2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Optimal PMU Placement for Voltage Estimation in Partially Known Power Networks

Aditya Mishra[1] and Raymond A. de Callafon[2]

Abstract— **Observability of all bus voltages in a power network enables overall monitoring and fault detection of power flow. Information on this voltage state is often a combination of voltage, current measurements obtained by Phasor Measurement Units (PMUs) and voltage state estimation via regression that uses admittance information of the connections between nodes within the power network. Voltage state estimation is a challenge for a power network in which data from limited PMUs is combined with partially known network admittance information. The challenge lies in choosing locations of PMUs such that full voltage state reconstruction is possible, despite the lack of complete knowledge on network admittance information. This paper proposes a methodology of placing PMUs across a network with incomplete network admittance information that guarantees comprehensive observability of the voltage states. The method separates network nodes in distinct nodal sets based on voltage, current and admittance information. The permutations of these nodal sets are used to establish the minimum number of PMUs required for full voltage state observability for a power network with partially known admittance information. Subsequently, an additional optimal placement can be used to minimize the variance of the estimated voltage states. The proposed PMU placement approach is tested on a modified IEEE-14 bus with incomplete network admittance information.**

I. INTRODUCTION

Continuous monitoring of the voltage of every node in a given power network is crucial in observing loading conditions in each node and currents in each line of the network. Since accurate monitoring of the network is directly related to the accuracy of voltage state measurements used for estimation [1], Phasor Measurement Units (PMUs) have been increasingly popular in providing network monitoring due to their higher sampling rate and accurate measurements over traditionally used supervisory control and data acquisition (SCADA) systems. PMUs provide both magnitude and phase of the measured voltage and current, unlike SCADA.

A trivial solution for complete observability of the voltage state is to place PMUs at every node. This solution is economically not feasible [2] as the cost of deploying the PMUs scales with the network size. However, PMUs also allow (multiple) current measurements which can be combined with admittance information to reconstruct or estimate voltages at neighboring nodes. If complete network admittance information is available, PMUs can be installed on a few nodes measuring nodal voltages, and the voltages of the rest of the nodes are obtained via state estimation [3], [4] using regression techniques. Unfortunately, the line admittance parameters that model the admittance connections between nodes in a power network may be unknown or subjected to uncertainty [5]. The placement of PMUs should ensure complete voltage state estimation, even when the power network admittance information is partially known.

Next to the problem of incomplete network admittance information, PMU measurements are typically subjected to noise [6] that can compromise the accuracy of the voltage state estimates. The accuracy of voltage estimates is important when estimating line impedance and detecting line faults [7]. To minimize the effect of PMU measurement noise on the voltage estimates, PMUs should be deployed strategically over the network [8]. Such placement is however subjected to the requirements of voltage state estimation in the presence of incomplete power network admittance information.

Approaches that solve the Optimal PMU Placement (OPP) problem often rely on Integer Linear Programming (ILP) that guarantees complete observability of voltage states [9]. However, the assumption made for state estimation in ILP is complete knowledge of network admittances. Recent work on state estimation in partially known power networks is demonstrated in [10] where a Kalman Filter - Simultaneous Input and State Estimation (KF-SISE) technique is proposed to estimate both the states and the unknown inputs of a partially known power network. A different method in [11] divides the network into known and unknown parts, and uses a greedy PMU placement algorithm to minimize the covariance of the estimation error and uses the KF-SISE for state estimation.

Earlier work on minimizing the variance of the estimated voltage states was presented in [8], but requires complete knowledge of network admittance information. This paper addresses the previously mentioned limitations of voltage state estimation in partially known networks by adding the following contributions in this paper:

- A new result on the minimum number of PMUs required for complete observability of the voltage state in a partially known network.
- an OPP solution for partially known network that minimizes the effect of PMU measurement noise on voltage estimates.

The voltage state estimation technique accommodates for data from previously installed SCADA systems at certain nodes for a given power network by categorizing them as nodes with known voltages in the formulation of the

*This work was supported by DERConnect, an NSF funded project

[1]Aditya Mishra is with the Mechanical and Aerospace Engineering, University of California San Diego, San Diego, CA 92093, USA amishra@eng.ucsd.edu

[2]Raymond A. de Callafon is with faculty of Mechanical and Aerospace Engineering, University of California San Diego, San Diego, CA 92093, USA callafon@ucsd.edu

regression problem. The proposed method is illustrated on a modified version of the IEEE-14 bus circuit [12] with assumptions of not knowing certain line admittance parameters. The results show that the proposed optimal placement method guarantees complete observability of voltage states, is obtained in a partially known network and minimizes variance on the estimated states.

II. DEFINITIONS

A. Network Definitions

Consider a network $\mathcal{N}$ consisting of set of enumerated nodes $\mathcal{V}$ with size N given by $\mathcal{V} = \{1, 2, ..., N\}$ and set of edges which indicate connections between nodes given by $\mathcal{E}$. The set of edges $\mathcal{E}$ is written as

$$\mathcal{E} = \{(i,j) | \exists \text{ connection between node } i \text{ and node } j\} \quad (1)$$

For a power network $\mathcal{N}$, the voltage at each node k is given by $V_k \in \mathbf{C}$ and a nodal current flowing in each node k is given by $I_k \in \mathbf{C}$. If an edge (i,j) exists between two nodes i and j of network $\mathcal{N}$ then the edge (i,j) represents connection via an admittance $y_{ij} \in \mathbf{C}$ where $|y_{ij}| \neq 0$. Note that for any given edge (i,j), the admittance y_{ij} may or may not be known and $y_{ij} = y_{ji}$. A compact notation for connection between nodes of the network is given by the adjacency matrix A where each element A_{ij} in A is defined by

$$A_{ij} = \begin{cases} 1, & \text{if } \exists\ y_{ij} \text{ s.t. } |y_{ij}| \neq 0 \\ 0, & \text{if } (i,j) \notin \mathcal{E} \end{cases} \quad (2)$$

The information on connection between nodes is used to define two additional sets of integers. For each node $k = 1, 2, 3, ..., N$ the two sets of integers are defined as

$$\mathcal{K}(k) = \{j | A_{kj} = 1\} \quad (3)$$

$$\mathcal{J}(k) = \{j | A_{kj} = 1 \text{ and } y_{kj} \text{ is known}\} \quad (4)$$

where the set $\mathcal{K}(k)$ denotes all nodes j connected to node k and $\mathcal{J}(k)$ denotes all nodes j connected to node k, for which the admittance y_{kj} of the connections is known. It is to be noted that the $d_k = \text{length}(\mathcal{K}(k))$ where d_k is the connection degree of the node i.e number of nodes to which the given node k is connected. Similarly we define $q_k = \text{length}(\mathcal{J}(k))$ where q_k is the knowledge degree of the node.

In general $\mathcal{J}(k) \subseteq \mathcal{K}(k)$ and the following special cases are distinguished.

- We use the notation $\mathcal{J}(k) = \mathcal{K}(k)$ to indicate that all edge admittances y_{kj}, $j \in \mathcal{K}(k)$ of node k are known.
- We use $\mathcal{J}(k) \subset \mathcal{K}(k)$ to indicate that some, but not all edge admittances y_{kj}, $j \in \mathcal{K}(k)$ are known.
- Finally $\mathcal{J}(k) = \Phi$ is used to denote that none of the edge admittance y_{kj}, $j \in \mathcal{K}(k)$ of node k are known.

We use the notation $j \in \mathcal{K}(k)$ to select elements of $\mathcal{K}(k)$.

B. Set Definitions

Based on the availability of knowledge on nodal voltage V_k, nodal current I_k and/or edge admittance y_{kj} we now define nodal sets that separate the nodes of the network similar to the nodal sets in [8]. The line set $\mathcal{L}_b$ consists of those branches where the branch currents are measured. This separation is needed to group and characterize the information available for each node that will lead to results on the minimum number of PMUs to estimate nodal voltage of all nodes in the network.

Definition 2.1 ($\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{L}_b, \overline{\mathcal{S}}_0, \overline{\mathcal{S}}_1, \overline{\mathcal{S}}_2, \overline{\mathcal{S}}_3, \mathcal{S}_r, \overline{\mathcal{S}}_r, \mathcal{S}, \overline{\mathcal{S}}$): To group the nodes having $\mathcal{J}(k) = \mathcal{K}(k)$, the following nodal sets are defined:

- $\mathcal{S}_0$: $\{k | \mathcal{J}(k) = \mathcal{K}(k), \text{and } V_k, I_k \text{ known}\}$
- $\mathcal{S}_1$: $\{k | \mathcal{J}(k) = \mathcal{K}(k), \text{and } V_k \text{ known and } I_k \text{ unknown}\}$
- $\mathcal{S}_2$: $\{k | \mathcal{J}(k) = \mathcal{K}(k), \text{and } V_k \text{ unknown and } I_k \text{ known}\}$
- $\mathcal{S}_3$: $\{k | \mathcal{J}(k) = \mathcal{K}(k), \text{and } V_k, I_k \text{ unknown}\}$

In addition we define $\overline{\mathcal{S}}_m$, $m = 0, 1, 2, 3$ for which $\mathcal{J}(k) \subset \mathcal{K}(k)$. We define each $\bar{\mathcal{S}}_m$ as

- $\overline{\mathcal{S}}_0$: $\{k | \mathcal{J}(k) \subset \mathcal{K}(k), \text{and } V_k, I_k \text{ known}\}$
- $\overline{\mathcal{S}}_1$: $\{k | \mathcal{J}(k) \subset \mathcal{K}(k), \text{and } V_k \text{ known and } I_k \text{ unknown}\}$
- $\overline{\mathcal{S}}_2$: $\{k | \mathcal{J}(k) \subset \mathcal{K}(k), \text{and } V_k \text{ unknown and } I_k \text{ known}\}$
- $\overline{\mathcal{S}}_3$: $\{k | \mathcal{J}(k) \subset \mathcal{K}(k), \text{and } V_k, I_k \text{ unknown}\}$

From the set definitions the nodes of the network are broadly classified into the following two sets

- $\mathcal{S}$: $\{k | \mathcal{J}(k) = \mathcal{K}(k)\ \forall k \in \mathcal{V}\}$
- $\overline{\mathcal{S}}$: $\{k | \mathcal{J}(k) \subset \mathcal{K}(k)\ \forall k \in \mathcal{V}\}$

The information on measured branch current is stored in line set:

- $\mathcal{L}_b$: $\{(i,j) | I_{ij} \text{ is measured}\}$

This implies that if $k \in \mathcal{S}_i$ for $i = 0, 1, 2, 3$ then $k \in \mathcal{S}$, similarly if $k \in \overline{\mathcal{S}}_i$ for $i = 0, 1, 2, 3$ then $k \in \overline{\mathcal{S}}$. The grouping of nodes in sets defined in 2.1 is determined by the placement parameter ρ.

C. Definition of Placement Parameter ρ

Knowledge on nodal voltage V_k is obtained by placing a PMU at a node k. The knowledge on nodal current I_k and branch currents in branches connected to node k depends on number of current measuring channels n_c available on the PMU. Given a finite number n of PMUs, there are $\frac{N!}{n!(N-n)!}$ possible combinations where PMUs could be placed in a network with N nodes. For notation convenience, we introduce a placement parameter ρ that defines the location of the PMUs and which node or branch current are being measured. The choice of placement parameter ρ will determine the sets $\mathcal{S}_k$ and $\overline{\mathcal{S}}_K$ for $k = 0, .., 3$ according to Definition 2.1. We use the notation ρ to indicate changes in the ordering of the enumeration of nodes of the network.

D. Admittance Matrix Definition

The line connected between any two buses m, $n \in \mathcal{V}$ where $(m,n) \in \mathcal{E}$ has the following admittance parameters

b_{mn}	line charging susceptance
y_{mn}	line admittance

The network admittance matrix $\boldsymbol{Y} \in \mathbb{C}^{N\times N}$ that comprises of all admittance information of all the lines in the network $\mathcal{N}$ is defined as

$$\boldsymbol{Y}_{ij} = \begin{cases} y_i + \sum_{k=1,k\neq i}^{N} y_{ik} & i = j \\ -y_{ij} & i \neq j \end{cases} \tag{5}$$

in which y_i term on the diagonal element is defined as

$$y_i = y_i^s + \sum_{k=1,k\neq i}^{N} \frac{jb_{ik}}{2} \tag{6}$$

where y_i^s is the shunt admittance at Bus i.

Using the IEEE π-model [13] of transmission line and Ohm's law, the current in line $(m,n) \in \mathcal{E}$ is formulated as

$$i_{mn} = \frac{jb_{mn}}{2} v_m + y_{mn}(v_m - v_n) \tag{7}$$

Using (7), an equation between network branch current vector $\boldsymbol{I}^b \in \mathbb{C}^{L\times 1}$, where L is the total number of lines in the network $\mathcal{N}$, and all the nodal voltages $\boldsymbol{V}$ is formulated based on Ohm's law. The network branch current vector $\boldsymbol{I}^b$ is given by

$$\boldsymbol{I}^b = \begin{bmatrix} i_{1m_1} & i_{1m_2} & . & . & i_{1m_k} & i_{2n_1} & . & i_{2n_k} & . & i_{Nl_k} \end{bmatrix}^T \tag{8}$$

where $(1,m_1),(1,m_2),..,(1,m_k),(2,n_1),..,(2,n_k),..,(N,l_k) \in \mathcal{E}$. Using Ohm's law the branch current vectors is equated to the nodal voltages by

$$\boldsymbol{I}^b = \boldsymbol{B}\boldsymbol{V} \tag{9}$$

where $\boldsymbol{B} \in \mathbb{C}^{L\times N}$ is the branch admittance matrix. The n-th row of branch admittance $\boldsymbol{B}$ that represents branch admittance information for the n-th line (m,n) corresponding to the current vector $I_{(m,n)}$ is given by

$$\boldsymbol{B}_{xy} = \begin{cases} \frac{jb_{mn}}{2} + y_{mn} & y = m \\ -y_{mn} & y = n \\ 0 & y \neq m, y \neq n \end{cases} \tag{10}$$

For simplicity, the admittance matrices $\boldsymbol{B}$ and $\boldsymbol{Y}$ and the complex vectors $\boldsymbol{I}^b$, $\boldsymbol{I}$ and $\boldsymbol{V}$ are considered single phase although these matrices and vectors can be extended to three-phase. For the three-phase case, the admittance matrices $\boldsymbol{Y}$ and $\boldsymbol{B}$ can be extended to accommodate three-phase admittances. The estimation can be extended for unsymmetrical operation, by modeling the unsymmetric loads as shunt resistance.

III. Voltage estimation with incomplete information

Using the set definitions of $\mathcal{S}$ and $\overline{\mathcal{S}}$ in Definition 2.1, the relationship between the nodal voltages and the nodal currents of these sets are written as

$$\begin{bmatrix} I_S \\ I_{\overline{S}} \end{bmatrix} = \begin{bmatrix} Y_{SS} & Y_{S\overline{S}} \\ Y_{\overline{S}S} & Y_{\overline{SS}} \end{bmatrix} \begin{bmatrix} V_S \\ V_{\overline{S}} \end{bmatrix} \tag{11}$$

where admittance submatrices Y_{SS}, $Y_{S\overline{S}}$, and $Y_{\overline{S}S}$ are completely known and $Y_{\overline{SS}}$ is partially known. This section gives an overview on how the permutation of nodal sets are used to compute the minimal number of PMUs for partially known network and formulate an optimal voltage state estimation.

A. Permutation of Network Admittance Matrix

Based on the placement parameter ρ, the nodes are distributes into nodal sets $\mathcal{S}_K$ or $\overline{\mathcal{S}}_k$ as defined in Definition 2.1. Taking the nodal sets into account, the network admittance matrix is permuted by the placement parameter ρ via $\tilde{\mathbf{Y}}(\rho) = U(\rho)\mathbf{Y}U(\rho)^T$ where $U(\rho)$ is the unitary permutation matrix. The permuted network admittance matrix is given by

$$\tilde{\mathbf{Y}}(\rho) = \begin{bmatrix} \mathbf{Y}_{00} & \mathbf{Y}_{01} & \mathbf{Y}_{02} & \mathbf{Y}_{03} & \mathbf{Y}_{0\overline{0}} & \mathbf{Y}_{0\overline{1}} & \mathbf{Y}_{0\overline{2}} & \mathbf{Y}_{0\overline{3}} \\ \mathbf{Y}_{10} & \mathbf{Y}_{11} & \mathbf{Y}_{12} & \mathbf{Y}_{13} & \mathbf{Y}_{1\overline{0}} & \mathbf{Y}_{1\overline{1}} & \mathbf{Y}_{1\overline{2}} & \mathbf{Y}_{1\overline{3}} \\ \mathbf{Y}_{20} & \mathbf{Y}_{21} & \mathbf{Y}_{22} & \mathbf{Y}_{23} & \mathbf{Y}_{2\overline{0}} & \mathbf{Y}_{2\overline{1}} & \mathbf{Y}_{2\overline{2}} & \mathbf{Y}_{2\overline{3}} \\ \mathbf{Y}_{30} & \mathbf{Y}_{31} & \mathbf{Y}_{32} & \mathbf{Y}_{33} & \mathbf{Y}_{3\overline{0}} & \mathbf{Y}_{3\overline{1}} & \mathbf{Y}_{3\overline{2}} & \mathbf{Y}_{3\overline{3}} \\ \mathbf{Y}_{\overline{0}0} & \mathbf{Y}_{\overline{0}1} & \mathbf{Y}_{\overline{0}2} & \mathbf{Y}_{\overline{0}3} & \mathbf{Y}_{\overline{00}} & \mathbf{Y}_{\overline{01}} & \mathbf{Y}_{\overline{02}} & \mathbf{Y}_{\overline{03}} \\ \mathbf{Y}_{\overline{1}0} & \mathbf{Y}_{\overline{1}1} & \mathbf{Y}_{\overline{1}2} & \mathbf{Y}_{\overline{1}3} & \mathbf{Y}_{\overline{10}} & \mathbf{Y}_{\overline{11}} & \mathbf{Y}_{\overline{12}} & \mathbf{Y}_{\overline{13}} \\ \mathbf{Y}_{\overline{2}0} & \mathbf{Y}_{\overline{2}1} & \mathbf{Y}_{\overline{2}2} & \mathbf{Y}_{\overline{2}3} & \mathbf{Y}_{\overline{20}} & \mathbf{Y}_{\overline{21}} & \mathbf{Y}_{\overline{22}} & \mathbf{Y}_{\overline{23}} \\ \mathbf{Y}_{\overline{3}0} & \mathbf{Y}_{\overline{3}1} & \mathbf{Y}_{\overline{3}2} & \mathbf{Y}_{\overline{3}3} & \mathbf{Y}_{\overline{30}} & \mathbf{Y}_{\overline{31}} & \mathbf{Y}_{\overline{32}} & \mathbf{Y}_{\overline{33}} \end{bmatrix} \tag{12}$$

where the matrix Y_{kj}, $k \neq j$, has the admittance of lines connecting nodes from nodal set $\mathcal{S}_k$ to nodal set $\mathcal{S}_j$, the matrix $Y_{k\overline{j}}$ has the admittance of lines connecting nodes from nodal set $\mathcal{S}_k$ to nodal set $\overline{\mathcal{S}}_j$, and the matrix Y_{kk} or the matrix $Y_{\overline{kk}}$ has the admittance of lines connecting the nodes inside the nodal set $\mathcal{S}_k$ or $\overline{\mathcal{S}}_k$ respectively.

B. Line currents and Branch Admittance Matrix

Since estimating voltage vectors $\mathbf{v}_2$, $\mathbf{v}_3$, $\overline{\mathbf{v}}_2$, and $\overline{\mathbf{v}}_3$ requires branch current and branch admittance information, the admittance matrix relating voltages and branch currents is formulated as a matrix of smaller admittance matrices corresponding to each set $\mathcal{S}$ and $\overline{\mathcal{S}}$ as defined in Definition 2.1. Depending on the nodes, the line currents be classified into the following:

- if the current in branch (i,j) of known admittance is measured and $i,\, j \in \mathcal{S}$ then $I_{ij} \in \mathbf{b}_0, \mathbf{b}_0 \in \mathbb{C}^{n_b\times 1}$. Note that the length of the array is n_b.
- if the current in branch (i,j) of known admittance is measured and if $i \wedge j \in \overline{\mathcal{S}}$ then the current in branch $I_{ij} \in \mathbf{b}_0 \in \mathbb{C}^{\overline{n}_b\times 1}$. Note that the length of the array is $\overline{n}_b$.

The array of measured currents $[\mathbf{b}_0\ \overline{\mathbf{b}}_0]^T$ is related to the node voltages by using the equation $[\mathbf{b}_0\ \overline{\mathbf{b}}_0]^T = \mathbf{B}(\rho)\mathbf{v}$, where the branch admittance matrix $\mathbf{B}(\rho)$ is given by

$$\begin{bmatrix} \mathbf{b}_0 \\ \overline{\mathbf{b}}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{B}_0 & \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \overline{\mathbf{B}}_0 & \overline{\mathbf{B}}_1 & \overline{\mathbf{B}}_2 & \overline{\mathbf{B}}_3 \\ \hat{\mathbf{B}}_0 & \hat{\mathbf{B}}_1 & \hat{\mathbf{B}}_2 & \hat{\mathbf{B}}_3 & \hat{\overline{\mathbf{B}}}_0 & \hat{\overline{\mathbf{B}}}_1 & \hat{\overline{\mathbf{B}}}_2 & \hat{\overline{\mathbf{B}}}_3 \end{bmatrix} \begin{bmatrix} V_s \\ V_{\overline{s}} \end{bmatrix} \tag{13}$$

where

$$\begin{bmatrix} V_s & V_{\overline{s}} \end{bmatrix}^T = \begin{bmatrix} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \overline{\mathbf{v}}_0 & \overline{\mathbf{v}}_1 & \overline{\mathbf{v}}_2 & \overline{\mathbf{v}}_3 \end{bmatrix}^T \tag{14}$$

The sub-matrices of branch admittance matrices are categorized into these four groups

- $\mathbf{B}_k, k = 0,1,2,3$ are the matrix coefficients for voltages $\mathbf{v}_k$ corresponding to branch currents in b_0.
- $\overline{\mathbf{B}}_k, k = 0,1,2,3$ are the matrix coefficients for voltages $\overline{\mathbf{v}}_k$ corresponding to branch currents in $\mathbf{b}_0$.

- $\hat{\mathbf{B}}_k, k = 0, 1, 2, 3$ are the matrix coefficients for voltages $\mathbf{v}_k$ corresponding to branch currents in $\overline{\mathbf{b}}_0$.
- $\hat{\overline{\mathbf{B}}}_k, k = 0, 1, 2, 3$ are the matrix coefficients for voltages $\overline{\mathbf{v}}_k$ corresponding to branch currents in $\overline{\mathbf{b}}_0$.

The defined sub-matrices of the admittance matrices will be used in the formulation of the regression equation to estimate the voltage vectors $\mathbf{v}_2$, $\mathbf{v}_3$, $\overline{\mathbf{v}}_2$, and $\overline{\mathbf{v}}_3$.

C. Voltage state estimation

For a given placement parameter ρ, the state estimation aims at estimating voltages $\mathbf{v}_2$, $\mathbf{v}_3$, $\overline{\mathbf{v}}_2$, and $\overline{\mathbf{v}}_3$ of the nodes in $\mathcal{S}_2$, $\mathcal{S}_3$, $\overline{\mathcal{S}}_2$, and $\overline{\mathcal{S}}_3$. To estimate the voltages the following regression is used

$$\mathbf{\Phi}_Y(\rho)\begin{bmatrix}\mathbf{v}_2 & \mathbf{v}_3 & \overline{\mathbf{v}}_2 & \overline{\mathbf{v}}_3\end{bmatrix}^T = \mathbf{I} \tag{15}$$

where vector $\mathbf{I}$ is given by

$$\mathbf{I} = \begin{bmatrix}\mathbf{b}_0 & \overline{\mathbf{b}}_0 & \mathbf{i}_0 & \mathbf{i}_2\end{bmatrix}^T - \mathbf{\Psi}_V(\rho)\begin{bmatrix}\mathbf{v}_0 & \mathbf{v}_1 & \overline{\mathbf{v}}_0 & \overline{\mathbf{v}}_1\end{bmatrix}^T \tag{16}$$

and the measured voltage-admittance matrix $\mathbf{\Psi}_V(\rho)$ is given by

$$\mathbf{\Psi}_V(\rho) = \begin{bmatrix}\mathbf{B}_0 & \mathbf{B}_1 & \overline{\mathbf{B}}_0 & \overline{\mathbf{B}}_1 \\ \hat{\mathbf{B}}_0 & \hat{\mathbf{B}}_1 & \hat{\overline{\mathbf{B}}}_0 & \hat{\overline{\mathbf{B}}}_1 \\ \mathbf{Y}_{00} & \mathbf{Y}_{01} & \mathbf{Y}_{0\overline{0}} & \mathbf{Y}_{0\overline{1}} \\ \mathbf{Y}_{20} & \mathbf{Y}_{21} & \mathbf{Y}_{2\overline{0}} & \mathbf{Y}_{2\overline{1}}\end{bmatrix} \tag{17}$$

The regressor matrix $\mathbf{\Phi}_Y(\rho)$ in (20) is given by

$$\mathbf{\Phi}_Y(\rho) = \begin{bmatrix}\mathbf{B}_2 & \mathbf{B}_3 & \overline{\mathbf{B}}_2 & \overline{\mathbf{B}}_3 \\ \hat{\mathbf{B}}_2 & \hat{\mathbf{B}}_3 & \hat{\overline{\mathbf{B}}}_2 & \hat{\overline{\mathbf{B}}}_3 \\ \mathbf{Y}_{02} & \mathbf{Y}_{03} & \mathbf{Y}_{0\overline{2}} & \mathbf{Y}_{0\overline{3}} \\ \mathbf{Y}_{22} & \mathbf{Y}_{23} & \mathbf{Y}_{2\overline{2}} & \mathbf{Y}_{2\overline{3}}\end{bmatrix} \tag{18}$$

where the regressor matrix $\mathbf{\Phi}_Y(\rho) \in \mathbf{C}^{(n_b+\overline{n}_b+n_0+n_2)\times(n_2+n_3+\overline{n}_2+\overline{n}_3)}$ and the voltage-admittance matrix $\mathbf{\Psi}_V(\rho) \in \mathbf{C}^{(n_b+\overline{n}_b+n_0+n_2)\times(n_0+n_1+\overline{n}_0+\overline{n}_1)}$.

To estimate the unmeasured voltages $\mathbf{v}_2$, $\mathbf{v}_3$, $\overline{\mathbf{v}}_2$, and $\overline{\mathbf{v}}_3$, a least squares problem is formulated as

$$\min_{[\mathbf{v}_2\ \mathbf{v}_3\ \overline{\mathbf{v}}_2\ \overline{\mathbf{v}}_3]^T} ||\mathbf{\Phi}_Y(\rho)[\mathbf{v}_2\ \mathbf{v}_3\ \overline{\mathbf{v}}_2\ \overline{\mathbf{v}}_3]^T - \mathbf{I}||_2^2 \tag{19}$$

where the regressor matrix $\mathbf{\Phi}_Y(\rho)$ is known. The solution to (19) is given as

$$\begin{bmatrix}\mathbf{v}_2 & \mathbf{v}_3 & \overline{\mathbf{v}}_2 & \overline{\mathbf{v}}_3\end{bmatrix}^T = \mathbf{\Phi}_Y(\rho)^\dagger \mathbf{I} \tag{20}$$

where $\mathbf{\Phi}_Y(\rho)^\dagger = (\mathbf{\Phi}_Y(\rho)^T\mathbf{\Phi}_Y(\rho))^{-1}\mathbf{\Phi}_Y(\rho)^T$. Unique determination of the voltage vectors $\mathbf{v}_2$, $\mathbf{v}_3$, $\overline{\mathbf{v}}_2$ and $\overline{\mathbf{v}}_3$ in (20) implies the regressor matrix $\mathbf{\Phi}_Y(\rho)$ having linearly independent columns. It should be noted that the size of $\mathbf{\Phi}_Y(\rho)$ depends on the number of PMUs available on the network, while rank properties depend on where PMUs are placed indicated by the placement parameter ρ.

D. Minimum number of PMUs

Consider a power network $\mathcal{N}$ with N nodes where n PMUs ($n < N$) are to be placed with each measuring n_c nodal/branch currents ($n_c \geq 1$).

Definition 3.1: The node voltages of the network are observable if the regression matrix $\mathbf{\Phi}_Y(\rho)$ in (15) has full column rank.

Lemma 3.1: For a given network $\mathcal{N}$, there exists a placement parameter ρ for minimum number of PMUs N_{pmu}^{min} which makes the voltage states of the network observable according to Definition 3.1 where

$$N_{pmu}^{min} = \min_{\rho}\lceil\frac{N - n_2(\rho) + \overline{n}_0(\rho)}{n_c + 1}\rceil \tag{21}$$

where $n_2(\rho)$ is the size of nodal set $\mathcal{S}_2$, $\overline{n}_0(\rho)$ is the size of nodal set $\overline{\mathcal{S}}_0$ and $\lceil.\rceil$ is defined as $\lceil x\rceil = a$ such that $a \in \mathbb{Z}^+$ and $x + 1 \geq a \geq x$.

Compared to the minimum PMU lemma in [8] for network with complete admittance information given by

$$N_{pmu}^{min} = \lceil\frac{N - n_2}{n_c + 1}\rceil \tag{22}$$

It is observed that more PMUs are required for the same network when partial admittance information is available.

E. Proof of Lemma 3.1

For complete observability, the regressor matrix $\Phi_{\mathbf{Y}}(\rho)$ should be full column rank from Definition 3.1. Since the regressor matrix $\mathbf{\Phi}_Y(\rho)$ is a function of the placement parameter ρ, we can choose a ρ such that the regressor matrix $\mathbf{\Phi}_Y(\rho)$ has full column rank for complete observability as per Definition 3.1. This implies that $n_2 + n_3 + \overline{n}_2 + \overline{n}_3$ columns of regressor matrix $\mathbf{\Phi}_Y$ are linearly independent having dimension $n_b + \overline{n}_b + n_0 + n_2$. This is possible when

$$n_b + \overline{n}_b + n_0 + n_2 \geq n_2 + n_3 + \overline{n}_2 + \overline{n}_3 \tag{23}$$

The PMU on buses in $\mathcal{S}_1$ and $\overline{\mathcal{S}}_1$ will measure n_c branch currents, and the PMU on buses in $\mathcal{S}_0$ and $\overline{\mathcal{S}}_0$ will measure $n_c - 1$ branch currents, hence the number of measured branch currents is given by

$$n_b = n_c \times (n_1 + \overline{n}_1) + (n_c - 1) \times (n_0 + \overline{n}_0) \tag{24}$$

Hence the inequality is expressed as

$$n_c \times (n_1 + \overline{n}_1) + (n_c - 1) \times (n_0 + \overline{n}_0) + n_0 + n_2 \geq n_2 + n_3$$

where the number of PMUs $N_{pmu} = n_1(\rho) + n_0(\rho) + \overline{n}_1(\rho) + \overline{n}_0(\rho)$ because the PMUs will either be installed on nodes in $\mathcal{S}_0$, $\mathcal{S}_1$, $\overline{\mathcal{S}}_0$, or $\overline{\mathcal{S}}_1$. Hence the inequality is written as

$$n_c \times N_{pmu} - n_0 - \overline{n}_0 + n_0 + n_2 \geq n_2 + n_3 + \overline{n}_2 + \overline{n}_3$$
$$n_c \times N_{pmu} + n_2 \geq N - (n_0 + n_1 + \overline{n}_0 + \overline{n}_1)$$
$$N_{pmu} \geq \frac{N - n_2 + \overline{n}_0}{n_c + 1}$$

Since the nodal sets $\mathcal{S}_2$ and $\overline{\mathcal{S}}_0$ are determined by the placement parameter ρ, the respective sizes of the nodal sets are a function of ρ as well and are denoted by $n_2(\rho)$ and $\overline{n}_0(\rho)$. Hence the minimum number of PMUs is given by

$$N_{pmu}^{min} = \min_{\rho} \lceil \frac{N - n_2(\rho) + \overline{n}_0(\rho)}{n_c + 1} \rceil \tag{25}$$

IV. Minimizing estimation variance

Although Lemma 3.1 states the minimum number of PMUs required for complete observability for any given network with partial network admittance information, it does not state where the PMUs should be placed. The goal is to have better estimates of the unmeasured voltages as the PMUs placed on nodes belonging to nodal sets $\mathcal{S}_0$, $\mathcal{S}_1$, $\overline{\mathcal{S}}_0$, and $\overline{\mathcal{S}}_1$ will have measurement noise.

The noise due to PMU measurements of voltages in $\mathbf{v}_0$, $\mathbf{v}_1$, $\overline{\mathbf{v}}_0$, and $\overline{\mathbf{v}}_1$ and currents in $\mathbf{b}_0$, $\overline{\mathbf{b}}_0$, $\mathbf{i}_0$ and $\mathbf{i}_2$ can result in inaccurate voltage estimates. Hence, it is important to minimize variance of voltages in $\mathbf{v}_2$, $\mathbf{v}_3$, $\overline{\mathbf{v}}_2$, and $\overline{\mathbf{v}}_3$ induced by PMU measurement noise. This is done by formulating the variance of the estimated states as a function of PMU measurement noise parameters and placement parameter ρ and then finding the optimal placement parameter ρ that minimizes the noise of the estimates.

A. PMU measurement noise model

As assumed in [8], all PMU measurement noise are zero mean and have a given variance uncorrelated to each other. Based on the assumptions, the measurement noise of voltages $\mathbf{v}_0$, $\mathbf{v}_1$, $\overline{\mathbf{v}}_0$, and $\overline{\mathbf{v}}_1$ has the following variance-covariance matrix

$$\begin{bmatrix} \mathbf{\Lambda}_{n_0\times n_0} & \mathbf{0}_{n_0\times n_1} & \mathbf{0}_{n_0\times \overline{n}_0} & \mathbf{0}_{n_0\times \overline{n}_1} \\ \mathbf{0}_{n_1\times n_0} & \mathbf{\Lambda}_{n_1\times n_1} & \mathbf{0}_{n_1\times \overline{n}_0} & \mathbf{0}_{n_1\times \overline{n}_1} \\ \mathbf{0}_{\overline{n}_0\times n_0} & \mathbf{0}_{\overline{n}_0\times n_1} & \mathbf{\Lambda}_{\overline{n}_0\times \overline{n}_0} & \mathbf{0}_{\overline{n}_0\times \overline{n}_1} \\ \mathbf{0}_{\overline{n}_1\times n_0} & \mathbf{0}_{\overline{n}_1\times n_1} & \mathbf{0}_{\overline{n}_1\times \overline{n}_0} & \mathbf{\Lambda}_{\overline{n}_1\times \overline{n}_1} \end{bmatrix} \tag{26}$$

where $\mathbf{\Lambda}_{k\times k}$ for $k = n_0, n_1, \overline{n}_0, \overline{n}_1$ is the diagonal variance matrices for nodal sets $\mathcal{S}_0$, $\mathcal{S}_1$, $\overline{\mathcal{S}}_0$, and $\overline{\mathcal{S}}_1$. respectively. The diagonal entries of $\mathbf{\Lambda}_{k\times k}$ for $k = n_0, n_1, \overline{n}_0, \overline{n}_1$ are $\lambda_i^{\mathcal{S}_0}$, $\lambda_i^{\mathcal{S}_1}$, $\lambda_i^{\overline{\mathcal{S}}_0}$, and $\lambda_i^{\overline{\mathcal{S}}_0}$ for nodal sets $\mathcal{S}_0$, $\mathcal{S}_1$, $\overline{\mathcal{S}}_0$, and $\overline{\mathcal{S}}_1$. respectively where i is the corresponding node in the nodal sets. In (26), the $\mathbf{0}_{m\times n}$ is a zero matrix of size $M \times N$. The measurement noise of current measurements $\mathbf{b}_0$, $\overline{\mathbf{b}}_0$, $\mathbf{i}_0$, and $\mathbf{i}_2$ has the same variance-covariance matrix structure as in (26) and is given by

$$\begin{bmatrix} \mu_{n_b\times n_b} & \mathbf{0}_{n_b\times \overline{n}_b} & \mathbf{0}_{n_b\times n_0} & \mathbf{0}_{n_b\times n_2} \\ \mathbf{0}_{\overline{n}_b\times n_b} & \mu_{\overline{n}_b\times \overline{n}_b} & \mathbf{0}_{\overline{n}_b\times n_0} & \mathbf{0}_{\overline{n}_b\times n_2} \\ \mathbf{0}_{n_0\times n_b} & \mathbf{0}_{n_0\times \overline{n}_b} & \mu_{n_0\times n_0} & \mathbf{0}_{n_0\times n_2} \\ \mathbf{0}_{n_2\times n_b} & \mathbf{0}_{n_2\times \overline{n}_b} & \mathbf{0}_{n_2\times n_0} & \mu_{n_2\times n_2} \end{bmatrix} \tag{27}$$

where the diagonal matrices $\mu_{n_b\times n_b}$, $\mu_{\overline{n}_b\times \overline{n}_b}$, $\mu_{n_0\times n_0}$, and $\mu_{n_2\times n_2}$ are the variance matrices of measured current vectors $\mathbf{b}_0$, $\overline{\mathbf{b}}_0$, $\mathbf{i}_0$ and $\mathbf{i}_2$ with diagonal entries $\mu_i^{\mathbf{b}_0}$, $\mu_i^{\overline{\mathbf{b}}_0}$, $\mu_i^{\mathcal{S}_0}$, and $\mu_i^{\mathcal{S}_2}$ where i is the corresponding node. It is assumed that all installed PMUs have the same noise characteristics with λ and $\mu \in \mathbb{R}$ as the voltage and current noise variance respectively. To find the optimal placement parameter ρ, the variance of the estimated voltages $\mathbf{v}_2$, $\mathbf{v}_3$, $\overline{\mathbf{v}}_2$, and $\overline{\mathbf{v}}_3$ has to be formulated as the function of the voltage noise variance λ and current noise variance μ.

B. Estimated Voltage Variance

The variance of the estimated voltages is given by

$$var(\begin{bmatrix}\mathbf{v}_2 & \mathbf{v}_3 & \overline{\mathbf{v}}_2 & \overline{\mathbf{v}}_3\end{bmatrix}^T) = \Phi_{\mathbf{Y}}(\rho)^{\dagger}(var(\mathbf{I}))(\Phi_{\mathbf{Y}}(\rho)^{\dagger})^T \tag{28}$$

where $var(.)$ represents the variance of the vector. The variance of $\mathbf{I}$ is written as

$$var(\begin{bmatrix}\mathbf{b}_0 \ \overline{\mathbf{b}}_0 \ \mathbf{i}_0 \ \mathbf{i}_2\end{bmatrix}^T) + \mathbf{\Psi}_V(var(\begin{bmatrix}\mathbf{v}_0 \ \mathbf{v}_1 \ \overline{\mathbf{v}}_0 \ \overline{\mathbf{v}}_1\end{bmatrix}^))\mathbf{\Psi}_V^T \tag{29}$$

since the noise model assumes no correlation between voltages and currents. Based on (26) and (27), the variance in (29) is formulated as the function voltage noise variance λ and current noise variance μ as the following

$$var(\mathbf{I}) = \mu\mathcal{I}_{n_b+\overline{n}_b+n_0+n_2} + \mathbf{\Psi}_V(\rho)(\lambda\mathcal{I}_{n_0+n_1+\overline{n}_0+\overline{n}_1})\mathbf{\Psi}_V(\rho)^T \tag{30}$$

where $\mathcal{I}_{m\times m}$ is an $m \times m$ identity matrix. Substituting (30) in (28), the variance of estimated voltage states is formulated as the function of noise in the following way

$$var(\begin{bmatrix}\mathbf{v}_2 & \mathbf{v}_3 & \overline{\mathbf{v}}_2 & \overline{\mathbf{v}}_3\end{bmatrix}^T) = \Phi_{\mathbf{Y}}(\rho)^{\dagger}(\mu\mathcal{I}_{n_b+\overline{n}_b+n_0+n_2}+ \\ \mathbf{\Psi}_V(\rho)(\lambda\mathcal{I}_{n_0+n_1+\overline{n}_0+\overline{n}_1})\mathbf{\Psi}_V(\rho)^T)(\Phi_{\mathbf{Y}}(\rho)^{\dagger})^T \tag{31}$$

In (31), the variance of the estimates is a function of the placement parameter ρ. To determine the optimal placement parameter ρ which will define optimal PMU placement to minimize the variance on the estimated states, the following optimization criterion is used

$$\begin{aligned} &\min_{\rho} \ \text{tr}(\Phi_{\mathbf{Y}}(\rho)^{\dagger}(\mu\mathcal{I}_{n_b++\overline{n}_b+n_0+n_2}+ \\ &\mathbf{\Psi}_V(\rho)(\lambda\mathcal{I}_{n_0+n_1+\overline{n}_0+\overline{n}_1})(\mathbf{\Psi}_V(\rho))^T)((\Phi_{\mathbf{Y}}(\rho))^{\dagger})^T) \\ &\text{s.t. rank}(\Phi_{\mathbf{Y}}(\rho)) \geq n_2 + n_3 + \overline{n}_2 + \overline{n}_3 \end{aligned} \tag{32}$$

where $tr(.)$ denotes the trace of the matrix and $rank(.)$ denotes the rank of the matrix. As given in Definition 3.1, the rank of $\Phi_{\mathbf{Y}}(\rho)$ must be at least $n_2+n_3+\overline{n}_2+\overline{n}_3$ for complete observability. Clearly, the constraint $\text{rank}(\Phi_{\mathbf{Y}}(\rho)) = n_2 + n_3 + \overline{n}_2 + \overline{n}_3$ in (32) gives the minimum number of PMUs for complete observability and their optimal placements for the least variance for those number of PMUs. Since both the regressor matrix $\Phi_{\mathbf{Y}}(\rho)$ and the voltage-admittance matrix $\mathbf{\Psi}_V(\rho)$ are a function of the placement parameter ρ, optimal placement parameter ρ will lead to the optimal placement of PMUs.

V. Application to IEEE-14 Bus

To demonstrate the OPP method, the IEEE-14 bus model [12] is chosen as an example which has no nodal current being injected at Bus 7 and the lines $(1,2), (4,7), (5,6), (6,12)$ and $(13,14)$ have unknown admittances. As given in [6], the values of the PMU measurement noise parameters, voltage noise variance and current noise variance are $\lambda = 3.8\times10^{-5}$ and $\mu = 2.7 \times 10^{-5}$ respectively. Single channel PMUs are considered for placement, which measure current in one line and voltage at a bus.

TABLE I

NODAL SETS DEFINED BY THE OPTIMAL PLACEMENT PARAMETER ρ_{opt}

$\mathcal{S}_0$	{ }	$\overline{\mathcal{S}}_0$	{ 7}
$\mathcal{S}_1$	{ 3,10}	$\overline{\mathcal{S}}_1$	{ 2, 5, 6, 12, 14}
$\mathcal{S}_2$	{ }	$\overline{\mathcal{S}}_2$	{ }
$\mathcal{S}_3$	{ 8, 9, 11}	$\overline{\mathcal{S}}_3$	{ 1,4,13}

A. Optimal placement of PMUs

Based on the optimization criterion in (32), the optimal placement parameter ρ_{opt} has defined the nodal sets and branch sets in Table I. The line currents being measured by the PMUs are given by the current vectors $\mathbf{b}_0 = [I_{10,11}]$ and $\overline{\mathbf{b}}_0 = [I_{2,4}, I_{3,4}, I_{5,1}, I_{6,11}, I_{7,8}, I_{12,13}, I_{14,9}]$. The placement parameter ρ_{opt} places the PMUs on eight buses $3, 10, 7, 2, 5, 6, 12, 14$ measuring lines $(3, 4), (10, 11), (7, 8), (2, 4), (5, 1), (6, 11), (12, 13), (14, 9)$ respectively. This confirms Lemma 3.1 of placing 7 PMUs.

B. Voltage state estimation

Based on (19), the estimates of $\mathbf{v}_3$ and $\overline{\mathbf{v}}_3$ are computed for the optimal placement ρ_{opt}. To illustrate how (32) minimizes the effect of PMU measurement noise on the estimates, a sub-optimal placement ρ_{sub_opt} is considered where $\mathcal{S}_0 = \{\}$, $\mathcal{S}_1 = \{3, 9, 10\}$, $\mathcal{S}_2 = \{\}$,$\mathcal{S}_3 = \{8, 11\}$,$\overline{\mathcal{S}}_0 = \{7\}$,$\overline{\mathcal{S}}_1 = \{1, 2, 6, 12\}$,$\overline{\mathcal{S}}_2 = \{\}$, and $\overline{\mathcal{S}}_3 = \{13, 4, 5, 14\}$ where 8 PMUs are deployed at buses $3, 9, 10, 7, 1, 2, 6$ and 12. It is to be noted that the placement ρ_{sub_opt} also guarantees complete observability of voltage states according to Definition 3.1. The plots in Fig.1 shows that the optimal placement significantly reduces the effect of PMU noise on the estimated voltage states. The average variance on the estimated voltage states for placement ρ_{opt} is 3.13×10^{-5} whereas for placement ρ_{sub_opt} is 3.45×10^{-5}. Hence the estimates optimal placement ρ_{opt} is 10% better than the sub-optimal placement ρ_{sub_opt}.

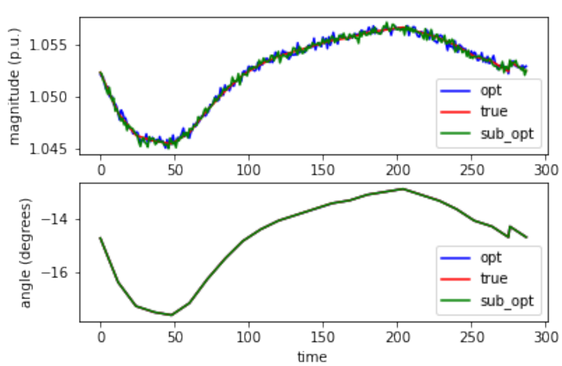

Fig. 1. Estimated V_{11} comparison for optimal placement ρ_{opt} in blue and sub-optimal placement ρ_{sub_opt} in green. For illustration purpose, the noise parameters λ and μ are multiplied by 10.

VI. CONCLUSIONS

This paper proposed a novel PMU placement method for power networks with partial admittance information that guarantees complete observability and minimizes the effect of PMU measurement noise on the estimated voltage states. Eight nodal sets are defined, categorizing the network nodes based on the information available on nodal voltage, nodal current, and the admittance information on the lines connected to the node. A defined placement parameter ρ determines the locations of the PMUs. The proposed least squares estimation solution estimates the unmeasured voltage where the regressor matrix and the voltage-admittance matrix are functions of the placement parameter ρ. Based on the definition of voltage state observability, a proposed lemma states the minimum number of PMUs required for complete observability for a given partially known network as function of the placement parameter ρ. The optimal placement for the minimum number of PMUs is the solution of an optimization criterion with the objective function as a function of the PMU measurement noise parameters. The proposed method determines the minimum number of single current measuring channel PMU on a modified IEEE-14 bus network to be 8. The optimal placement of the PMUs reduced the variance on the estimates by 10% compared to a sub-optimal placement of the PMUs.

REFERENCES

[1] M. S. Thomas and J. D. McDonald, *Power system SCADA and smart grids*. CRC press, 2017.

[2] A. Almunif and L. Fan, "Optimal pmu placement for modeling power grid observability with mathematical programming methods," *International Transactions on Electrical Energy Systems*, vol. 30, no. 2, p. e12182, 2020.

[3] M. Meriem, C. Bouchra, B. Abdelaziz, S. O. B. Jamal, E. M. Faissal, and C. Nazha, "Study of state estimation using weighted-least-squares method (wls)," in *2016 International Conference on Electrical Sciences and Technologies in Maghreb (CISTEM)*, 2016, pp. 1–5.

[4] X. Zhou, Z. Liu, Y. Guo, C. Zhao, J. Huang, and L. Chen, "Gradient-based multi-area distribution system state estimation," *IEEE Transactions on Smart Grid*, vol. 11, no. 6, pp. 5325–5338, 2020.

[5] Y. Yuan, S. H. Low, O. Ardakanian, and C. J. Tomlin, "Inverse power flow problem," *IEEE Transactions on Control of Network Systems*, 2022.

[6] M. Brown, M. Biswal, S. Brahma, S. J. Ranade, and H. Cao, "Characterizing and quantifying noise in pmu data," in *2016 IEEE Power and Energy Society General Meeting (PESGM)*. IEEE, 2016, pp. 1–5.

[7] A. Mishra and R. A. de Callafon, "Recursive estimation of three phase line admittance in electric power networks," *IFAC-PapersOnLine*, vol. 53, no. 2, pp. 34–39, 2020.

[8] ——, "Algebraic approach to pmu placement for minimum variance linear state estimation in power networks," *IEEE Transactions on Power Systems*, pp. 1–10, 2022.

[9] X. Chen, L. Sun, T. Chen, Y. Sun, K. J. Tseng, K. V. Ling, W. K. Ho, and G. A. Amaratunga, "Full coverage of optimal phasor measurement unit placement solutions in distribution systems using integer linear programming," *Energies*, vol. 12, no. 8, p. 1552, 2019.

[10] M. A. Abooshahab, M. Hovd, and R. R. Bitmead, "Disturbance and state estimation in partially known power networks," in *2019 IEEE Conference on Control Technology and Applications (CCTA)*, 2019, pp. 98–105.

[11] M. Ali Abooshahab, M. Hovd, and G. Valmorbida, "Optimal sensor placement for partially known power system dynamic estimation," in *2021 IEEE PES Innovative Smart Grid Technologies Europe (ISGT Europe)*, 2021, pp. 1–6.

[12] "Power systems test case archive." [Online]. Available: https://labs.ece.uw.edu/pstca/pf14/pg_tca14bus.htm

[13] H. Safar, "Power transmission line analysis using exact, nominal , and modified models," in *2010 The 2nd International Conference on Computer and Automation Engineering (ICCAE)*, vol. 5, 2010, pp. 128–134.