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Relaxation systems and cyclic monotonicity

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Abstract— **It is shown that an LTI system is a relaxation system if and only if its Hankel operator is cyclic monotone. Cyclic monotonicity of the Hankel operator implies the existence of a storage function whose gradient is the Hankel operator. This storage is a function of past inputs alone, is independent of the state space realization, and admits a generalization to nonlinear circuit elements.**

I. Introduction

Relaxation systems are a class of LTI systems which first arose in the study of relaxation phenomena in viscoelastic materials, and, in the finite dimensional case, correspond to RC and RL circuits [1]. Relaxation systems are highly structured. They correspond to systems with completely monotonic impulse responses, with transfer functions which are sums of first order lags [1]–[3] and it was shown by Willems [4] that they admit state space realizations which are both externally symmetric, corresponding to the circuit property of reciprocity, and internally symmetric, encoding the fact that all the energy storage elements are of the same type. There has been a recent revival of interest in relaxation systems [5]–[12]. For example, it was observed by Pates *et al.* [5], [6] that they admit very simple H_∞-optimal controllers, with highly structured circuit realizations.

Dissipativity theory [13] connects the circuit theory of passivity to the dynamical systems theory of stability via the *storage function*, which represents the energy stored in a system. For a relaxation system, there exists a storage function which is completely determined by the Hankel operator, that is, the future output in response to a past input [4]. Relaxation systems therefore represent a class of systems for which the storage can be defined externally, as a function of past input only.

Existing characterizations of relaxation systems rely on linearity and time invariance. We are motivated by a characterization that is not limited to LTI systems. This paper presents some preliminary steps in this direction, through connections to monotone operator theory. The property of monotonicity was originally introduced in efforts to generalize the property of passivity to networks of nonlinear resistors [14]–[17]. Monotone operator theory now forms a pillar of convex optimization theory [18]–[21], owing to the fact that the gradient of a convex function is a monotone operator.

An early question in the theory of monotone operators was when the converse is true, when is a monotone operator the gradient of a convex function? This question was answered by Rockafellar [22], [23], who showed that a stronger property than monotonicity is required: cyclic monotonicity.

In this paper, we reconnect the property of cyclic monotonicity with its circuit theoretic origins, showing that cyclic monotonicity corresponds precisely to relaxation, that is, to circuits with a single type of energy storage element. Our main result shows that an equivalent characterization of relaxation is that a system's Hankel operator is cyclic monotone. For single input, single output LTI operators, this equivalence was shown independently in the recent work of Yafaev [10], [11]. Our proof is MIMO, and uses a state space representation. Cyclic monotonicity of the Hankel operator implies that it is the gradient of some convex functional, and we show that this convex functional is precisely the intrinsic storage of a relaxation system observed by Willems. Because cyclic monotonicity is not restricted to linear systems, our characterization opens the way to a nonlinear concept of relaxation.

Cyclic monotonicity has previously been studied in the context of Lur'e systems [24], [25], multi-agent systems [26] and recently in the context of incrementally port-Hamiltonian systems [27], where it was shown that a port-Hamiltonian system with a maximal cyclic monotone Dirac structure may be defined in terms of a convex function of the state and input. In contrast, we consider cyclic monotonicity of an *external* map, the Hankel operator, that maps past inputs to future outputs.

The remainder of this paper is structured as follows. In Section II, we introduce the necessary preliminary material from the theory of passivity and monotone operators. In Section III, we give the first of our main results, that relaxation is equivalent to cyclic monotonicity of the Hankel operator. In Section IV, we introduce a new notion of an intrinsic storage functional and show that the convex functional whose gradient is the Hankel operator is the intrinsic storage of Willems. Conclusions and directions for future work are given in Section V.

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II. Preliminaries

A. State space systems and Hankel operators

We study linear, time-invariant state space systems of the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t),\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$. A system is said to be *stable* if A is Hurwitz, and *minimal* if (A, B) is controllable and (A, C) is observable. The transfer function of system (1) is given by $H(s) := C(sI - A)^{-1}B + D$, and the impulse response is given by $h(t) := D\delta(t) + Ce^{At}B$, where $\delta(t)$ denotes the Dirac delta. We also define $g(t) := Ce^{At}B$ to be the impulse response of the system with no feedthrough term.

A complete inner product space is called a Hilbert space. The space $L_2(\mathbb{R}, \mathbb{R}^n)$ is the set of signals $u : \mathbb{R} \to \mathbb{R}^n$ such that

$$\int_{-\infty}^{\infty} u(t)^\top u(t)\ \mathrm{d}t < \infty.$$

This space forms a Hilbert space of equivalence classes of functions when equipped with the inner product

$$\langle u, y\rangle := \int_{-\infty}^{\infty} u(t)^\top y(t)\ \mathrm{d}t,$$

which induces the norm $\|u\| := \sqrt{\langle u, u\rangle}$. We define $L_2(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ and $L_2(\mathbb{R}_{\leq 0}, \mathbb{R}^n)$ similarly, but with time axes of $[0, \infty)$ and $(-\infty, 0]$, respectively. We will use the shorthand notation L_2^n for $L_2(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$.

A stable system admits a *Hankel operator*, which maps an input on $L_2(\mathbb{R}_{\leq 0}, \mathbb{R})$ to the corresponding output on $L_2(\mathbb{R}_{\geq 0}, \mathbb{R})$, assuming zero input from time 0. Given an impulse response h and input $\bar{u} \in L_2(\mathbb{R}_{\leq 0}, \mathbb{R})$, the output of the Hankel operator Γ_h at time t is given by

$$y(t) = \int_{-\infty}^{0} h(t - \tau)\bar{u}(\tau)\ \mathrm{d}\tau.$$

Letting $u(t) := \bar{u}(-t)$, the Hankel operator has the expression

$$(\Gamma_h u)(t) := \int_{0}^{\infty} h(t + \tau)u(\tau)\ \mathrm{d}\tau,$$

and defines an operator on $L_2(\mathbb{R}_{\geq 0}, \mathbb{R})$. If the system is stable, the Hankel operator is continuous [28, Prop. 4.1].

For the remainder of this paper, we will consider systems which are *square*, that is, the input dimension m is equal to the output dimension p.

B. Passivity, reciprocity and relaxation

Passivity is a formalization of the notion that a system can be realized without any internal power source. Central to the theory of passivity is the storage function, which represents the energy stored within a system. We recall the following definition of passivity.

Definition 1 ([29, Def. 5])**.** A system of the form (1) is said to be *passive* if, for any input/output trajectory (u, y) of the system and $t_0 \in \mathbb{R}$, there exists a $K \in \mathbb{R}$ such that, if $(\hat{u}, \hat{y})$ is also an input/output trajectory of the system and $(\hat{u}(t), \hat{y}(t)) = (u(t), v(t))$ for all $t < t_0$, then

$$-\int_{t_0}^{t_1} \hat{u}(t)^\top \hat{y}(t)\ \mathrm{d}t \leq K$$

for all $t_1 \geq t_0$. ⌟

It is shown in [29, Thm. 13] that, for a (not necessarily minimal) system of the form (1), Definition 1 is equivalent to the existence of a matrix $Q = Q^\top \succeq 0$ satisfying the linear matrix inequality

$$\begin{pmatrix} A^\top Q + QA & QB - C^\top \\ B^\top Q - C & -D - D^\top \end{pmatrix} \preceq 0. \tag{2}$$

This is precisely the condition given by [4, Thm. 3] in the context of minimal LTI state space systems.

A signature matrix is a diagonal matrix whose diagonal entries are either 1 or -1.

Definition 2. A system of the form (1) is said to be *(externally) reciprocal* with respect to the signature matrix Σ_e if $\Sigma_e H(s) = \Sigma_e H(s)^\top$, where $H(s)$ is the transfer matrix of (1). ⌟

Reciprocal systems admit internally reciprocal state space realizations.

Theorem 1 ([4, Thm. 6])**.** *A system of the form* (1) *is reciprocal if and only if it admits a state space realization* (A, B, C, D) *such that*

$$\begin{pmatrix} \Sigma_i & 0 \\ 0 & \Sigma_e \end{pmatrix}\begin{pmatrix} -A & -B \\ C & D \end{pmatrix} = \begin{pmatrix} -A^\top & C^\top \\ -B^\top & D^\top \end{pmatrix}\begin{pmatrix} \Sigma_i & 0 \\ 0 & \Sigma_e \end{pmatrix},$$

where Σ_i *is a signature matrix.*

We now define relaxation systems, the main subject of this paper.

Definition 3. A system of the form (1) is said to be a *relaxation system* if $D = D^\top \succeq 0$ and $g(t) = Ce^{At}B$ is a completely monotonic function for $t \in [0, \infty)$:

$$g(t) = g(t)^\top \text{ for all } t \geq 0,$$
$$(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} g(t) \succeq 0 \text{ for all } k = 1, 2, \ldots \text{ and } t \geq 0. \quad ⌟$$

Relaxation systems first arose in the context of viscoelastic materials [1], and, in the context of electrical circuits, correspond to the impedances of RC circuits and the admittances of RL circuits. Several equivalent characterizations of relaxation systems are known in the literature [1], [2], [4], [30]–[32], which we collect in the following theorem.

Theorem 2. *Consider a system of the form* (1)*. Then the following are equivalent:*

1) *the system is a relaxation system.*
2) $H(s)$ *admits the form*

$$H(s) = G_0 + \frac{G_1}{s} + \sum_{i=2}^{n} \frac{G_i}{s + \lambda_i},$$

where $G_i = G_i^\top \succeq 0$ for all i and $0 \leq \lambda_0 < \lambda_1 < \ldots < \lambda_N$, for some $N \in \mathbb{Z}_{\geq 0}$.

3) *$H(s)$ admits a minimal state space realization (A_1, B_1, C_1, D_1) such that*

$$\begin{aligned} A_1 &= A_1^\top \preceq 0 \\ B_1 &= C_1^\top \\ D_1 &= D_1^\top \succeq 0. \end{aligned}$$

4) $D \succeq 0$,

$$\begin{pmatrix} CB & CAB & \ldots & CA^{n-1}B \\ CAB & CA^2B & \ldots & CA^nB \\ \vdots & \vdots & \ddots & \vdots \\ CA^{n-1}B & CA^nB & \ldots & CA^{2n-2}B \end{pmatrix} \succeq 0$$

$$\begin{pmatrix} CAB & CA^2B & \ldots & CA^nB \\ CA^2B & CA^3B & \ldots & CA^{n+1}B \\ \vdots & \vdots & \ddots & \vdots \\ CA^nB & CA^{n+1}B & \ldots & CA^{2n-1}B \end{pmatrix} \preceq 0,$$

and all three of these matrices are symmetric.

C. Cyclic monotonicity

In this section, we introduce the notions of monotonicity and cyclic monotonicity, for operators on a Hilbert space $\mathcal{H}$.

Definition 4. Given an operator $A : \mathcal{H} \to \mathcal{H}$, the *graph of* A is the set $\text{gra}\,(A) \subseteq \mathcal{H} \times \mathcal{H}$ defined by

$$\text{gra}\,(A) := \{(u, y) \mid u \in \mathcal{H}, y = A(u)\}. \quad ⌟$$

Definition 5. An operator $A : \mathcal{H} \to \mathcal{H}$ is said to be *monotone* if, for all $u_1, u_2 \in \mathcal{H}, y_1 = A(u_1), y_2 = A(u_2)$,

$$\langle u_1 - u_2, y_1 - y_2 \rangle \geq 0. \tag{3}$$

If $\text{gra}\,(A)$ is not properly contained within the graph of any other monotone operator, A is said to be *maximal monotone*. ⌟

Definition 6. An operator $A : \mathcal{H} \to \mathcal{H}$ is said to be *n-cyclic monotone* if, for all sets of input/output pairs $\{(u_i, y_i) \mid u_i \in \mathcal{H}, y_i = A(u_i), i = 0, \ldots, n\}$,

$$\langle y_0, u_0 - u_1 \rangle + \langle y_1, u_1 - u_2 \rangle + \ldots + \langle y_n, u_n - u_0 \rangle \geq 0.$$

If A is n-cyclic monotone for all $n \geq 1$, A is said to be *cyclic monotone*. If $\text{gra}\,(A)$ is not contained within the graph of any other monotone operator, A is said to be *maximal cyclic monotone*. ⌟

Maximality is guaranteed for continuous operators [19, Cor. 20.25], so the Hankel operators associated with the stable linear operators considered in this paper are automatically maximal.

Definition 7. An operator $A : \mathcal{H} \to \mathcal{H}$ is said to be self-adjoint if, for all $u, y \in \mathcal{H}$,

$$\langle A(u), y \rangle = \langle u, A(y) \rangle. \quad ⌟$$

Asplund [33] gives the following characterization of the cyclic monotonicity of a linear operator. Given a linear operator $A : \mathcal{H} \to \mathcal{H}$, we define the *complexification of* A, denoted A_c, by

$$A_c(u + jw) := A(u) + jA(w).$$

This operates on the complexification of $\mathcal{H}$, denoted $\mathcal{H}_c$. We endow this space with the inner product

$$\langle u + jw, y + jv \rangle_c := \langle u, y \rangle + \langle w, v \rangle + j(\langle w, y \rangle - \langle u, v \rangle).$$

The *numerical range* of an operator A_c on $\mathcal{H}_c$ is defined as

$$W(A_c) := \left\{ \frac{\langle A_c(z), z \rangle_c}{\|z\|} \;\middle|\; z \in \text{dom}\,(A_c)\,, \|z\| \neq 0 \right\}.$$

Theorem 3 (Asplund [33, Thm. 3])**.** *A linear operator A on $\mathcal{H}$ is n-cyclic monotone if and only if, for all $z \in W(A_c)$, $\arg z \leq \pi/n$.*

For the limiting case of cyclic monotonicity, we have the following corollary.

Corollary 1. *A linear operator A on $\mathcal{H}$ is cyclic monotone if and only if it is self-adjoint and, for all $u \in \text{dom}\,(A)$, $\langle A(u), u \rangle \geq 0$.*

Proof. n-cyclic monotonicity for all n implies that $\arg z = 0$ for all $z \in W(A_c)$. Equivalently, $\arg \langle A_c(z), z \rangle = 0$ for all $z = u + jw \in \text{dom}\,(A_c)\,, \|z\| \neq 0$. Expanding the inner product:

$$\begin{aligned} &\arg(\langle u, A(u) \rangle + \langle w, A(w) \rangle + \\ &\quad j(\langle w, A(u) \rangle - \langle u, A(w) \rangle)) = 0 \\ &\quad \text{so } \langle u, A(u) \rangle + \langle w, A(w) \rangle \geq 0 \\ &\text{and } \langle A(w), u \rangle = \langle w, A(u) \rangle. \end{aligned} \quad \square$$

Definition 8. A function $f : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ is said to be *proper* if its value is never $-\infty$ and is finite somewhere, *closed* if its epigraph is closed and *convex* if, for all $x, y \in \mathcal{H}$ and $\vartheta \in (0, 1)$,

$$f(\vartheta x + (1 - \vartheta)y) \leq \vartheta f(x) + (1 - \vartheta) f(y). \quad ⌟$$

Our interest in cyclic monotonicity stems from the following theorem of Rockafellar.

Theorem 4 (Rockafellar's theorem [22], [23])**.** *A continuous operator $A : \mathcal{H} \to \mathcal{H}$ is maximal cyclic monotone if and only if it is the gradient of a closed, convex and proper function from $\mathcal{H}$ to $(-\infty, \infty]$. Moreover, this function is uniquely determined by A up to an additive constant.*

III. Relaxation and cyclic monotonicity

In this section, we establish the relationship between relaxation systems and cyclic monotone operators, and add a fifth equivalence to Theorem 2: relaxation is equivalent to cyclic monotonicity of the Hankel operator. The following theorem generalizes [10, Cor. 1.2] to multiple input, multiple output operators, assuming a finite-dimensional state space realization.

Theorem 5. *Consider the system* (1) *and assume that A is Hurwitz. The system is a relaxation system if and only if its Hankel operator Γ_h is cyclic monotone and $D = D^\top \succeq 0$.*

Proof. We begin by showing that relaxation implies cyclic monotonicity of the Hankel operator (the condition on D being immediate from the definition of relaxation). By Corollary 1, cyclic monotonicity of Γ_h is equivalent to the following two conditions, for all $u, w \in L_2^m$:

$$\langle \Gamma_h w, u\rangle = \langle w, \Gamma_h u\rangle \tag{4}$$
$$\langle u, \Gamma_h u\rangle \geq 0. \tag{5}$$

We begin by showing (4). Note that relaxation implies reciprocity with respect to $\Sigma_e = I$, and this in turn implies symmetry of the impulse responses $h(t)$ and $g(t)$.

We also note that, for any $u, w \in L_2^m$,

$$\begin{aligned}&\int_0^\infty u(t)^\top \left(\int_0^\infty h(t+\tau)w(\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}t\\ &= \int_0^\infty u(t)^\top \left(\int_0^\infty g(t+\tau)w(\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}t. \end{aligned}\tag{6}$$

Indeed,

$$\begin{aligned}&\int_0^\infty u(t)^\top \left(\int_0^\infty h(t+\tau)w(\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}t\\ &= \int_0^\infty u(t)^\top \int_0^\infty Ce^{At}e^{A\tau}Bw(\tau) + Dw(\tau)\delta(t+\tau)\,\mathrm{d}\tau\,\mathrm{d}t\\ &= \int_0^\infty u(t)^\top \left(\int_0^\infty Ce^{At}e^{A\tau}Bw(\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}t\\ &\quad + \int_0^\infty u(t)^\top D\bar{w}(t)\,\mathrm{d}t, \end{aligned}\tag{7}$$

where

$$\bar{w}(t) := \begin{cases} w(t) & t = 0\\ 0 & \text{otherwise.}\end{cases}$$

We then have

$$\int_0^\infty u(t)^\top D\bar{w}(t)\,\mathrm{d}t = 0,$$

so (7) implies (6). Using symmetry of the inner product, (4) is equivalent to

$$\begin{aligned}&\int_0^\infty u(t)^\top \left(\int_0^\infty g(t+\tau)w(\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}t\\ &= \int_0^\infty w(t)^\top \left(\int_0^\infty g(t+\tau)u(\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}t. \end{aligned}\tag{8}$$

To show that $g(t) = g(t)^\top$ implies (8), take the left hand side of (8), transpose and apply Fubini's theorem:

$$\begin{aligned}&\int_0^\infty u(t)^\top \left(\int_0^\infty g(t+\tau)w(\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}t\\ &= \int_0^\infty \left(\int_0^\infty w(\tau)^\top g(t+\tau)^\top\,\mathrm{d}\tau\right) u(t)\,\mathrm{d}t\\ &= \int_0^\infty \left(\int_0^\infty w(\tau)^\top g(t+\tau)^\top u(t)\,\mathrm{d}t\right)\,\mathrm{d}\tau\\ &= \int_0^\infty w(t)^\top \left(\int_0^\infty g(t+\tau)u(t)\,\mathrm{d}t\right)\,\mathrm{d}\tau. \end{aligned}$$

We next show that relaxation implies (5). Let (A_1, B_1, C_1, D_1) be a state space realization of the form of Theorem 2, 3), with impulse response $h(t)$. Then, using (6),

$$\begin{aligned}\langle u, \Gamma_h u\rangle &= \int_0^\infty u^\top(t)\int_0^\infty h(t+\tau)u(\tau)\,\mathrm{d}\tau\,\mathrm{d}t\\ &= \int_0^\infty u^\top(t)\int_0^\infty C_1 e^{A_1 t}e^{A_1\tau}B_1 u(\tau)\,\mathrm{d}\tau\,\mathrm{d}t \quad (9)\\ &= \left(\int_0^\infty e^{A_1 t}B_1 u(t)\,\mathrm{d}t\right)^\top \int_0^\infty e^{A_1 t}B_1 u(t)\,\mathrm{d}t\\ &\geq 0.\end{aligned}$$

This establishes that relaxation implies cyclic monotonicity of the Hankel operator.

We now show the converse, that cyclic monotonicity of the Hankel operator and $D = D^\top \succeq 0$ together imply relaxation. We begin by showing that (8) implies symmetry of $g(t)$ for all $t \geq 0$. Indeed, let $v(\tau) = \delta(\tau)e_j$ and $u(t) = \delta(t - t_0)e_i$, where $t_0 \in [0, \infty)$, e_i denotes the i^{th} canonical basis vector of $\mathbb{R}^n$ and δ denotes the Dirac delta. Substituting these signals into (8) gives

$$e_i^\top g(t_0)e_j = e_j^\top g(t_0)e_i,$$

that is, $g(t_0)$ is symmetric for all $t_0 \in [0, \infty)$, which is equivalent to symmetry of $h(t)$ under the assumption $D = D^\top$. This in turn is equivalent to reciprocity with respect to $\Sigma_e = I$.

Finally, we show that reciprocity, (5) and $D = D^\top \succeq 0$ imply relaxation. Let be a stable system with $\hat{D} = \hat{D}^\top \succeq 0$ and Hankel operator Γ_h which satisfies (5) and (8). Let $(\hat{A}, \hat{B}, \hat{C}, D)$ be a minimal system with transfer function equal to $D\delta(t) + Ce^{At}B$. By reciprocity, it follows from [4, Lem 3] that there exists a unique, invertible, symmetric matrix T such that

$$\begin{aligned}\hat{A}^\top T &= T\hat{A}\\ T\hat{B} &= \hat{C}^\top.\end{aligned}$$

We claim that $T \geq 0$. Suppose, on the contrary, that T has a negative eigenvalue. Let x_0 be a corresponding eigenvector. Let $\bar{u} : (-\infty, 0] \to \mathbb{R}^n$ be an input that drives the system from $x = 0$ at $t = -\infty$ to $x(0) = x_0$. Such an input exists, as $(\hat{A}, \hat{B})$ is controllable. Let $u(t) = \bar{u}(-t)$. By positivity of Γ_h, we have

$$\begin{aligned}0 &\leq \langle u, \Gamma_h u\rangle\\ &= \int_0^\infty u(t)^\top \int_0^\infty \hat{C}e^{\hat{A}(t+\tau)}\hat{B}u(\tau)\,\mathrm{d}\tau\,\mathrm{d}t\\ &= \int_0^\infty u(t)^\top \hat{C}e^{\hat{A}t}\int_{-\infty}^0 e^{-\hat{A}\tau}\hat{B}\bar{u}(\tau)\,\mathrm{d}\tau\,\mathrm{d}t\\ &= \int_0^\infty u(t)^\top \hat{C}e^{\hat{A}t}x_0\,\mathrm{d}t\\ &= \int_0^\infty u(t)^\top \hat{B}^\top Te^{\hat{A}t}x_0\,\mathrm{d}t\\ &= \int_{-\infty}^0 \bar{u}(t)^\top \hat{B}^\top e^{-\hat{A}^\top t}\,\mathrm{d}tTx_0\\ &= x_0^\top Tx_0 < 0,\end{aligned}$$

which is a contradiction. Hence $T \geq 0$. It follows from Lemma 3 in the Appendix that the system is passive. It then follows from [4, Thm. 7] that there exists a minimal realization (A_1, B_1, C_1, D_1) of the system which satisfies

$$\begin{aligned}\Sigma_i A_1 &= A_1^\top \Sigma_i \\ C_1^\top &= -\Sigma_i B_1 \\ D_1 &= D_1^\top \succeq 0,\end{aligned}$$

where Σ_i is a signature matrix. It follows from Equation (9) and positivity of Γ_h that

$$\int_0^\infty u(t)^\top C_1 e^{At}\,\mathrm{d}t \int_0^\infty e^{A_1\tau} B_1 u(\tau)\,\mathrm{d}\tau \geq 0 \quad (10)$$

for all u. Hence

$$-\int_0^\infty u(t)^\top B_1^\top e^{A_1^\top t}\,\mathrm{d}t \Sigma_i \int_0^\infty e^{A_1\tau} B_1 u(\tau)\,\mathrm{d}\tau \geq 0$$

for all u. Suppose that Σ_i has entry (j, j) equal to 1. By controllability of (A_1, B_1), we can choose an input such that

$$\int_0^\infty e^{A_1\tau} B_1 u(\tau)\,\mathrm{d}\tau = e_j.$$

But then $-e_j^\top \Sigma_i e_j \prec 0$, which contradicts (10). Hence $\Sigma_i = -I$, so the system is of the relaxation type. □

IV. Intrinsic storages for relaxation systems

Theorem 5 establishes the equivalence of relaxation and cyclic monotonicity of the Hankel operator. It then follows from Rockafellar's theorem that the Hankel operator is the gradient of a closed, convex and proper functional mapping $L_2^m \to \mathbb{R}$. It turns out that this convex functional is precisely the input/output storage observed by Willems [4, §10].

Before formalizing this result, we show that passivity is guaranteed by the existence of a nonnegative functional of the past input to the system. We call this object an *intrinsic storage functional*. We then give a simple, illustrative example.

Proposition 1. *Consider a system of the form* (1). *Given a signal $u \in L_2(\mathbb{R}, \mathbb{R}^m)$ and time $t \in \mathbb{R}$, denote by u_t the truncation of u to the time axis $(-\infty, t]$. If there exists a functional V mapping a truncated signal u_t into $\mathbb{R}_{\geq 0}$ and satisfying*

$$\frac{\mathrm{d}V}{\mathrm{d}t}(u_t) \leq u(t)^\top y(t), \quad (11)$$

for all $t \in \mathbb{R}$ and input/output trajectories (u, y) of the system, then the system is passive.

Proof. Let $t_0, t_1 \in \mathbb{R}$, $t_1 \geq t_0$. Integrating (11) from t_0 to t_1 gives

$$V(u_{t_0}) - V(u_{t_1}) \geq -\int_{t_0}^{t_1} u(t)^\top y(t)\,\mathrm{d}t.$$

Passivity then follows from nonnegativity of $V(u_{t_1})$, with K in Definition 1 equal to $V(u_{t_0})$. □

Example 1. Consider the linear RC circuit shown in Figure 1. Denoting the voltage on the capacitor by v_c, we have the following state space model for the impedance of the circuit:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} v_c &= \frac{-1}{R_1 C} v_c + \begin{pmatrix} \frac{1}{C} & \frac{1}{C}\end{pmatrix}\begin{pmatrix} i_1 \\ i_2\end{pmatrix}, \\ \begin{pmatrix} v_1 \\ v_2\end{pmatrix} &= \begin{pmatrix} 1 \\ 1\end{pmatrix} v_2 + \begin{pmatrix} 0 & 0 \\ 0 & R_2\end{pmatrix}\begin{pmatrix} i_1 \\ i_2\end{pmatrix}.\end{aligned}$$

We consider the following experiment: time-varying current

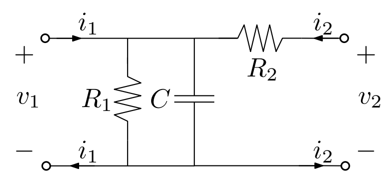

Fig. 1. A two-port RC circuit.

sources, $\bar{i}_{1t}(\cdot)$ and $\bar{i}_{2t}(\cdot)$, are attached to the ports from time $-\infty$, when there is no charge on the capacitor, to time $t \in \mathbb{R}$. The current sources are then replaced by voltmeters, which read voltages $\bar{v}_1(\cdot)$ and $\bar{v}_2(\cdot)$. We define $i_{nt}(\tau) = \bar{i}_{nt}(t - \tau)$ and $v_n(\zeta) = \bar{v}_n(\zeta + t)$ for $n = 1, 2$. Define $v = \begin{pmatrix} v_1 & v_2\end{pmatrix}^\top$ and $i_t = \begin{pmatrix} i_{1t} & i_{2t}\end{pmatrix}^\top$. Solving the state space model gives the Hankel operator

$$v(\zeta) = \int_0^\infty \begin{pmatrix} 1 \\ 1\end{pmatrix} e^{\frac{-1}{R_1 C}(\zeta + \tau)} \begin{pmatrix} \frac{1}{C} & \frac{1}{C}\end{pmatrix} i_t(\tau)\,\mathrm{d}\tau,$$

plus an additional term $R_2 i_{2t}(0)$ when $\zeta = 0$. Computing the inner product $(1/2)\langle i_t, v\rangle$ over L_2 gives

$$\begin{aligned}&\frac{1}{2}\int_0^\infty i_t(\zeta)^\top \int_0^\infty \begin{pmatrix} 1 \\ 1\end{pmatrix} e^{\frac{-1}{R_1 C}(\zeta+\tau)} \begin{pmatrix} \frac{1}{C} & \frac{1}{C}\end{pmatrix} i_t(\tau)\,\mathrm{d}\tau\,\mathrm{d}\zeta \\ &= \frac{1}{2C}\left(\int_0^\infty (i_{1t}(\zeta) + i_{2t}(\zeta)) e^{\frac{-1}{R_1 C}\zeta}\,\mathrm{d}\zeta \right. \\ &\qquad \left. \int_0^\infty (i_{1t}(\tau) + i_{2t}(\tau)) e^{\frac{-1}{R_1 C}\tau}\,\mathrm{d}\tau\right) \\ &= \frac{1}{2C} q_c(0)^2,\end{aligned}$$

where $q_c = \frac{1}{C} v_c$ is the charge on the capacitor and the last line follows by solving the state space equations with zero initial condition. This expression is the energy stored in the capacitor at time $\tau = 0$. Taking the derivative with respect to time gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\langle i_t, v\rangle = \frac{1}{C} q_c(0) \frac{\mathrm{d}}{\mathrm{d}t} q_c(0).$$

Let $\eta(t, \tau) := t - \tau$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} i_{nt} = \frac{\mathrm{d}}{\mathrm{d}t}\bar{i}_{nt}(\eta(t,\tau)) = \frac{\mathrm{d}\bar{i}_{nt}}{\mathrm{d}\eta}\frac{\mathrm{d}\eta}{\mathrm{d}t} = \bar{i}'_{nt}(t - \tau)$$

and $$\frac{\mathrm{d}}{\mathrm{d}\tau} i_{nt} = \frac{\mathrm{d}}{\mathrm{d}\tau}\bar{i}_{nt}(\eta(t,\tau)) = \frac{\mathrm{d}\bar{i}_{nt}}{\mathrm{d}\eta}\frac{\mathrm{d}\eta}{\mathrm{d}\tau} = -\bar{i}'_{nt}(t - \tau),$$

so $\frac{\mathrm{d}}{\mathrm{d}t} i_{nt} = -\frac{\mathrm{d}}{\mathrm{d}\tau} i_{nt}$. We then have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} q_c(0) &= \int_0^\infty e^{\frac{-1}{R_1 C}\tau} \frac{\mathrm{d}}{\mathrm{d}t}(i_{1t}(\tau) + i_{2t}(\tau))\,\mathrm{d}\tau \\ &= -\int_0^\infty e^{\frac{-1}{R_1 C}\tau} \frac{\mathrm{d}}{\mathrm{d}\tau}(i_{1t}(\tau) + i_{2t}(\tau))\,\mathrm{d}\tau.\end{aligned}$$

Integrating by parts then gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}q_c(0) &= -\left[e^{\frac{-1}{R_1C}\tau}(i_{1t}(\tau)+i_{2t}(\tau))\right]_0^\infty \\ &\quad -\frac{1}{R_1C}\int_0^\infty e^{\frac{-1}{R_1C}\tau}(i_{1t}+i_{2t})(\tau)\,\mathrm{d}\tau \\ &= i_{1t}(0)+i_{2t}(0)-\frac{1}{R_1C}q_c(0), \text{ so} \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\langle i_t, v\rangle &= v_c(0)(i_{1t}(0)+i_{2t}(0)) - \frac{1}{R_1C^2}q_c(0)^2 \\ &\le v_c(0)(i_{1t}(0)+i_{2t}(0)) + R_2 i_{2t}(0)^2 \\ &= \bar{v}(t)\bar{i}_t(t).\end{aligned}$$

The variables $\bar{v}$ and $\bar{i}_t$ correspond to a particular experiment, however, the right hand side of this dissipation inequality only involves the value of $\bar{i}_t$ and $\bar{v}$ at time t, the instant in the experiment when both the current source and the voltmeter are connected. These can thus be considered samples of an arbitrary current/voltage trajectory. The functional $(1/2)\langle i_t, v\rangle$ is thus an intrinsic storage functional for the system, and is expressed purely in terms of the input i and output v. Furthermore, the derivative of this functional with respect to i_t is the Hankel operator of the system. The quantity $(1/R_1C^2)q_c(0)^2$ is the instantaneous power dissipated by the resistor R_1. ⌟

In order to generalize the construction of the intrinsic storage in Example 1 to arbitrary relaxation systems, we require a notion of gradient on L_2^m. This is given by the functional derivative, $\partial V/\partial u$, which we define via the first variation:

$$\left\langle \frac{\partial V}{\partial u}, \phi \right\rangle := \left[\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(V(u+\varepsilon\phi)\right)\right]_{\varepsilon=0}.$$

Lemma 1. *Let h be the impulse response of a relaxation system, and Γ_h be the corresponding Hankel operator. Then Γ_h is the functional derivative of*

$$V(u) := \frac{1}{2}\langle u, \Gamma_h u\rangle.$$

Proof. Computing the functional derivative gives

$$\begin{aligned}\left\langle \frac{\partial V}{\partial u}, \phi \right\rangle &= \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\langle u+\varepsilon\phi, \Gamma_h(u+\varepsilon\phi)\rangle\right]_{\varepsilon=0} \\ &= \frac{1}{2}\langle \phi, \Gamma_h u\rangle + \frac{1}{2}\langle u, \Gamma_h \phi\rangle \\ &= \langle \Gamma_h u, \phi\rangle,\end{aligned}$$

where the final inequality follows from self-adjointness of Γ_h. It then follows that $\partial V/\partial u = \Gamma_h$. □

The following theorem establishes that the function of Lemma 1 is in fact an intrinsic storage functional.

Theorem 6. *Let h be the impulse response of a relaxation system, and Γ_h be the corresponding Hankel operator. Then the system is passive with intrinsic storage functional*

$$V(u) := \frac{1}{2}\langle u, \Gamma_h u\rangle.$$

The proof of Theorem 6 makes use of the following lemma, which establishes a recursive property of relaxation systems with respect to the derivative. This is a generalization of the fact that the power dissipated by the resistor R_1 in Example 1 is positive.

Lemma 2. *Let $g(t) = Ce^{At}B$ be the impulse response of a relaxation system, without the direct component $D\delta(t)$. Then any system with impulse response $-\frac{\mathrm{d}}{\mathrm{d}t}g$ is also a relaxation system.*

Proof. By Definition 3, g is completely monotonic, so

$$(-1)^k\frac{\mathrm{d}^k}{\mathrm{d}t^k}g(t) \ge 0$$

for all $k = 1, 2, \ldots$ This implies complete monotonicity of $-\frac{\mathrm{d}}{\mathrm{d}t}g$. □

Proof of Theorem 6. Nonnegativity of V follows from positivity of Γ_h (Theorem 5). It remains to show that V satisfies the dissipation inequality (11). Let the input trajectory be $\bar{u} \in L_2(\mathbb{R}, \mathbb{R}^m)$ and define the past input corresponding to time $t \in \mathbb{R}$ by

$$u_t(\tau) := \bar{u}(t-\tau), \quad \tau \in [0, \infty).$$

Let $\eta(t,\tau) := t - \tau$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}u_t(\tau) = \frac{\mathrm{d}}{\mathrm{d}t}\bar{u}_t(\eta(t,\tau)) = \frac{\mathrm{d}\bar{u}_t}{\mathrm{d}\eta}\frac{\mathrm{d}\eta}{\mathrm{d}t} = \bar{u}_t'(t-\tau)$$

and $$\frac{\mathrm{d}}{\mathrm{d}\tau}u_t = \frac{\mathrm{d}}{\mathrm{d}\tau}\bar{u}_t(\eta(t,\tau)) = \frac{\mathrm{d}\bar{u}_t}{\mathrm{d}\eta}\frac{\mathrm{d}\eta}{\mathrm{d}\tau} = -\bar{u}_t'(t-\tau),$$

so $$\frac{\mathrm{d}}{\mathrm{d}t}u_t = -\frac{\mathrm{d}}{\mathrm{d}\tau}u_t. \tag{12}$$

We then have

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}(u_t) &= \left\langle \frac{\partial V}{\partial u}(u_t), \frac{\partial u_t}{\partial t}\right\rangle \\ &= \int_0^\infty\int_0^\infty u_t(\tau)^\top h(\zeta+\tau)\,\mathrm{d}\tau\frac{\partial u_t}{\partial t}(\zeta)\,\mathrm{d}\zeta \\ &= -\int_0^\infty y(\zeta)^\top\frac{\partial u_t}{\partial\zeta}(\zeta)\,\mathrm{d}\zeta,\end{aligned}$$

where the final line uses Lemma 1 and Equation (12). Integration by parts then gives

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}(u_t) &= -\left[y(\zeta)^\top u_t(\zeta)\right]_0^\infty + \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}\zeta}y(\zeta)^\top u_t(\zeta)\,\mathrm{d}\zeta \\ &= y(0)^\top u_t(0) + \\ &\int_0^\infty\int_0^\infty u_t(\tau)^\top\frac{\mathrm{d}h}{\mathrm{d}\zeta}(\tau+\zeta)\,\mathrm{d}\tau u_t(\zeta)\,\mathrm{d}\zeta,\end{aligned} \tag{13}$$

where (13) uses (6) in the proof of Thm. 5. Denote $\mathrm{d}g/\mathrm{d}\zeta$ by g'. Then the rightmost term in (13) can be written as

$$-\left\langle \Gamma_{(-g')}u_t, u_t\right\rangle \le 0, \tag{14}$$

where the inequality follows from the fact that that $\Gamma_{(-g')}$ is the Hankel operator of a relaxation system (Lemma 2), hence cyclic monotone (Theorem 5). Substituting in (13) gives

$$\frac{\mathrm{d}V}{\mathrm{d}t}(u_t) \le u_t(0)y(0) = \bar{u}_t(t)^\top\bar{y}(t). \qquad \square$$

A consequence of Rocakfellar's theorem is that the storage $V(u_t)$ is uniquely determined by the Hankel operator Γ_h, up to an additive constant. It was observed in [4] that this same storage is also uniquely determined by the requirements of passivity and internal reciprocity.

V. Conclusions

We have shown that a system being of the relaxation type is equivalent to cyclic monotonicity of the Hankel operator. Rockafellar's theorem allows us to construct a convex storage functional, whose gradient is the Hankel operator, which is completely determined by input/output measurements.

Cyclic monotonicity is equally well-defined for the Hankel operators of nonlinear systems, and this allows us to construct intrinsic storages for nonlinear systems. This will be a topic of future research.

Appendix

The proof of the following lemma can be found in the arxiv version of this paper.

Lemma 3. *Consider a stable system of the form* (1)*. Suppose that* $D = D^\top \succeq 0$ *and there exists a matrix* $T = T^\top \succeq 0$ *such that*

$$A^\top T = TA$$
$$TB = C^\top.$$

Then the system is passive.

References

[1] J. Meixner, "On the theory of linear passive systems," *Archive for Rational Mechanics and Analysis*, vol. 17, no. 4, pp. 278–296, 1964. DOI: 10.1007/BF00282291.

[2] S. Bernstein, "Sur les fonctions absolument monotones," *Acta Mathematica*, vol. 52, 1928.

[3] D. V. Widder, *The Laplace Transform* (Princeton Mathematical Series 6). Princeton University Press, 1941.

[4] J. C. Willems, "Dissipative dynamical systems Part II: Linear systems with quadratic supply rates," *Archive for Rational Mechanics and Analysis*, vol. 45, no. 5, pp. 352–393, 1972. DOI: 10.1007/BF00276494.

[5] R. Pates, C. Bergeling, and A. Rantzer, "On the Optimal Control of Relaxation Systems," in *2019 IEEE 58th Conference on Decision and Control (CDC)*, 2019, pp. 6068–6073. DOI: 10.1109/CDC40024.2019.9029933.

[6] R. Pates, "Passive and Reciprocal Networks: From Simple Models to Simple Optimal Controllers," *IEEE Control Systems Magazine*, vol. 42, no. 3, pp. 73–92, 2022. DOI: 10.1109/MCS.2022.3157116.

[7] C. Grussler, T. Damm, and R. Sepulchre, "Balanced truncation of *k*-positive systems," *IEEE Transactions on Automatic Control*, vol. 67, no. 1, pp. 526–531, 2022, ISSN: 1558-2523. DOI: 10.1109/TAC.2021.3075319.

[8] C. Grussler and R. Sepulchre, "Variation diminishing linear time-invariant systems," *Automatica*, vol. 136, p. 109 985, 2022, ISSN: 0005-1098. DOI: 10.1016/j.automatica.2021.109985.

[9] E. Bar-Shalom, O. Dalin, and M. Margaliot, "Compound matrices in systems and control theory: A tutorial," *Mathematics of Control, Signals, and Systems*, 2023. DOI: 10.1007/s00498-023-00351-8.

[10] D. R. Yafaev, "On finite rank Hankel operators," *Journal of Functional Analysis*, vol. 268, no. 7, pp. 1808–1839, 2015. DOI: 10.1016/j.jfa.2014.12.005.

[11] D. Yafaev, "Criteria for Hankel operators to be sign-definite," *Analysis & PDE*, vol. 8, no. 1, pp. 183–221, 2015. DOI: 10.2140/apde.2015.8.183.

[12] M. Margaliot and E. D. Sontag, "Revisiting totally positive differential systems: A tutorial and new results," *Automatica*, vol. 101, pp. 1–14, 2019. DOI: 10.1016/j.automatica.2018.11.016.

[13] J. C. Willems, "Dissipative dynamical systems Part I: General theory," *Archive for Rational Mechanics and Analysis*, vol. 45, pp. 321–351, 1972.

[14] R. J. Duffin, "Nonlinear networks. I," *Bulletin of the American Mathematical Society*, vol. 52, no. 10, pp. 833–839, 1946. DOI: 10.1090/S0002-9904-1946-08650-4.

[15] G. J. Minty, "Monotone networks," *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, vol. 257, no. 1289, pp. 194–212, 1960. DOI: 10.1098/rspa.1960.0144.

[16] G. J. Minty, "Solving Steady-State Nonlinear Networks of 'Monotone' Elements," *IRE Transactions on Circuit Theory*, vol. 8, no. 2, pp. 99–104, 1961. DOI: 10.1109/TCT.1961.1086765.

[17] G. J. Minty, "Monotone (nonlinear) operators in Hilbert space," *Duke Mathematical Journal*, vol. 29, no. 3, pp. 341–346, 1962. DOI: 10.1215/S0012-7094-62-02933-2.

[18] R. T. Rockafellar, "Monotone operators and the proximal point algorithm," *SIAM Journal on Control and Optimization*, vol. 14, no. 5, pp. 877–898, 1976. DOI: 10.1137/0314056.

[19] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces* (CMS Books in Mathematics). New York, NY: Springer New York, 2011. DOI: 10.1007/978-1-4419-9467-7.

[20] P. L. Combettes, "Monotone operator theory in convex optimization," *Mathematical Programming*, vol. 170, no. 1, pp. 177–206, 2018. DOI: 10.1007/s10107-018-1303-3.

[21] E. K. Ryu and W. Yin, *Large-Scale Convex Optimization: Algorithms & Analysis via Monotone Operators*. Cambridge: Cambridge University Press, 2022, ISBN: 978-1-00-916085-8. DOI: 10.1017/9781009160865.

[22] R. Rockafellar, "Characterization of the subdifferentials of convex functions," *Pacific Journal of Mathematics*, vol. 17, no. 3, pp. 497–510, 1966. DOI: 10.2140/pjm.1966.17.497.

[23] R. Rockafellar, "On the maximal monotonicity of subdifferential mappings," *Pacific Journal of Mathematics*, vol. 33, no. 1, pp. 209–216, 1970. DOI: 10.2140/pjm.1970.33.209.

[24] S. Adly, A. Hantoute, and B. K. Le, "Maximal monotonicity and cyclic monotonicity arising in nonsmooth Lur'e dynamical systems," *Journal of Mathematical Analysis and Applications*, vol. 448, no. 1, pp. 691–706, 2017. DOI: 10.1016/j.jmaa.2016.11.025.

[25] C. W. Scherer, "Dissipativity, Convexity and Tight O'Shea-Zames-Falb Multipliers for Safety Guarantees," *IFAC-PapersOnLine*, 25th International Symposium on Mathematical Theory of Networks and Systems MTNS 2022, vol. 55, no. 30, pp. 150–155, 2022, ISSN: 2405-8963. DOI: 10.1016/j.ifacol.2022.11.044.

[26] M. Sharf and D. Zelazo, "Analysis and Synthesis of MIMO Multi-Agent Systems Using Network Optimization," *IEEE Transactions on Automatic Control*, vol. 64, no. 11, pp. 4512–4524, 2019. DOI: 10.1109/TAC.2019.2908258.

[27] M. K. Camlibel and A. J. Van Der Schaft, "Port-Hamiltonian Systems Theory and Monotonicity," *SIAM Journal on Control and Optimization*, vol. 61, no. 4, pp. 2193–2221, 2023. DOI: 10.1137/22M1503749.

[28] J. R. Partington, *An Introduction to Hankel Operators* (London Mathematical Society Student Texts 13). Cambridge ; New York: Cambridge University Press, 1988, 103 pp., ISBN: 978-0-521-36611-3 978-0-521-36791-2.

[29] T. H. Hughes, "A theory of passive linear systems with no assumptions," *Automatica*, vol. 86, pp. 87–97, 2017. DOI: 10.1016/j.automatica.2017.08.017.

[30] J. C. Willems, "Realization of systems with internal passivity and symmetry constraints," *Journal of the Franklin Institute*, vol. 301, no. 6, pp. 605–621, 1976. DOI: 10.1016/0016-0032(76)90081-8.

[31] S. Marcus and J. Willems, "Nonstationary network synthesis via state-space techniques," *IEEE Transactions on Circuits and Systems*, vol. 22, no. 9, pp. 713–720, 1975. DOI: 10.1109/TCS.1975.1084120.

[32] M. Aissen, A. Edrei, I. J. Schoenberg, and A. Whitney, "On the Generating Functions of Totally Positive Sequences," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 37, no. 5, pp. 303–307, 1951.

[33] E. Asplund, "A monotone convergence theorem for sequences of nonlinear mappings," in *Nonlinear Functional Analysis, Proceedings of Symposia in Pure Mathematics*, vol. 18, American Mathematical Society, 1970.