2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Maximizing the Exponential Decay Rate for Finite Dimensional Bilinear Systems using Passivity-Based Controllers

Vlad Mihaly, *Member, IEEE*, Mircea Şuşcă, *Member, IEEE* and Petru Dobra, *Member, IEEE*

***Abstract*— Passivity-based controllers which ensure the asymptotic stability of the closed-loop system can be developed with the aid of the Krasovskii passivity notion. However, there are no available design procedures to impose a set of performances. The main focus of the current paper is to formulate an optimization problem whose solution provides the parameters of a Krasovskii passivity-based controller (K-PBC) which ensures local exponential stability with a maximized lower bound of the decay rate. After a convexification procedure we obtain a linear programming problem with linear matrix inequality constraints. The numerical example underlines the advantage of the proposed method by emphasizing the difference between an optimized and a default controller obtained without optimization.**

I. Introduction

A. Literature Review

The class of bilinear systems represents a natural generalization of linear systems, representing a gateway between linear and nonlinear systems. Moreover, this framework is able to provide a more accurate model in various domains, such as transmission and power systems, or is used to approximate the behaviour of switching systems [1]. Even if the framework of bilinear systems has been studied since 1960s, there are recent papers which deal with this class of problems as an intermediate step for approaching the more general case of input-affine nonlinear systems [2]. Other bilinear system applications can be found in [3] and [4].

The problem of finding a state feedback controller which guarantees global asymptotic stability (GAS) has been studied for a particular case of bilinear systems in [5], while for the more general drift-free case the problem has been solved in [6] using linear matrix inequalities (LMIs). Another set of conditions to find a static and a dynamic feedback controller which ensures GAS of multi-input multi-output (MIMO) bilinear systems with undamped natural response has been developed in [7]. The problem of quadratic optimal control for both finite-time and infinite-time cases are studied in [8]. A state-dependent switching controller for MIMO bilinear systems with constant delays using Lyapunov-Krasovskii functions has been proposed in [9]. The problem of uniform exponential stabilization of finite and infinite bilinear systems with multiplicative control inputs was addressed in [10]. Furthermore, in [11] the authors propose a method to find the least restrictive requirements to interconnect systems such that the result maintains stability and dissipativity.

For the purpose of this paper we consider the passivity notion to design a controller which ensures local exponential stability. The passivity concept is a particular case of dissipativity and is used to develop the so-called passivity-based controllers (PBCs). The monograph [13] presents several feedback passivation techniques to impose the passivity property using analytic and geometric tools. A compact overview of L_2-gain theory and passivity concepts for a general class of nonlinear systems is presented in [14]. The recent paper [15] presents a method to find PBC using the concept of Krasovskii passivity [16], and presents the relations between this type of passivity and shifted passivity, differential passivity [17], and incremental passivity [18]. A less conservative set of sufficient conditions for guaranteeing Krasovskii passivity for the case of input-affine systems with polytopic cover has been showed in [19]. A passity index-based approach to study and impose passivity of connected two input feed-forward output-feedback systems is presented in [20], while the letter [21] addresses the problem of maximizing the passivity level of the closed-loop system using specified controller structures through numeric optimization. Their solution works for both continuous and discrete cases.

B. Research Gaps and Contributions

The notion of Krasovskii passivity has been introduced in the above mentioned papers, along with a possibility to construct first-order and second-order Krasovskii passivity-based controllers. However, from our findings, the problem of designing the K-PBC's parameters to impose a set of performance indices has not been addressed. In [22] the authors studied the problem in an preliminary version for the first-order controller case using ad hoc root-locus-type analysis and a metaheuristic approach. In the same paper it was additionally proved that, even if the asymptotic stability can be guaranteed, the state trajectories could be slower and more oscillating if the controller's parameters are not well calibrated. As such, the main goal of the current paper is to present a method to design the controller's parameters which guarantees exponential stability with an optimized decay rate. The main contributions of the current paper are:

(i) to formulate an optimization problem in terms of the K-PBC's parameters for the general case of multi-input

This research was supported by the project New smart and adaptive robotics solutions for personalized minimally invasive surgery in cancer treatment - ATHENA, funded by European Union – NextGenerationEU and Romanian Government, under National Recovery and Resilience Plan for Romania, contract no. 760072/23.05.2023, code CF 116/15.11.2022, through the Romanian Ministry of Research, Innovation and Digitalization, within Component 9, investment I8.

All authors are with the Department of Automation, Technical University of Cluj-Napoca, Romania (emails: {vlad.mihaly, mircea.susca, petru.dobra}@aut.utcluj.ro).

and multi-output bilinear systems in lower linear fractional transform interconnection with first- and second-order Krasovskii passivity-based controllers such that the local exponential stability can be ensured and the convergence rate is maximized;

(ii) to convexify the above mentioned optimization problem using a variable change such that the final problems from Section III become linear programming (LP) types with LMI constraints;

(iii) to underline the importance of the proposed method on a numerical example adapted from the literature by comparing the behaviour of the closed-loop system obtained using the controller by the proposed method against possible behaviours without the optimization.

C. Paper Structure and Notations

The paper is organized as follows: in Section II some preliminary results from the literature regarding Krasovskii passivity are presented; Section III formulates the problem of designing first-order and second-order Krasovskii controller parameters to ensure exponential stability, along with a step-by-step convexification procedure to formulate a convex optimization problem for maximizing the convergence rate; the main theoretical contributions of the paper are illustrated through a numerical example in Section IV, while Section V presents the conclusions.

Notations: $\mathbb{S}_n^+$ is the set of symmetrical and positive definite real matrices of order n. For $P \in \mathbb{S}_n^+$ we consider the induced norm $\|\mathbf{x}\|_P = \sqrt{\mathbf{x}^\top P\mathbf{x}}$. $\mathrm{Sym}(A) = \frac{1}{2}(A + A^\top)$ is the symmetrical part of a matrix $A \in \mathbb{R}^{n\times n}$.

II. Krasovskii Passivity for Bilinear Systems

In this section we briefly overview the main results regarding Krasovskii passivity presented in [15] for the particularized case of bilinear systems, with a less conservative set of conditions for Krasovskii passivity underlined in [19].

Definition 1: An input-affine nonlinear system of order n having the state vector $\mathbf{x} \in \mathbb{R}^n$:

$$\begin{cases} \dot{\mathbf{x}} = g_0(\mathbf{x}) + \sum_{i=1}^{n_u} g_i(\mathbf{x})u_i \equiv f(\mathbf{x}, \mathbf{u}); \\ \mathbf{y} = h(\mathbf{x}), \end{cases} \tag{1}$$

is called *bilinear* if the functions $g_i : \mathbb{R}^n \to \mathbb{R}^n$ can be written as $g_i(\mathbf{x}) = A_i\mathbf{x}+b_i$, $A_i\in\mathbb{R}^{n\times n}$ and $b_i\in\mathbb{R}^n$, $i=\overline{1, n_u}$.

Passivity is a particular case of the dissipativity notion introduced in [12], and consists in finding a storage function $S : \mathbb{R}^n \to \mathbb{R}_+$ of class C^1 such that:

$$\frac{\partial S(\mathbf{x})}{\partial \mathbf{x}} f(\mathbf{x}, u) \le \mathbf{y}^\top \cdot \mathbf{u}, \quad (\forall)\ (\mathbf{x}, \mathbf{u}) \in \mathcal{D}_\mathbf{x} \times \mathcal{D}_\mathbf{u}, \tag{2}$$

where $\mathcal{D}_\mathbf{x}$ is the reachable domain of the state vector $\mathbf{x}$ and $\mathcal{D}_\mathbf{u}$ is the admissible inputs domain. However, there are cases when the input-output pairs for which the system is passive do not include control inputs, so the passivity-based controller cannot be computed. A solution consists in extending the system as follows:

$$\begin{cases} \dot{\mathbf{x}} = g_0(\mathbf{x}) + \sum_{i=1}^{n_u} g_i(\mathbf{x}) \cdot u_i; \\ \dot{\mathbf{u}} = \mathbf{u}_d; \\ \mathbf{y}_d = h_K(\tilde{\mathbf{x}}), \end{cases} \tag{3}$$

resulting in a new state vector $\tilde{\mathbf{x}} = \begin{pmatrix}\mathbf{x}^\top & \mathbf{u}^\top\end{pmatrix}^\top \in \mathbb{R}^{n+n_u}$, while $h_K : \mathbb{R}^n \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ is an auxiliary output function of class C^1 and $\mathbf{u}_d$ is the new input vector. The extended system has been used to define the Krasovskii passivity of system (1) in terms of the passivity of the extended system:

Definition 2 ([15]): System (1) is called Krasovskii passive at a given forced equilibrium point $(\mathbf{x}^\star, u^\star)$ if there exists a class C^1 storage function $S_K : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}_+$ such that $S_K(\mathbf{x}^\star, u^\star) = 0$ and:

$$\frac{\partial S_K(\mathbf{x}, u)}{\partial \mathbf{x}} f(\mathbf{x}, u) + \frac{\partial S_K(\mathbf{x}, u)}{\partial u} u_d \le u_d \cdot y_d, \tag{4}$$

for each $(\mathbf{x}, u) \in \mathcal{D}_x \times \mathcal{D}_u$ and $u_d \in \mathcal{D}_{u_d}$.

A set of necessary and sufficient conditions for a system to be Krasovskii passive consists in:

Lemma 1 ([15]): System (1) is Krasovskii passive at a given forced equilibrium point $(\mathbf{x}^\star, u^\star)$ if and only if there exists a class C^1 function $S_K : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}_+$ such that $S_K(\mathbf{x}^\star, u^\star) = 0$ and:

$$\begin{cases} \dfrac{\partial S_K(\mathbf{x}, \mathbf{u})}{\partial \mathbf{x}} f(\mathbf{x}, u) \le 0 \\ \dfrac{\partial S_K(\mathbf{x}, \mathbf{u})}{\partial \mathbf{u}} = h_K^\top(\mathbf{x}) \end{cases}, \quad \forall (\mathbf{x}, \mathbf{u}) \in \mathcal{D}_\mathbf{x} \times \mathcal{D}_\mathbf{u}. \tag{5}$$

For the special case of the storage function:

$$S_K(\mathbf{x}, \mathbf{u}) = \|f(\mathbf{x}, u)\|_Q^2 = \|\dot{\mathbf{x}}\|_Q^2, \quad Q \in \mathbb{S}_n^+, \tag{6}$$

then the output function h_K ensures the passivity of the extended system (3) and, by default, the Krasovskii passivity of the system (1), can be constructed as follows:

$$h_K(\mathbf{x}, \mathbf{u}) = \begin{pmatrix} \mathbf{x}^\top A_1^\top + b_1^\top \\ \vdots \\ \mathbf{x}^\top A_{n_u}^\top + b_{n_u}^\top \end{pmatrix} \cdot Q \cdot \dot{\mathbf{x}}, \tag{7}$$

if the matrix Q satisfies the first condition from (5). For the case of bilinear systems, the said condition is equivalent to the following infinite set of LMIs:

$$QA_0 + A_0^\top Q + \sum_{i=1}^{n_u} \left(QA_i + A_i^\top Q\right) u_i \le 0, \quad (\forall)\ \mathbf{u} \in \mathcal{D}_\mathbf{u}. \tag{8}$$

However, in the practical case of considering a bounded input domain $\mathcal{D}_u = [\underline{u}_1, \overline{u}_1]\times\cdots\times[\underline{u}_{n_u}, \overline{u}_{n_u}]$, the previous infinite set of LMIs can be converted, due to its convex nature, into the following 2^{n_u} LMIs:

$$QA_0 + A_0^\top Q + \sum_{i=1}^{n_u} \left(e_i \left(QA_i + A_i^\top Q\right) \underline{u}_i + (1 - e_i) \left(QA_i + A_i^\top Q\right) \overline{u}_i\right) \le 0, \tag{9}$$

for each $\mathbf{e} = \begin{pmatrix}e_1 & \ldots & e_{n_u}\end{pmatrix} \in \mathbb{Z}_2^{n_u}$.

Considering this type of passivity, a methodology to design first- and second-order type Krasovksii passivity-based controllers (K-PBCs) has been introduced in [15]. The state vector is given by $\mathbf{x}_c = \mathbf{u} - \mathbf{u}^\star$. The first-order structure is:

$$\mathbf{y}_c = \dot{\mathbf{x}}_c = -\left(K_1\mathbf{x}_c + K_2\mathbf{u}_c\right) \equiv f_{c,1}(\mathbf{x}_c, \mathbf{u}_c), \qquad (10)$$

where $\mathbf{u}_c$ and $\mathbf{y}_c$ are the input and output signals, respectively. For the second-order type K-PBC we have:

$$\mathbf{y}_c = \ddot{\mathbf{x}}_c = -(K_3\dot{\mathbf{x}}_c + K_1(\mathbf{u} - \mathbf{u}^\star) + K_2\mathbf{u}_c) \equiv f_{c,2}(\mathbf{x}_c, \mathbf{u}_c). \qquad (11)$$

In both cases, the terms $K_1, K_2, K_3 \in \mathbb{S}^+_{n_u}$ are the controller parameters which ensure their Krasovskii passivity. By considering the classical lower linear fractional transform (LLFT) interconnection between the plant and the controller, i.e. $\mathbf{y}_c \equiv \mathbf{u}_d$ and $\mathbf{u}_c \equiv \mathbf{y}_d$, the resulting closed-loop system is also Krasovskii passive.

It was proved in [15] that the Krasovskii passivity along with detectability of the closed-loop with two sets of outputs system are sufficient conditions for asymptotic stability. Recall one of the main results from [15]. The above mentioned output signals are:

$$\begin{cases} \mathbf{y}_1 = \frac{\partial S_K(\mathbf{x},\mathbf{u})}{\partial \mathbf{x}} f(\mathbf{x}, \mathbf{u}); \\ \mathbf{y}_2 = K_1(\mathbf{u} - \mathbf{u}^\star) + \mathbf{y}_d. \end{cases} \qquad (12)$$

Lemma 2: The detectability condition for the particular case of bilinear systems is equivalent to having the matrix $A_0 + \sum_{i=1}^{n_u} A_i u_i^\star$ Hurwitz.

Proof: From $\mathbf{y}_2 \equiv 0$ and $\mathbf{u} = \mathbf{u}^\star$ we have $\mathbf{y}_d = h_K(\mathbf{x}, \mathbf{u}^\star) = 0$. Moreover, for considering $\mathbf{u} = \mathbf{u}^\star$ we have:

$$\begin{aligned} \dot{\mathbf{x}} - 0 =& (A_0\mathbf{x} + b_0) + \sum_{i=1}^{n_u} (A_i\mathbf{x} + b_i)\, u_i^\star - \\ &- (A_0\mathbf{x}^\star + b_0) - \sum_{i=1}^{n_u} (A_i\mathbf{x}^\star + b_i)\, u_i^\star, \end{aligned}$$

which implies:

$$\dot{\mathbf{x}} = \left(A_0 + \sum_{i=1}^{n_u} A_i u_i^\star\right)(\mathbf{x} - \mathbf{x}^\star),$$

and the condition $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^\star$ is equivalent to having the matrix $\left(A_0 + \sum_{i=1}^{n_u} A_i u_i^\star\right)$ Hurwitz. ■

III. Exponential Stability Analysis

Definition 3: A nonlinear system having the form (1) is *global exponentially stable* at $\mathbf{x}^\star$ if there exist two real constants $\alpha, \beta > 0$ such that, for each initial point $\mathbf{x}_0 \in \mathbb{R}^n$, the state trajectory starting from $\mathbf{x}_0$, denoted by $\xi(\mathbf{x}_0, t)$, fulfills the inequality:

$$\|\xi(\mathbf{x}_0, t) - \mathbf{x}^\star\| \le \alpha \cdot e^{-\beta \cdot t}\|\mathbf{x}(0) - \mathbf{x}^\star\|. \qquad (13)$$

Furthermore, the term β is called the *exponential decay rate*.

However, sometimes the behaviour is valid in a certain region around the equilibrium point, the system being *locally exponentially stable*. For such cases, the domain of the initial points for which this property is valid should be estimated.

Definition 4: The initial points for which the inequality (13) is valid is called the *region of attraction* (RoA) of rate β and is denoted by $\mathcal{D}_{\mathbf{x}}^{(\beta)}$.

According to Lemma 3.2 from [12], $\mathcal{D}_{\mathbf{x}}^{(\beta)}$ is an open, connected and invariant set. However, each RoA can be generally estimated instead of being explicitly computed. In order to study the local exponential stability at $\mathbf{x}^\star$ we use the following result:

Theorem 1: A system (1) is locally exponentially stable at $\mathbf{x}^\star$ with respect to a RoA $\mathcal{D}_{\mathbf{x}}$ if there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ and three constants $\underline{a}, \overline{a}, a_3 > 0$ such that:

$$\underline{a}\|\mathbf{x}\|^2 \le V(\mathbf{x}) \le \overline{a}\|\mathbf{x}\|^2, \quad \forall \mathbf{x} \in \mathcal{D}_{\mathbf{x}}, \qquad (14a)$$

$$\dot{V}(\mathbf{x}) < -a_3\|\mathbf{x}\|^2, \quad \forall \mathbf{x} \in \mathcal{D}_{\mathbf{x}}, \qquad (14b)$$

the exponential decay rate being lower bounded by $\frac{a_3}{2\overline{a}}$.

A. First-Order Controller

Considering the bilinear system (1) in the extended form (3) and the first-order K-PBC from (10), the closed-loop system can be written as follows:

$$\begin{cases} \dot{\mathbf{x}} = (A_0\mathbf{x} + b_0) + \sum_{i=1}^{n_u} (A_i\mathbf{x} + b_i)\, u_i; \\ \dot{\mathbf{u}} = -K_1\mathbf{u} - K_2 \begin{pmatrix} \mathbf{x}^\top A_1^\top + b_1^\top \\ \vdots \\ \mathbf{x}^\top A_{n_u}^\top + b_{n_u}^\top \end{pmatrix} \cdot Q \cdot \dot{\mathbf{x}} + K_1\mathbf{u}^\star. \end{cases} \qquad (15)$$

For the remainder of the paper, consider the next shorthand notation to consistently describe the functions g_i, $i = \overline{1, n_u}$:

$$\mathbf{g}(\mathbf{x}) = \begin{pmatrix} A_1\mathbf{x} + b_1 & \dots & A_{n_u}\mathbf{x} + b_{n_u} \end{pmatrix}. \qquad (16)$$

The gradient of the closed-loop state function (15) is:

$$\frac{\partial f_o}{\partial \mathbf{x}_o} = \left(\begin{array}{c|c} A_0 + \sum_{i=1}^{n_u} A_i u_i & \mathbf{g}(\mathbf{x}) \\ \hline -K_2 \cdot \mathbf{a}^\top(Q) & -K_1 - K_2\Pi(Q) \end{array}\right), \qquad (17)$$

where:

$$\Pi(Q) = \left((A_i\mathbf{x} + b_i)^\top Q (A_j\mathbf{x} + b_j)\right)_{1 \le i,j \le n_u} \qquad (18)$$

and

$$\mathbf{a}(Q) = \begin{pmatrix} a_1(Q) & \dots & a_{n_u}(Q) \end{pmatrix}, \qquad (19)$$

$$\begin{aligned} a_i(Q) = & A_i^\top Q(A_0\mathbf{x} + b_0) + A_0^\top Q(A_i\mathbf{x} + b_i) + \\ & \sum_{j=1}^{n_u} \left(A_j^\top Q(A_i\mathbf{x} + b_i) + A_i^\top Q(A_j\mathbf{x} + b_j)\right) u_j. \end{aligned} \qquad (20)$$

To study the exponential stability of the closed-loop system, we consider the function $V(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|^2_P$ as a Lyapunov function, having the following structure:

$$P = \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix}, \quad P_{11} \in \mathbb{S}^+_n, P_{22} \in \mathbb{S}^+_{n_u}. \qquad (21)$$

$$\text{Sym}\left\{\left(\begin{array}{c|c} P_{11}\left(A_0+\sum_{i=1}^{n_u}A_iu_i\right) & P_{11}\mathbf{g}(\mathbf{x}) \\ \hline -P_{22}K_2\cdot\mathbf{a}^\top(Q) & -P_{22}K_1-P_{22}K_2\Pi(Q)\end{array}\right)\right\}<-\frac{1}{2}a_3I,\quad \forall\mathbf{x}\in\mathcal{D}_\mathbf{x}. \tag{22}$$

$$\text{Sym}\left\{\left(\begin{array}{c|c} P_{11}\left(A_0+\sum_{i=1}^{n_u}A_iu_i\right) & P_{11}\mathbf{g}(\mathbf{x}) \\ \hline -\widetilde{K_2}\cdot\mathbf{a}^\top(Q_0) & -\widetilde{K_1}-\widetilde{K_2}\Pi(Q_0)\end{array}\right)\right\}<-\frac{1}{2}a_3I,\quad \forall\mathbf{x}\in\mathcal{V}\left\{\mathcal{D}_\mathbf{x}\right\}. \tag{23}$$

The second condition of the exponential stability of the closed-loop system presented in Theorem 1 is shown in (22).

But, relation (22) represents an infinite set of nonlinear linear matrix inequalities. Therefore, a convexification procedure becomes necessary. First, it can be noticed that the following substitution should be performed to remove a part of the nonlinearities:

$$\widetilde{K}_1=P_{22}K_1\in\mathbb{S}^+_{n_u}\quad\text{and}\quad\widetilde{K}_2=P_{22}K_2\in\mathbb{S}^+_{n_u}. \tag{24}$$

To remove the remaining nonlinearities, i.e. those that appear in $\widetilde{K_2}\cdot\mathbf{a}^\top(Q)$ and $\widetilde{K_2}\Pi(Q)$, consider matrix Q as the solution Q_0 of the LMI feasibility problem described in (9). Using the above mentioned variable changes and solution Q_0 instead of Q, the nonlinear infinite set of matrix inequalities from (22) becomes an infinite set of LMIs. To solve the final difficulty of having an infinite set, consider only the vertices $\mathcal{V}\left\{\mathcal{D}_x\right\}$ of the RoA $\mathcal{D}_\mathbf{x}$. Now, the second condition of the exponential stability can be written as in (23).

After all these considerations, the following linear programming optimization problem with LMI constraints can be formulated:

Problem 1: Given a bilinear system with a forced equilibrium point $(\mathbf{x}^\star,\mathbf{u}^\star)$, a desired region of attraction $\mathcal{D}_\mathbf{x}$, and two real parameters $0<\underline{a}<\overline{a}$, the exponential decay rate can be maximized by solving the following problem:

$$\begin{aligned}&\max\quad a_3 \\ &\text{s.t.}\quad (23)\quad\text{and} \\ &\qquad \underline{a}I\le P_{11}\le\overline{a}I,\ \widetilde{K_1},\widetilde{K_2}\in\mathbb{S}^+_{n_u},\ a_3>0.\end{aligned} \tag{25}$$

B. Second-Order Controller

In a similar manner with the case of the first-order controller type, this subsection extends the results for the second-order type controller. The resulting closed-loop system in this case can be written as:

$$\begin{cases}\dot{\mathbf{x}}=(A_0\mathbf{x}+b_0)+\sum_{i=1}^{n_u}(A_i\mathbf{x}+b_i)u_i; \\ \dot{\mathbf{u}}=\mathbf{u}_d; \\ \dot{\mathbf{u}}_d=-K_3\mathbf{u}_d-K_1\mathbf{u}-K_2\cdot\mathbf{g}^\top(\mathbf{x})\cdot Q\cdot\dot{\mathbf{x}}+K_1\mathbf{u}^\star.\end{cases} \tag{26}$$

The gradient of the closed-loop state function is given by:

$$\frac{\partial f_o}{\mathbf{x}_o}=\left(\begin{array}{c|c|c} A_0+\sum_{i=1}^{n_u}A_iu_i & \mathbf{g}(\mathbf{x}) & O \\ \hline O & O & I \\ \hline -K_2\cdot\mathbf{a}^\top(Q) & -K_1-K_2\Pi(Q) & -K_3\end{array}\right), \tag{27}$$

where $\Pi(Q)$ and $\mathbf{a}^\top(Q)$ are defined in (18) and (19), respectively. For the control Lyapunov function we consider the following structure $V(\mathbf{x})=\frac{1}{2}\|\mathbf{x}\|^2_P$, with:

$$P=\begin{pmatrix}P_{11} & O & O \\ O & P_{22} & O \\ O & O & P_{33}\end{pmatrix},\quad P_{11}\in\mathbb{S}^+_n, P_{22},P_{33}\in\mathbb{S}^+_{n_u}, \tag{28}$$

and the second condition of the exponential stability can be written in a similar manner to (22) in relation (29).

Therefore, we obtain another infinite set of nonlinear matrix inequalities, which requires a convexification procedure. This procedure can be performed in a similar way by considering the following variable changes:

$$\widetilde{K_1}=P_{33}K_1,\quad\widetilde{K_2}=P_{33}K_2,\quad\text{and}\quad\widetilde{K_3}=P_{33}K_3. \tag{31}$$

Additionally, the variable Q will be set to the solution Q_0 of the LMI feasibility problem (9), which guarantees the Krasovskii passivity of the bilinear system. The issue of having an infinite set of LMI conditions can be fixed by considering the vertices of the region of attraction of the equilibrium point $\mathcal{D}_\mathbf{x}$. As such, the second condition of the local exponential stability from Theorem 1 consists in having a solution of the LMI problem (30).

The problem of maximizing the lower bound of the convergence decay rate can be formulated as a linear programming problem having constraints expressed in terms of linear matrix inequalities:

Problem 2: Considering a bilinear system with a forced equilibrium point $(\mathbf{x}^\star,\mathbf{u}^\star)$, a desired region of attraction $\mathcal{D}_\mathbf{x}$, and two real parameters $0<\underline{a}<\overline{a}$, the exponential decay rate can be maximized by solving the following problem:

$$\begin{aligned}&\max\quad a_3 \\ &\text{s.t.}\quad (30)\quad\text{and} \\ &\qquad \underline{a}I\le P_{11},P_{22}\le\overline{a}I,\ \widetilde{K_1},\widetilde{K_2},\widetilde{K_3}\in\mathbb{S}^+_{n_u},\ a_3>0.\end{aligned} \tag{32}$$

IV. Numerical Example

To illustrate the relevance of the proposed analysis, we consider the following bilinear system with two inputs:

$$\begin{cases}\dot{x}_1=-x_2+x_2u_2+u_1; \\ \dot{x}_2=x_1-x_1u_2-x_3u_2; \\ \dot{x}_3=-x_4+x_2u_2+u_1; \\ \dot{x}_4=x_3-x_4.\end{cases} \tag{33}$$

The forced equilibrium point considered for the experiments presented in this section is:

$$\mathbf{x}^\star=\begin{pmatrix}0.25 & 0.5 & 0.5 & 0.5\end{pmatrix}^\top\ \text{and}\ \mathbf{u}^\star=\begin{pmatrix}\frac{1}{3} & \frac{1}{3}\end{pmatrix}^\top. \tag{34}$$

$$\mathrm{Sym}\left\{\left(\begin{array}{c|c|c} P_{11}\left(A_0+\sum_{i=1}^{n_u}A_iu_i\right) & P_{11}\mathbf{g}(\mathbf{x}) & O \\ \hline O & O & P_{22} \\ \hline -P_{33}K_2\cdot\mathbf{a}^\top(Q) & -P_{33}K_1-P_{33}K_2\Pi(Q) & -P_{33}K_3 \end{array}\right)\right\} < -\frac{1}{2}a_3I, \quad \forall \mathbf{x}\in\mathcal{D}_{\mathbf{x}}. \tag{29}$$

$$\mathrm{Sym}\left\{\left(\begin{array}{c|c|c} P_{11}\left(A_0+\sum_{i=1}^{n_u}A_iu_i\right) & P_{11}\mathbf{g}(\mathbf{x}) & O \\ \hline O & O & P_{22} \\ \hline -\widetilde{K_2}\cdot\mathbf{a}^\top(Q_0) & -\widetilde{K_1}-\widetilde{K_2}\Pi(Q_0) & -\widetilde{K_3} \end{array}\right)\right\} < -\frac{1}{2}a_3I, \quad \forall \mathbf{x}\in\mathcal{V}\left\{\mathcal{D}_{\mathbf{x}}\right\}. \tag{30}$$

In order to prove that the given bilinear system is Krasovskii passive at $(\mathbf{x}^\star,\mathbf{u}^\star)$, we consider the local input domain $\mathcal{D}_{\mathbf{u}^\star}=[0,0.7]\times[0,0.7]$. By solving the set of 4 LMIs from (9), the storage function which guarantees the Krasovskii passivity is:

$$S_K(\mathbf{x},\mathbf{u})=\frac{1}{2}\|\dot{\mathbf{x}}\|_Q, \quad Q=3424.85\cdot I_4. \tag{35}$$

Due to the homogeneity of the LMIs set (9), we can also select the matrix $Q=I_4$. In this case, the output function $h_K(\mathbf{x},\mathbf{u})$ can be written as:

$$h_K(\mathbf{x},\mathbf{u})=\begin{pmatrix}1 & 0 & 1 & 0\\ x_2 & -(x_1+x_3) & x_2 & 0\end{pmatrix}\cdot\dot{\mathbf{x}}. \tag{36}$$

Moreover, the given bilinear system is detectable at $(\mathbf{x}^\star,\mathbf{u}^\star)$, according to Lemma 2 being sufficient to have $A_0+A_1u_1^\star+A_2u_2^\star$ Hurwitz. Therefore, the closed-loop system is asymptotically stable at the forced equilibrium point $(\mathbf{x}^\star,\mathbf{u}^\star)$ for each $K_1,K_2,K_3\in\mathbb{S}_2^+$. To underline the relevance of the proposed method, we impose several configurations for the K-PBC's parameters obtained with the random generator mechanism from MATLAB. Figure 1 are depicts the four closed-loop trajectories obtained using the randomly generated matrices K_1 and K_2 against those obtained with the optimal values $K_1^\star$ and $K_2^\star$. As noticed, the optimized trajectories present a significantly faster convergence rate, even though the asymptotic stability is guaranteed by all controller configurations.

To find the parameters $K_1^\star$ and $K_2^\star$ which maximize the lower bound of the convergence rate, Problem 1 has been solved using the `mincx` from the LMI Solvers contained in the Robust Control Toolbox in MATLAB. The following hyperparameters were used: $\underline{a}=10^{-3}$, $\overline{a}=10^9$ and the RoA $\mathcal{D}_{\mathbf{x}}=[0.2,0.3]\times[0.3,0.7]\times[0.3,0.7]\times[0.3,0.7]$. The Lyapunov function which guarantees the local exponential stability of the forced equilibrium point is:

$$P_{11}=10^8\times\begin{pmatrix}2.1353 & -0.4589 & 0.1515 & 0.6211\\ -0.4589 & 2.4671 & -1.4968 & 0.6062\\ 0.1515 & -1.4968 & 1.4747 & -0.6276\\ 0.6211 & 0.6062 & -0.6276 & 0.8518\end{pmatrix}, \tag{37}$$

while for P_{22} we consider the identity matrix times the maximum eigenvalue of P_{11} to obtain $K_1^\star=\widetilde{K_1}^\star/\lambda_{\max}(P_{11})$ and $K_2^\star=\widetilde{K_2}^\star/\lambda_{\max}(P_{11})$. The optimal parameters are:

$$K_1^\star=\begin{pmatrix}1.8217 & 0.7233\\ 0.7233 & 1.0783\end{pmatrix} \tag{38}$$

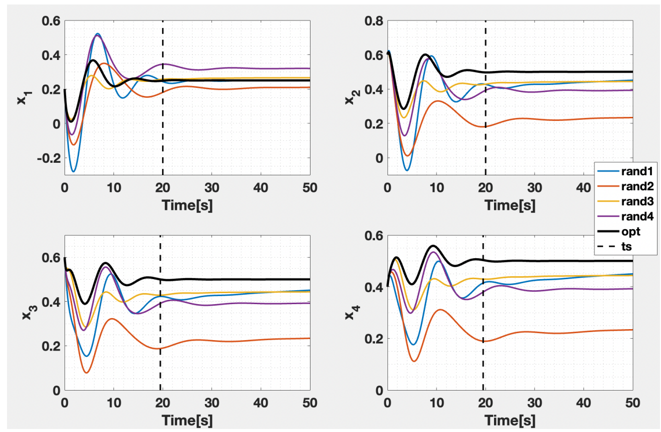

Fig. 1. The influence of the controller parameters to the convergence rate of the closed-loop system: the trajectories with blue, red, yellow and purple are represented with random parameter configurations, which still guarantee the asymptotic stability, while the black cases denote the optimized trajectories in terms of convergence rate.

and

$$K_2^\star=\begin{pmatrix}0.3321 & 0.4234\\ 0.4234 & 0.5479\end{pmatrix}. \tag{39}$$

The resulting lower bound of the convergence rate is 0.0167. The settling time of the optimized trajectories is about $20[s]$, as noticed in Figure 1. Moreover, four simulations considering multiple starting points from the admissible input domains are illustrated in Figure 2, where the settling time limit is depicted with a vertical strip.

A natural question is about the robustness of the proposed controller in terms of equilibrium point changes. We consider several equilibrium points for performing said analysis:

$$(\mathbf{x}^{(1)},\mathbf{u}^{(1)})=\begin{pmatrix}0.16 & 0.4 & 0.4 & 0.4 & 0.286 & 0.286\end{pmatrix}^\top;$$
$$(\mathbf{x}^{(2)},\mathbf{u}^{(2)})=\begin{pmatrix}0.2025 & 0.45 & 0.45 & 0.45 & 0.31 & 0.31\end{pmatrix}^\top;$$
$$(\mathbf{x}^{(4)},\mathbf{u}^{(4)})=\begin{pmatrix}0.3025 & 0.55 & 0.55 & 0.55 & 0.355 & 0.355\end{pmatrix}^\top;$$
$$(\mathbf{x}^{(5)},\mathbf{u}^{(5)})=\begin{pmatrix}0.36 & 0.6 & 0.6 & 0.6 & 0.375 & 0.375\end{pmatrix}^\top,$$

while $(\mathbf{x}^{(3)},\mathbf{u}^{(3)})\equiv(\mathbf{x}^\star,\mathbf{u}^\star)$. As noticed, pairs $(\mathbf{x}^{(1)},\mathbf{u}^{(1)})$ and $(\mathbf{x}^{(5)},\mathbf{u}^{(5)})$ are not in the imposed RoA $\mathcal{D}_{\mathbf{x}}$ of the forced equilibrium point. The set $\mathcal{D}_{\mathbf{x}}$ contains a smaller RoA for the forced equilibrium points $(\mathbf{x}^{(2)},\mathbf{u}^{(2)})$ and $(\mathbf{x}^{(4)},\mathbf{u}^{(4)})$, all of them containing the starting point:

$$(\mathbf{x}_0,\mathbf{u}_0)=\begin{pmatrix}0.2 & 0.6 & 0.6 & 0.4 & 0 & 0\end{pmatrix}^\top.$$

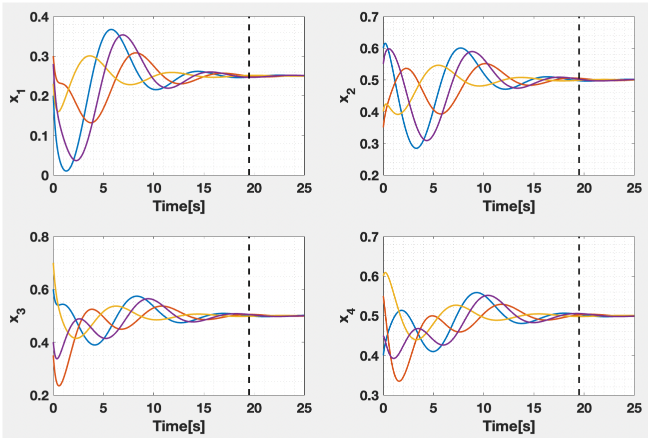

Fig. 2. Multiple starting points for the closed-loop system simulated with the optimized first-order K-PBC. The settling time limit is depicted with the black vertical strip.

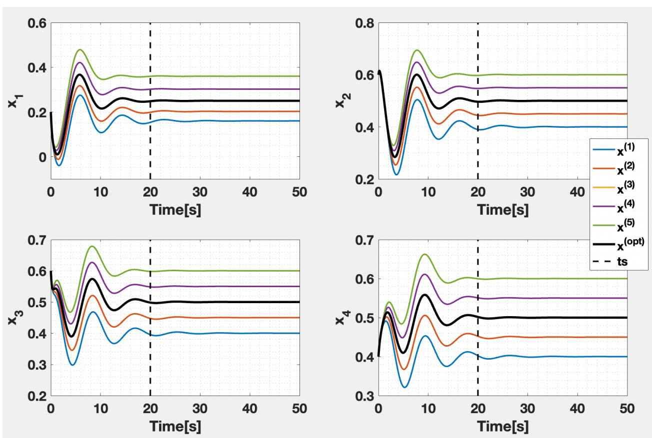

Fig. 3. The robustness of the proposed controllers in terms of variations of the forced equilibrium points: $\mathcal{D}_{\mathbf{x}}$ is a RoA for $((\mathbf{x}^{(i)}, \mathbf{u}^{(i)}))$, $i = \overline{2,4}$, which contains the starting point, while $(\mathbf{x}^{(1)}, \mathbf{u}^{(1)}), (\mathbf{x}^{(5)}, \mathbf{u}^{(5)}) \notin \mathcal{D}_{\mathbf{x}}$. The lower bound of the convergence rate is checked by $((\mathbf{x}^{(i)}, \mathbf{u}^{(i)})) \in \mathcal{D}_{\mathbf{x}}$, $i = \overline{2,4}$, and, additionally, for $(\mathbf{x}^{(5)}, \mathbf{u}^{(5)}) \notin \mathcal{D}_{\mathbf{x}}$.

The closed-loop trajectories starting from $(\mathbf{x}_0, \mathbf{u}_0)$ are depicted in Figure 3. As noticed, the exponential stability is fulfilled with the same convergence rate for the equilibrium points $(\mathbf{x}^{(2)}, \mathbf{u}^{(2)})$, $(\mathbf{x}^{(3)}, \mathbf{u}^{(3)})$, $(\mathbf{x}^{(4)}, \mathbf{u}^{(4)})$, and additionally for $(\mathbf{x}^{(5)}, \mathbf{u}^{(5)})$ (without having mathematical guarantees), while for $(\mathbf{x}^{(1)}, \mathbf{u}^{(1)})$ the lower bound estimation is not satisfied. As such, if a larger RoA is considered, the robustness problem can be addressed in terms of variations of the forced equilibrium points.

V. Conclusions and Future Work

The problem of designing a Krasovskii passivity-based controller which ensures the asymptotic stability has been already addressed in the literature. The current paper proposes a mechanism to formulate the problem of designing K-PBC parameters which ensure the local exponential stability of a finite dimensional MIMO bilinear system having an optimized lower bound of the convergence decay rate. The initial infinite set of nonlinear matrix inequality constraints have been successfully converted into a finite set of LMI-based constraints, resulting a linear programming optimization problem with LMI constraints, which is convex. Moreover, the proposed controller ensures robustness in terms of variations of the considered forced equilibrium point.

The main research directions consist in (i) analyzing more general input-affine nonlinear system classes, starting from our previous results on local polytopic bounded systems [19]; (ii) to encompass uncertainties; (iii) to design an observer for obtaining the auxiliary output function $h_K(\mathbf{x})$.

References

[1] D.L. Elliott, *Bilinear Control Systems*, Springer, 2009.
[2] H. Omran, L. Hetel, M. Petreczky, J.-P. Richard, F.L.-Lagarrigue, Stability analysis of some classes of input-affine nonlinear systems with aperiodic sampled-data control, *Automatica*, vol. 70, pp. 266–274, August 2016, doi:10.1016/j.automatica.2016.02.013.
[3] R. Cisneros, M. Pirro, G. Bergna, R. Ortega, G. Ippoliti, M. Molinas, Global Tracking Passivity-based PI Control of Bilinear Systems and its Application to the Boost and Modular Multilevel Converters, IFAC-PapersOnLine, vol. 48, no. 11, pp. 420–425, 2015.
[4] B. Zitte, B. Hamroun, D. Astolfi, F. Couenne, Robust Control of a Class of Bilinear Systems by Forwarding: Application to Counter Current Heat Exchanger, 21st IFAC World Congress: Berlin, Germany, 11–17 July 2020, doi:10.1016/j.ifacol.2020.12.603.
[5] H. Jerbi, Global Feedback Stabilization of New Class of Bilinear Systems, *Systems and Control Letters*, vol. 42, pp. 313–320, 2001.
[6] F. Amato, C. Cosentino, A. Merola, Stabilization of Bilinear Systems via Linear State Feedback Control, *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 56(1), pp. 76-80, 2009.
[7] G. Lu, Y. Zheng, C. Zhang, Dynamical Output Feedback Stabilization of MIMO Bilinear Systems with Undamped Natural Response, *Asian Journal of Control*, vol. 5(2), pp. 251–260, 2003.
[8] S. Yahyaoui, M. Ouzahra, Quadratic optimal control and feedback stabilization of bilinear systems, *Opt. Ctrl. App. Meth.*, pp. 1–13, 2021.
[9] T. Sanchez, A. Polyakov, E. Fridman, L. Hetel, A Switching Controller for a Class of MIMO Bilinear Systems With Time Delay, *IEEE Transactions on Automatic Control*, vol. 65(5), 2020.
[10] M. Ouzahra, Exponential Stabilization of Unstable Bilinear Systems in Finite- and Infinite-Dimensional Spaces, *IEEE Transactions on Automatic Control*, vol. 66(12), pp. 5982-5989, 2021.
[11] A.J. Malan, P. Jané-Soneira, S. Hohmann, Constructive Analysis and Design of Interconnected Krasovskii Passive and Quadratic Dissipative Systems, *IEEE Conference on Decision and Control (CDC)*, Cancun, Mexico, December 2022, doi:10.1109/CDC51059.2022.9992956.
[12] H. Khalil. *Nonlinear Systems.* Pearson Education Limited, 2015.
[13] R. Sepulchre, M. Jankovic, P.V. Kokotovici, *Constructive Nonlinear Control*, Sppringer-Verlag London, 1997.
[14] A. van der Schaft, *L_2-Gain and Passivity Techniques in Nonlinear Control*, Springer International Publishing, 2017.
[15] Y. Kawano, K.C. Kosaraju, J.M.A. Scherpen, Krasovskii and Shifted Passivity-Based Control, *IEEE Trans. on Aut. Ctrl.*, vol. 66(10), 2021.
[16] K.C. Kosaraju, Y. Kawano, J.M.A. Scherpen. Krasovskii's Passivity. In Proceeding of the *11th IFAC Symposium on Nonlinear Control Systems NOLCOS*, vol. 52, pp. 466–471, 2019.
[17] A.J. van der Schaft, On differential passivity. In *IFAC Proceeding*, vol. *46*, pp. 21–25, 2013.
[18] Pavlov, A.; Marconi, L. Incremental passivity and output regulation, In the Proceedings of the 45th *IEEE Conference on Decision and Control*, San Diego, CA, USA, 2006.
[19] V. Mihaly, M. Şuşcă, D. Morar, P. Dobra, Polytopic Robust Passivity Cascade Controller Design for Nonlinear Systems. In *Proceedings of the European Control Conference*, London, UK, 2022.
[20] F. Zhu, M. Xia, P.J. Antsaklis, Passivity analysis and passivation of feedback systems using passivity indices, *IEEE American Control Conference (ACC)*, Portland, OR, USA, 04-06 June 2014.
[21] L. Su, V. Gupta, P. Antsaklis, Feedback Passivation of Linear Systems With Fixed-Structure Controllers, *IEEE Control Systems Letters*, vol. 4, number 2, April 2020, doi:10.1109/LCSYS.2019.2960502.
[22] V. Mihaly, M. Şuşcă, D. Morar, P. Dobra, Sensitivity Analysis of Krasovskii Passivity-Based Controllers. *Mathematics*, vol. 10(20):3750, 2022.