}$$

Notice by construction that $\widehat{Q}(u|\xi_i,\xi_{-i}) = \widetilde{Q}(u|\overline{\xi}_i,\xi_{-i})$ holds. Hence, summing $J(\widetilde{Q};D)$ over ξ_i and substituting $\widehat{Q}$ in place of $\widetilde{Q}$:

$$\begin{aligned} J(\widetilde{Q};D) = \sum_{\xi_W,w_{-i},u_i} &\mathbb{P}(\xi_W)\ell(u,\xi_W)[\lambda_i\widetilde{Q}(u|\xi_W,\xi_{-i})\\ &+ (1-\lambda_i)\widetilde{Q}(u|\overline{\xi}_W,\xi_{-i})]\\ = \sum_{\xi_W,w_{-i},u_i} &\mathbb{P}(\xi_W)\ell(u,\xi_W)[\lambda_i\widehat{Q}(u|\overline{\xi}_W,\xi_{-i})\\ &+ (1-\lambda_i)\widehat{Q}(u|\xi_W,\xi_{-i})] \end{aligned} \tag{15}$$

Define $p_{ij} := (\lambda_i+\lambda_j-1)/(2\lambda_i-1)$. Since $1/2 \leq \lambda_j \leq \lambda_i$, it holds that $0 \leq p_{ij} \leq 1$. Also then, $1-p_{ij} = (2\lambda_j-1)/(2\lambda_i-1) \in [0,1]$. It is easy to verify

$$\begin{aligned} p_{ij}\lambda_i + (1-p_{ij})(1-\lambda_i) &= \lambda_j\\ p_{ij}(1-\lambda_i) + (1-p_{ij})(\lambda_i) &= 1-\lambda_j \end{aligned} \tag{16}$$

Now, define $Q'(u|\xi) = p_{ij}\widehat{Q}(u|\xi) + (1-p_{ij})\widetilde{Q}(u|\xi)$. By definition, $Q' \in \mathrm{conv}\mathcal{Q}_{\mathsf{P}'}$. Now we have from construction of $\widehat{Q}$,

$$\begin{aligned}&\widehat{Q}(u_i = a, u_j = b, u_{-\{i,j\}} | \xi_i = x, \xi_j = y, \xi_{-\{i,j\}})\\ &= \delta(u_j = a, u_{\{-K,j\}}, 0) \operatorname{Tr}(\rho_K P_b^i(y) \bigotimes_{r \in K \setminus i} P_{u_r}^r(\xi_r))\\ &= Q(u_i = b, u_j = a, u_{-\{i,j\}} | \xi_i = y, \xi_j = x, \xi_{-\{i,j\}}) \quad (17)\end{aligned}$$

and permutation symmetry of the cost with respect to the agents,

$$\begin{aligned}&\ell(u_i = a, u_j = b, u_{-\{i,j\}}, \xi_W)\\ &\qquad = \ell(u_j = a, u_i = b, u_{-\{i,j\}}, \xi_W). \quad (18)\end{aligned}$$

Now, from equations (14), (15) and (16), we have $J(Q'; D) =$

$$\sum_{\xi_W, \xi_{-i}, u_i} \mathbb{P}(\xi_W) \ell(u, \xi_W) [\lambda_j Q(u|\xi_W, \xi_{-i}) + (1 - \lambda_j) Q(u|\overline{\xi}_W, \xi_{-i})]$$

Exchanging i and j in (19), it holds from equations (17) and (18) that $J(Q'; D)$

$$\begin{aligned}&= \sum_{\xi_W, \xi_{-j}, u_j} \mathbb{P}(\xi_W) \ell(u, \xi_W) [\lambda_j \widehat{Q}(u|\xi_W, \xi_{-j})\\ &\qquad + (1 - \lambda_j) \widehat{Q}(u|\overline{\xi}_W, \xi_{-j})] = J(Q; D). \quad (19)\end{aligned}$$

Since $Q' \in \operatorname{conv} \mathcal{Q}_{\mathsf{P}'}$, it holds because of linearity of the cost in Q' that $\exists Q'' \in \mathcal{Q}_{\mathsf{P}'}$ such that $J(Q''; D) \leq J(Q; D)$. This completes the proof. ∎

We now show our main quantum resource allocation result.

Proposition 4.2 (Resource Allocation): Let $\mathsf{P}^* = \{\{1, \cdots, k\}, \{k+1\}, \cdots, \{n\}\}$ be a partition on $\mathcal{N}$ where $K^* := \{1, \cdots, k\}$ is the set of agents that share a k-partite quantum resource. Then $J^*_{\mathcal{Q}_{\mathsf{P}^*}}(D) = J^*_{k\text{-}\mathcal{Q}}(D)$.

Proof: Let P be a partition with $n-k$ singleton elements and $K \in \mathsf{P}$ such that it contains the other k elements of $\mathcal{N}$. Indeed then $|K| = k$. Hence $\mathcal{Q}_{\mathsf{P}} \in k\text{-}\mathcal{Q}$. Now notice that since $|K^*| = |K| = k$ and K^* is the set of k minimum elements in $\mathcal{N}$, we construct a bijection $g : K \to K^*$ such that $g(i) \leq i$, i.e. $\lambda_i \leq \lambda_{g(i)}$ for all $i \in K$, and $g(i) = i$ for all $i \in K \cap K^*$. Denote $\kappa := K \setminus (K^* \cap K)$ and $\kappa^* := K^* \setminus (K^* \cap K)$. Let $g(i) = i$ for all $i \in K^* \cap K$. Order κ and κ^* in ascending order such that $\kappa := \{i^{(1)}, \cdots, i^{(m)}\}$ and $\kappa^* := \{j^{(1)}, \cdots, j^{(m)}\}$ and let $g(i^{(r)}) = j^{(r)}$ for all $r \in \{1, \cdots, m\}$. This defines the desired bijection g.

Define $K_0 = K$, and $K_r = (K_{r-1} \setminus \{i^{(r)}\}) \cup \{g(i^{(r)})\}$ for $r \leq m$. Clearly, $K_m = K^*$. Also define $\mathsf{P}^{(r)}$ such that $\mathcal{Q}_{\mathsf{P}^{(r)}} \in k\text{-}\mathcal{Q}$ with $K_r \in \mathsf{P}^{(r)}$. Now apply Lemma 4.1 by assigning the following values to variables in its statement : $i = g(i^{(r)})$, $j = g(i^{(r)})$, $\mathsf{P} = \mathsf{P}^{(r-1)}$ and $\mathsf{P}^* = \mathsf{P}^{(r)}$. The lemma asserts that $J^*_{\mathcal{Q}_{\mathsf{P}^{(r)}}}(D) \leq J^*_{\mathcal{Q}\mathsf{P}^{(r-1)}}(D)$ holds for each $r \in 1, \cdots, m$. Inductively, it holds that $J^*_{\mathcal{Q}_{\mathsf{P}'}} = J^*_{\mathcal{Q}_{\mathsf{P}(m)}}(D) \leq J^*_{\mathcal{Q}_{\mathsf{P}}}(D)$. Since P was arbitrary, the inequality holds for all partitions with $n-k$ elements and the proposed is established. ∎

Thus, the maximum advantage possible over all quantum resources shared amongst arbitrary subsets of k agents is availed when the k most informed agents share these resources. Although the above result talks of the *lowest* cost over *all* resources, from the proof of Lemma 4.1 it can be seen that the same can be claimed for any fixed resource.

V. Concluding Remarks

We initiated a study on allocation of limited quantum resources amongst a team of n agents constrained by a static information structure. We showed that when agents are informed of a binary source through independent binary symmetric channels, and faced with homogeneous estimation error, any k-partite quantum resource must be allocated to the top k most informed players, i.e., the players with the channels with top k least noisy channels. We also showed *cut-offs*, that is ranges on the reliabilities of the channels beyond which the allocation of the resource produces no advantage over classical strategies. Extending this result to more general costs is a problem for the future.

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