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On Internal Model Compensation for General Regulator Problems

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***Abstract*— The paper studies a general regulator problem with an internal model in a subset of measurement channels. It proposes a procedure to reduce the stabilization problem for an augmented system (the plant plus internal model) to an equivalent one based on a process without the internal model and having the complexity of the plant. A key idea is to introduce stable *internal model compensation* (IMC) elements to the controller, which are, in a sense, dual to the dead-time compensators used in the control of delay systems. Closed-form state-space expressions for such IMC elements and the resulted equivalent plant are derived. It is shown that the complexity of the resulted overall controller is lower than in approaches based on the augmented plant.**

***Index Terms*— Internal model principle, regulator problem, linear systems.**

I. Introduction

Consider a continuous-time linear time-invariant (LTI) plant $P : u \mapsto y$ under a control input $u(t) \in \mathbb{R}^m$ and a measured output $y(t) \in \mathbb{R}^p$. By the regulator problem we understand the problem of designing a stabilizing feedback controller $R : y \mapsto u$, which asymptotically rejects effects of persistent disturbances and / or reference signals of a known class on a part of the measured output, referred to as the *regulated signal*. We assume that the regulated signal is

$$e(t) = Ey(t) \in \mathbb{R}^{p_e}$$

for some $E \in \mathbb{R}^{p_e \times p}$ assumed to be normalized, i.e. such that $EE' = I$ (a typical choice would be $E = \begin{bmatrix} I & 0 \end{bmatrix}$).

The regulator problem is conventionally addressed via the Internal Model Principle [1], by including a model of persistent exogenous signals into the controller. A possible configuration of the controller for the regulated signal as above is (paraphrased from [2, Sec. 4.4])

$$R = R_{\mathrm{s}}(E'ME + I - E'E), \tag{1}$$

where M is a $p_e \times p_e$ internal model, whose (pure imaginary) poles model exogenous signals, and R_{s} is a design parameter, dubbed *stabilizer* or post-processor, whose goal is to stabilize the resulted closed-loop system and take care of transients and other performance aspects. We assume the standard feedback configuration, with the loop $RP = R_{\mathrm{s}}(E'ME + I - E'E)P$. The robust regulation requires each pole of M to have the geometric multiplicity p_e, see [2, 3]. The internal model is normally fixed as a part of regulation considerations and the stabilizer is designed for the augmented plant

$$P_{\mathrm{aug}} := (E'ME + I - E'E)P. \tag{2}$$

A stabilizing R_{s} can always be designed, provided the two terms in the right-hand side of (2) have no unstable cancellations. Controller (1) solves then the regulator problem under mild technical assumptions on P for fairly general disturbance attenuation and tracking setups.

Yet, despite its conceptual simplicity, the procedure outlined above has its own catches. The obvious one is the inflation of dimensions when the complexity of the internal model increases. An extreme example of that is repetitive control [4] handling arbitrary periodic exogenous signals, whose model is infinite dimensional and the design of a stabilizer for which is highly nontrivial. Moreover, addressing closed-loop performance for a high-dimensional P_{aug}, which then has several undamped resonances, might not be quite simple. In many cases the choice is to resort to a low-gain R_{s}, at least if P is stable itself. Another shortcoming of designing R_{s} for P_{aug} is a complex dependence of the parameters of the stabilizer on those of the internal model. This implies that adjusting R_{s} to changes in M might be very involved.

In this paper we put forward an alternative approach to design internal model controllers. The idea is to introduce fixed stable *internal model compensation* (IMC) elements, which reduce the stabilization of P_{aug} to that of a system having the same complexity as the non-augmented plant P. In some situations, this could even be the stabilization of P itself. This approach is motivated by the delay compensation in repetitive control [5] and that for a general internal model in the state-feedback case with a control channel model [6]. Below we extend these ideas to a fairly general class of output-feedback systems with internal models. In situations when the regulated signal e is a proper subset of measured signal y, i.e. when $p_e < p$, the proposed IMC is nontrivially different from those in [5, 6] in the need to include two IMC elements. One of those elements is in parallel to the "central controller" in the regulation channel, similarly to that in earlier results. But another one, connecting two measurement channels, has no counterparts there. Explicit state-space construction of stable IMC elements is proposed.

Notation: The closed right half of the complex plane is denoted $\bar{\mathbb{C}}_0$. The complex-conjugate transpose of a matrix A is denoted by A'. The notation $\mathrm{spec}(A)$ stands for the matrix spectrum when A is a square matrix or for the set of poles if A is an LTI system. By H_∞ we denote the set of holomorphic and bounded functions in the open right-half plane. The notation $\epsilon \downarrow 0$ reads "ϵ approaches zero from above."

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We use the compact notation

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := D + C(sI - A)^{-1}B$$

for transfer functions in terms of their state realizations.

II. Internal Model Compensation

We study the problem of designing a stabilizer R_s in controller (1) with a given internal model M for a plant P having a proper transfer function $P(s)$. We assume that

$\mathcal{A}_1$: $\operatorname{spec}(M) \in \bar{\mathbb{C}}_0$, $M^{-1} \in H_\infty$, and $M(\infty) = I$,

$\mathcal{A}_2$: $p_e = m$ and $EP(s)$ has full normal rank,

$\mathcal{A}_3$: $\operatorname{spec}((EP)^{-1}) \cap \operatorname{spec}(M) = \varnothing$.

The first part of $\mathcal{A}_1$ is standard, there is no need for stable poles in internal models, whose role is to generate persistent signals of a given class. The second part of that assumption does not entail any restriction on the class of controllers, and their zeros in $\bar{\mathbb{C}}_0$ or at the infinity could be easily introduced via R_s, if required for whatever reason. Likewise, the normalization of $M(\infty)$ can always be ensured by R_s. Assumption $\mathcal{A}_2$ says that the regulated channel, $u \mapsto e$, is neither underactuated, which is necessary for any regulator problem, nor has redundancies. The latter simplifies our arguments, although is not necessary for them and can be relaxed at the expense of bulkier technicalities. Finally, $\mathcal{A}_3$, together with the second part of $\mathcal{A}_1$, ensures that there are no unstable cancellations in (2).

We start with a technical result, which plays a key role in our developments and whose implications will be discussed later on. We say that R_s internally stabilizes $P_{\rm aug}$ if the system (the gang of four)

$$T_{4,\rm aug} := \begin{bmatrix} I \\ -R_s \end{bmatrix} (I - P_{\rm aug} R_s)^{-1} \begin{bmatrix} I & P_{\rm aug} \end{bmatrix} \tag{3}$$

is stable (i.e. its transfer function belongs to H_∞). Introduce also the matrix $E_\perp \in \mathbb{R}^{(p-p_e)\times p}$ as any matrix satisfying

$$E'_\perp E_\perp = I - E'E.$$

Clearly, if $p > p_e$, then every such $E_\perp$ has full row rank and satisfies $E_\perp \begin{bmatrix} E' & E'_\perp \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$.

Theorem 1: R_s internally stabilizes $P_{\rm aug}$ defined by (2) iff

$$R_s = \bar{R}(I + E'_\perp \Upsilon_2 E) - \Upsilon_1 E \tag{4}$$

for some $\bar{R}$ internally stabilizing

$$\bar{P} := (I + E'_\perp \Upsilon_2 E) P_{\rm aug} (I + \Upsilon_1 E P_{\rm aug})^{-1} \tag{5}$$

and $\Upsilon_1, \Upsilon_2 \in H_\infty$ and such that $I + \Upsilon_1 E P_{\rm aug}$ is invertible.

Proof: The invertibility of $I + E'_\perp \Upsilon_2 E = (I - E'_\perp \Upsilon_2 E)^{-1}$ implies that any stabilizer R_s is produced by the unique $\bar{R} = R_s(I - E'_\perp \Upsilon_2 E) + \Upsilon_1 E$ and we may consider R_s in form (4) without loss of generality. Thus, we only need to show the equivalence of the stability of $T_{4,\rm aug}$ and that of

$$\bar{T}_4 := \begin{bmatrix} I \\ -\bar{R} \end{bmatrix} (I - \bar{P}\bar{R})^{-1} \begin{bmatrix} I & \bar{P} \end{bmatrix}, \tag{6}$$

which is the counterpart of $T_{4,\rm aug}$ for the interconnection of $\bar{P}$ and $\bar{R}$. These two systems are related as

$$T_{4,\rm aug} = \begin{bmatrix} I - E'_\perp \Upsilon_2 E & 0 \\ \Upsilon_1 E & I \end{bmatrix} \bar{T}_4 \begin{bmatrix} I + E'_\perp \Upsilon_2 E & 0 \\ -\Upsilon_1 E & I \end{bmatrix}, \tag{7}$$

which follows by the relation

$$P_{\rm aug} = (I - E'_\perp \Upsilon_2 E)(I - \bar{P}\Upsilon_1 E)^{-1} \bar{P}$$

and straightforward albeit lengthy algebra. Because

$$\begin{bmatrix} I + E'_\perp \Upsilon_2 E & 0 \\ -\Upsilon_1 E & I \end{bmatrix} = \begin{bmatrix} I - E'_\perp \Upsilon_2 E & 0 \\ \Upsilon_1 E & I \end{bmatrix}^{-1}$$

is bi-stable, we have that $T_{4,\rm aug} \in H_\infty \iff \bar{T}_4 \in H_\infty$, which completes the proof. ∎

Theorem 1 says that the stabilization of $P_{\rm aug}$ can be solved via that of $\bar{P}$ by introducing *internal model compensators* (IMC) Υ_1 and Υ_2 into the controller as in (4). This result is reminiscent of [5, Thm. 1] and [6, Thm. 1], where the stabilization of augmented plants is also reduced to that of a plant free of internal models via the use of IMC elements. However, the compensation is now qualitatively different, the stabilizer in (4) uses not only the parallel element $-\Upsilon_1 E$, similar to those in [5, 6], but also the cascade block $I + E'_\perp \Upsilon_2 E$. The latter connects the regulated measurement e with its complement in y. This is a consequence of the use of only a part of the measured signal for the internal model. If $e = y$, then $E_\perp$ is void, so is Υ_2, and (4) has the same structure as the controllers of [5, 6].

A. Complexity of $\bar{P}$

Stabilizing $\bar{P}$ as a means to stabilize the augmented plant makes sense only if $\bar{P}$ is simpler than $P_{\rm aug}$. In this subsection we show, by qualitative arguments, that stable Υ_1 and Υ_2 for which the order of $\bar{P}$ is the same as that of P can always be found. Concrete choices are then discussed in Section III.

As a first step, we find a more informative relation between $\bar{P}$ and IMC elements Υ_1 and Υ_2. To this end, rewrite (5), using (2) and relations between E and $E_\perp$, in the form

$$\begin{bmatrix} E \\ E_\perp \end{bmatrix} \bar{P} = \begin{bmatrix} MEP \\ E_\perp P + \Upsilon_2 MEP \end{bmatrix} (I + \Upsilon_1 MEP)^{-1}.$$

Equivalently,

$$\begin{bmatrix} E\bar{P} \\ E_\perp \bar{P} \end{bmatrix} (I + \Upsilon_1 MEP) = \begin{bmatrix} MEP \\ E_\perp P + \Upsilon_2 MEP \end{bmatrix}. \tag{8}$$

By $\mathcal{A}_1$ and $\mathcal{A}_2$, EP and M are invertible. Hence, post-multiplying the relation above by $(EP)^{-1}M^{-1}$ does not change it and we have

$$\begin{bmatrix} E\bar{P} \\ E_\perp \bar{P} \end{bmatrix} ((EP)^{-1}M^{-1} + \Upsilon_1) = \begin{bmatrix} I \\ E_\perp P (EP)^{-1} M^{-1} + \Upsilon_2 \end{bmatrix}.$$

The first row above implies that $E\bar{P}$ is invertible as well, so all we need is to construct a "simple" $\bar{P}$ such that

$$\begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \end{bmatrix} = \begin{bmatrix} I \\ E_\perp \bar{P} \end{bmatrix} (E\bar{P})^{-1} - \begin{bmatrix} I \\ E_\perp P \end{bmatrix} (EP)^{-1} M^{-1} \tag{9}$$

are stable and $I + \Upsilon_1 MEP$ is invertible (the latter is always true if $EP(s)$ is strictly proper).

Two observations are important to understand implications of (9). First, the logic of choosing $(E\bar{P})^{-1}$ and $E_{\perp}\bar{P}(E\bar{P})^{-1}$ is to match unstable, including non-proper, parts of $(EP)^{-1}M^{-1}$ and $E_{\perp}P(EP)^{-1}M^{-1}$, respectively. Second, M^{-1} is itself stable, by $\boldsymbol{\mathcal{A}_1}$, so instabilities above are related only to the plant P, without the internal model. These observations suggest that the complexity of $\bar{P}$ shall not exceed that of P. If EP is stably invertible, then the obvious choice is $\bar{P} = P$. Otherwise, the dependence of $\bar{P}$ on P is normally more involved. Still, the logic of constructing the two IMC elements is simple and Section III presents state-space formulae, in which the order of $\bar{P}$ equals that of P and the orders of Υ_1 and Υ_2 equal that of M^{-1}.

Remark 1 (if $p_e < m$): The reasoning above still applies if the regulator problem is over-actuated. What changes in that case is the need to replace $(EP)^{-1}$ as the right multiplier in processing (8) with a nonsingular $m \times m$ system $\begin{bmatrix} (EP)^{\#} & P_{\perp} \end{bmatrix}$, where $(EP)^{\#}$ is a right inverse of EP and $P_{\perp}$ is its complement such that $EPP_{\perp} = 0$. This would still lead us to (9), modulo the replacement of $(EP)^{-1}$ with $(EP)^{\#}$, and additional equations $E\bar{P}P_{\perp} = 0$ and $E_{\perp}\bar{P}P_{\perp} = E_{\perp}PP_{\perp}$ independent of Υ_1 and Υ_2. ▽

B. Closed-loop systems

The result of Theorem 1 addresses only the internal stability issue. It is naturally also important to understand the effect of applying the IMC elements on closed-loop systems of interest. In the context of internal-model principle, those are mainly the closed-loop sensitivity, S, and disturbance sensitivity, T_{d}, functions

$$\begin{bmatrix} S & T_{\mathrm{d}} \end{bmatrix} := (I - PR)^{-1} \begin{bmatrix} I & P \end{bmatrix}. \tag{10}$$

We are interested to understand relations between them and the corresponding closed-loop systems associated with $\bar{P}$ and $\bar{R}$ in (4) and (5), i.e. $\begin{bmatrix} \bar{S} & \bar{T}_{\mathrm{d}} \end{bmatrix} := (I - \bar{P}\bar{R})^{-1} \begin{bmatrix} I & \bar{P} \end{bmatrix}$. The relation is given by the result below.

Proposition 1: If Υ_1 and Υ_2 are given by (9), then

$$T_{\mathrm{d}} = P(EP)^{-1}M^{-1}E\bar{T}_{\mathrm{d}} \tag{11a}$$

and

$$S = I - P(EP)^{-1}E + P(EP)^{-1}M^{-1}E \times \big(\bar{S}(I - P(EP)^{-1}E) + \bar{T}_{\mathrm{d}}(EP)^{-1}E\big). \tag{11b}$$

Proof: It is readily seen that

$$\begin{bmatrix} S & T_{\mathrm{d}} \end{bmatrix} = \begin{bmatrix} M_0^{-1} & 0 \end{bmatrix} T_{4,\mathrm{aug}} \begin{bmatrix} M_0 & 0 \\ 0 & I \end{bmatrix},$$

where $T_{4,\mathrm{aug}}$ is defined by (3) and $M_0 := E'ME + E'_{\perp}E_{\perp}$ is the factor containing the internal model in controller (1). Taking into account (7), we then have the relation

$$\begin{bmatrix} S & T_{\mathrm{d}} \end{bmatrix} = (M_0^{-1} - E'_{\perp}\Upsilon_2 E) \begin{bmatrix} \bar{S} & \bar{T}_{\mathrm{d}} \end{bmatrix} \begin{bmatrix} M_0 + E'_{\perp}\Upsilon_2 ME & 0 \\ -\Upsilon_1 ME & I \end{bmatrix}$$

between the functions of interest. It then follows from (9) that

$$\begin{bmatrix} M_0 + E'_{\perp}\Upsilon_2 ME \\ -\Upsilon_1 ME \end{bmatrix} = \begin{bmatrix} I - PP_e^{-1}E \\ P_e^{-1}E \end{bmatrix} + \begin{bmatrix} \bar{P} \\ -I \end{bmatrix} \bar{P}_e^{-1}ME,$$

where $P_e = EP$ and $\bar{P}_e = E\bar{P}$, so that

$$\begin{bmatrix} S & T_{\mathrm{d}} \end{bmatrix} = (M_0^{-1} - E'_{\perp}\Upsilon_2 E) \begin{bmatrix} \bar{S} & \bar{T}_{\mathrm{d}} \end{bmatrix} \begin{bmatrix} I - PP_e^{-1}E & 0 \\ P_e^{-1}E & I \end{bmatrix}.$$

Using the expression for Υ_2 from (9), it can be shown that

$$(M_0^{-1} - E'_{\perp}\Upsilon_2 E)\bar{S} = I - \bar{P}\bar{P}_e^{-1}E + PP_e^{-1}M^{-1}E\bar{S},$$

from which (11a) follows by $\bar{T}_{\mathrm{d}} = \bar{S}\bar{P}$ and (11b) is derived using the relation $(I - \bar{P}\bar{P}_e^{-1}E)(I - PP_e^{-1}E) = I - PP_e^{-1}E$, which is readily verified. ■

It follows from Proposition 1 that

$$E \begin{bmatrix} S & T_{\mathrm{d}} \end{bmatrix} = M^{-1}E \begin{bmatrix} \bar{S} & \bar{T}_{\mathrm{d}} \end{bmatrix} \begin{bmatrix} I - P(EP)^{-1}E & 0 \\ (EP)^{-1}E & I \end{bmatrix},$$

meaning that unstable poles of $M(s)$ are zeros of both $ES(s)$ and $ET_{\mathrm{d}}(s)$, as expected from the internal model principle.

III. State-Space Construction of Υ_1, Υ_2, and $\bar{P}$

To derive state-space expressions for the systems in Theorem 1, bring in state-space realizations of the plant P and the internal model M,

$$P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \quad \text{and} \quad M(s) = \left[\begin{array}{c|c} A_{\mathrm{m}} & B_{\mathrm{m}} \\ \hline C_{\mathrm{m}} & I \end{array}\right],$$

whose state dimensions are n and n_{m}, respectively. To simplify formulae, we assume throughout this section that

$\boldsymbol{\mathcal{A}_4}$: $ED = 0$,

i.e. that $P_e(s)$ is strictly proper, and that

$\boldsymbol{\mathcal{A}_5}$: zeros of $M(s)$ and $P_e(s)$ are disjoint.

Choosing (stable) zeros of $M(s)$ to be different from those of $P_e(s)$ is not restrictive, this can always be compensated by R_{s}. We also need matrices $B^{\#} \in \mathbb{R}^{m \times n}$ and $B^{\perp} \in \mathbb{R}^{(n-m) \times n}$ such that

$$\begin{bmatrix} B^{\perp} \\ B^{\#} \end{bmatrix} B = \begin{bmatrix} 0 \\ I \end{bmatrix} \quad \text{and} \quad \det \begin{bmatrix} B^{\perp} \\ B^{\#} \end{bmatrix} \neq 0.$$

They exist whenever B has full column rank, which is guaranteed by $\boldsymbol{\mathcal{A}_2}$ and $\boldsymbol{\mathcal{A}_4}$. The following result can then be formulated.

Proposition 2: If $\boldsymbol{\mathcal{A}_{1\text{–}5}}$ hold true, then the generalized Sylvester equation

$$\begin{bmatrix} B^{\perp} \\ 0 \end{bmatrix} X(A_{\mathrm{m}} - B_{\mathrm{m}}C_{\mathrm{m}}) - \begin{bmatrix} B^{\perp}A \\ -EC \end{bmatrix} X = \begin{bmatrix} 0 \\ C_{\mathrm{m}} \end{bmatrix}, \tag{12}$$

has a unique bounded solution $X \in \mathbb{R}^{n \times n_{\mathrm{m}}}$ and

$$\begin{bmatrix} \Upsilon_1(s) \\ \Upsilon_2(s) \end{bmatrix} = \left[\begin{array}{c|c} A_{\mathrm{m}} - B_{\mathrm{m}}C_{\mathrm{m}} & B_{\mathrm{m}} \\ \hline C_0 & 0 \\ E_{\perp}CX + E_{\perp}DC_0 & 0 \end{array}\right], \tag{13}$$

where $C_0 := B^{\#}X(A_{\mathrm{m}} - B_{\mathrm{m}}C_{\mathrm{m}}) - B^{\#}AX$, and

$$\bar{P}(s) = \left[\begin{array}{c|c} A + XB_{\mathrm{m}}EC & B \\ \hline C & D \end{array}\right] \tag{14}$$

satisfy (9).

Proof: We start with (12). It is a generalized Sylvester equation, known [7] to be solvable if the pencils

$$\begin{bmatrix} B^{\perp}A \\ -EC \end{bmatrix} - s\begin{bmatrix} B^{\perp} \\ 0 \end{bmatrix} \quad \text{and} \quad sI-(A_{\mathrm{m}}-B_{\mathrm{m}}C_{\mathrm{m}}) \tag{15}$$

are regular and have no common roots. The second pencil above is obviously regular. To see whether this is the case for the first pencil of (15), rewrite it as

$$\begin{bmatrix} B^{\perp}A \\ -EC \end{bmatrix} - s\begin{bmatrix} B^{\perp} \\ 0 \end{bmatrix} = \begin{bmatrix} B^{\perp} & 0 \\ 0 & -I \end{bmatrix}\begin{bmatrix} A-sI \\ EC \end{bmatrix}.$$

Because adding zero columns does not change the rank, the rank of the matrix above is equivalent to the rank of

$$\begin{bmatrix} B^{\perp} & 0 \\ 0 & -I \end{bmatrix}\begin{bmatrix} A-sI & 0 \\ EC & 0 \end{bmatrix} = \begin{bmatrix} B^{\perp} & 0 \\ 0 & -I \end{bmatrix}\begin{bmatrix} A-sI & B \\ EC & 0 \end{bmatrix}$$

for all $s \in \mathbb{C}$. The last factor in the right-hand side above is the Rosenbrock system matrix associated with P_e, so it has full normal row rank by $\boldsymbol{\mathcal{A}_2}$. The regularity of the first pencil of (15) follows then by the full row rank of $B^{\perp}$. Now, by $\boldsymbol{\mathcal{A}_5}$ zeros of $P_e(s)$, which are the roots of the first pencil of (15), are assumed to be different from the zeros of $M(s)$, which are the roots of the second pencil of (15). Thus, those pencils are regular and have no common roots, which, in turn, proves the first statement of the Lemma.

Because $P_e(s)$ is strictly proper, its inverse does not have a standard state-space realization. To avoid bulky technicalities of moving to the descriptor formalism, we consider a perturbed version of (9), viz.

$$\begin{bmatrix} \Upsilon_{1\epsilon} \\ \Upsilon_{2\epsilon} \end{bmatrix} = \begin{bmatrix} I \\ E_{\perp}\bar{P}_{\epsilon} \end{bmatrix}(E\bar{P}_{\epsilon})^{-1} - G_{\epsilon}, \tag{9_ϵ}$$

where

$$G_{\epsilon} := \begin{bmatrix} I \\ E_{\perp}P \end{bmatrix}(\epsilon I + P_e)^{-1}M^{-1},$$

for some $\epsilon > 0$. If we find $\bar{P}_{\epsilon}$ and $\Upsilon_{1\epsilon}, \Upsilon_{2\epsilon} \in H_{\infty}$ satisfying this equation and if these systems are well defined as $\epsilon \downarrow 0$, then their limits solve (9).

It is readily verified that

$$\begin{bmatrix} M(s)(\epsilon I + P_e(s)) \\ E_{\perp}P(s) \end{bmatrix} = \left[\begin{array}{cc|c} A_{\mathrm{m}} & B_{\mathrm{m}}EC & \epsilon B_{\mathrm{m}} \\ 0 & A & B \\ \hline C_{\mathrm{m}} & EC & \epsilon I \\ 0 & E_{\perp}C & E_{\perp}D \end{array}\right]. \tag{16}$$

Furthermore, using the idea of [8, §III-C], the realization

$$G_{\epsilon}(s) = \left[\begin{array}{cc|c} A_{\mathrm{m}}-B_{\mathrm{m}}C_{\mathrm{m}} & 0 & B_{\mathrm{m}} \\ -\epsilon^{-1}BC_{\mathrm{m}} & A-\epsilon^{-1}BEC & \epsilon^{-1}B \\ \hline -\epsilon^{-1}C_{\mathrm{m}} & -\epsilon^{-1}EC & \epsilon^{-1}I \\ -\epsilon^{-1}E_{\perp}DC_{\mathrm{m}} & E_{\perp}(I-\epsilon^{-1}DE)C & \epsilon^{-1}E_{\perp}D \end{array}\right]$$

is obtained by swapping the input and the first output signals of the system in (16). The eigenvalues of $A_{\mathrm{m}} - B_{\mathrm{m}}C_{\mathrm{m}}$ and $A-\epsilon^{-1}BEC$ are zeros of $M(s)$ and $\epsilon I + P_e(s)$, respectively. By $\boldsymbol{\mathcal{A}_5}$ they are disjoint for all sufficiently small ϵ. As such, the Sylvester equation

$$X_{\epsilon}(A_{\mathrm{m}}-B_{\mathrm{m}}C_{\mathrm{m}}) - (A-\epsilon^{-1}BEC)X_{\epsilon} = \epsilon^{-1}BC_{\mathrm{m}} \tag{12_ϵ}$$

has a unique solution X_{ϵ}. Applying a similarity transformation with the matrix $\left[\begin{smallmatrix} I & 0 \\ X_{\epsilon} & I \end{smallmatrix}\right]$ to the realization of G_{ϵ} above, we end up with

$$G_{\epsilon}(s) = \left[\begin{array}{c|c} A-\epsilon^{-1}BEC & \epsilon^{-1}B + X_{\epsilon}B_{\mathrm{m}} \\ \hline -\epsilon^{-1}EC & \epsilon^{-1}I \\ E_{\perp}(I-\epsilon^{-1}DE)C & \epsilon^{-1}E_{\perp}D \end{array}\right] - \left[\begin{array}{c|c} A_{\mathrm{m}}-B_{\mathrm{m}}C_{\mathrm{m}} & B_{\mathrm{m}} \\ \hline \epsilon^{-1}(C_{\mathrm{m}}-ECX_{\epsilon}) & 0 \\ E_{\perp}CX_{\epsilon} + \epsilon^{-1}E_{\perp}D(C_{\mathrm{m}}-ECX_{\epsilon}) & 0 \end{array}\right].$$

The second term above is assumed to be stable, so it does not need to be canceled by $\bar{P}_{\epsilon}$ terms. Thus, we may take

$$\begin{bmatrix} \Upsilon_{1\epsilon}(s) \\ \Upsilon_{2\epsilon}(s) \end{bmatrix} = \left[\begin{array}{c|c} A_{\mathrm{m}}-B_{\mathrm{m}}C_{\mathrm{m}} & B_{\mathrm{m}} \\ \hline C_{\epsilon} & 0 \\ E_{\perp}CX_{\epsilon} + E_{\perp}DC_{\epsilon} & 0 \end{array}\right].$$

where $C_{\epsilon} := \epsilon^{-1}(C_{\mathrm{m}} - ECX_{\epsilon}) = B^{\#}X_{\epsilon}(A_{\mathrm{m}} - B_{\mathrm{m}}C_{\mathrm{m}}) - B^{\#}AX_{\epsilon}$ and the second equality follows by (12_ϵ). In this case we just need to find $\bar{P}_{\epsilon}$ such that

$$\begin{bmatrix} I \\ E_{\perp}\bar{P}_{\epsilon}(s) \end{bmatrix}(E\bar{P}_{\epsilon}(s))^{-1} = \left[\begin{array}{c|c} A-\epsilon^{-1}BEC & \epsilon^{-1}B + X_{\epsilon}B_{\mathrm{m}} \\ \hline -\epsilon^{-1}EC & \epsilon^{-1}I \\ E_{\perp}(I-\epsilon^{-1}DE)C & \epsilon^{-1}E_{\perp}D \end{array}\right]$$

to cancel all potential instabilities. To this end we again swap the input and the first output and end up with

$$\begin{bmatrix} E \\ E_{\perp} \end{bmatrix}\bar{P}_{\epsilon}(s) = \begin{bmatrix} E \\ E_{\perp} \end{bmatrix}\left[\begin{array}{c|c} A + X_{\epsilon}B_{\mathrm{m}}EC & B + \epsilon X_{\epsilon}B_{\mathrm{m}} \\ \hline C & D + \epsilon E' \end{array}\right].$$

This solves (9_ϵ).

Consider now (12_ϵ). It is obviously equivalent to

$$\begin{bmatrix} B^{\perp} \\ B^{\#} \end{bmatrix}(X_{\epsilon}(A_{\mathrm{m}}-B_{\mathrm{m}}C_{\mathrm{m}}) - AX_{\epsilon}) = \epsilon^{-1}\begin{bmatrix} 0 \\ C_{\mathrm{m}} - ECX_{\epsilon} \end{bmatrix}.$$

Multiplying the second block row above by $\epsilon > 0$, we have

$$\begin{bmatrix} B^{\perp} \\ \epsilon B^{\#} \end{bmatrix}X_{\epsilon}(A_{\mathrm{m}}-B_{\mathrm{m}}C_{\mathrm{m}}) - \begin{bmatrix} B^{\perp}A \\ \epsilon B^{\#}A - EC \end{bmatrix}X_{\epsilon} = \begin{bmatrix} 0 \\ C_{\mathrm{m}} \end{bmatrix}.$$

Because this is a linear equation in X_{ϵ}, the latter is continuous as a function of ϵ and then $\lim_{\epsilon\downarrow 0} X_{\epsilon} = X$, the solution of (12). Then $\bar{P}$, Υ_1, and Υ_2 are the limited cases of $\bar{P}_{\epsilon}$, $\Upsilon_{1\epsilon}$, and $\Upsilon_{2\epsilon}$. This completes the proof. ■

Curiously, the invariant zeros of this $\bar{P}$ coincide with those of P. This is seen from the relation

$$\begin{bmatrix} A + XB_{\mathrm{m}}EC - sI & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & XB_{\mathrm{m}}E \\ 0 & I \end{bmatrix}\begin{bmatrix} A-sI & B \\ C & D \end{bmatrix}$$

between their Rosenbrock matrices. By similar arguments, the invariant zeros of $E\bar{P}$ coincide with those of EP.

The overall controller (1) is then

$$R(s) = \begin{bmatrix} \bar{R}(s) & B^{\#} \end{bmatrix}\left[\begin{array}{c|c} A_{\mathrm{m}} & B_{\mathrm{m}}E \\ \hline CX & I \\ AX - X(A_{\mathrm{m}}-B_{\mathrm{m}}C_{\mathrm{m}}) & 0 \end{array}\right] \tag{17}$$

where $\bar{R}$ is a controller stabilizing $\bar{P}$. The order of this controller is the sum of those of $\bar{R}$ and the model M.

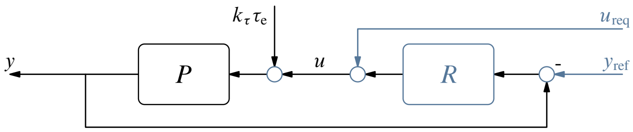


Fig. 1: 2DOF control system

If the former is an observer-based controller for $\bar{P}$, the controller order is $n + n_\mathrm{m}$. This is a clear advantage over the conventional design for the augmented plant P_aug in (2), where the controller order would be $n + 2n_\mathrm{m}$ in the observer-based case.

IV. Illustrative Example

Consider an armature-controlled DC motor connected to a rigid mechanical load, see [9, Sec. 6.5] for details. We assume that both the shaft angle θ_sh and its angular velocity ω_sh are measurable, i.e. that $y = \left[\begin{smallmatrix}\theta_\mathrm{sh}\\ \omega_\mathrm{sh}\end{smallmatrix}\right]$, and the control input is the armature voltage u. The controlled plant is

$$P(s) = \begin{bmatrix} P_\theta(s) \\ P_\omega(s) \end{bmatrix} = \begin{bmatrix} 1/s \\ 1 \end{bmatrix} \frac{K_\mathrm{m}}{(Js+f)R_\mathrm{a} + K_\mathrm{m}^2},$$

where K_m is the motor (torque) coefficient, R_a is the armature resistance (the inductance is neglected), and J and f are the moment of inertia and viscous friction coefficient of the rigid load, respectively. The disturbance signal is an external torque τ_e applied to the load, which is equivalent to the load (input) disturbance $k_\tau \tau_\mathrm{e}$, where $k_\tau := R_\mathrm{a}/K_\mathrm{m}$. The regulated variable is the shaft angle θ_sh, for which $E = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

We use the 2-degrees-of-freedom (2DOF) control architecture in the form depicted in Fig. 1. The signals y_ref and u_req represent the nominal command following requirements and R is a feedback controller. The closed-loop relations in this case are

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} y_\mathrm{ref} \\ u_\mathrm{req} \end{bmatrix} + \begin{bmatrix} S \\ RS \end{bmatrix} \bigl(P k_\tau \tau_\mathrm{e} - y_\mathrm{ref} + P u_\mathrm{req}\bigr),$$

where S is the sensitivity function defined in (10). If y_ref and u_req are chosen consistently, so that $y_\mathrm{ref} = P u_\mathrm{req}$, and there are no disturbances, i.e. $\tau_\mathrm{e} = 0$, then the perfect tracking condition $y = y_\mathrm{ref}$ holds regardless the choice of the feedback controller R. Modelling uncertainty and disturbances change this. But with an appropriate choice of R the effect of those factors on y (and u, but this is less relevant for our discussion) can be reduced. Specifically, if at some ω_i

$$ES(\mathrm{j}\omega_i) = 0 \quad \text{and} \quad ET_\mathrm{d}(\mathrm{j}\omega_i) = 0 \tag{18}$$

where S and T_d are as in (10), then the corresponding harmonic of τ_e, y_ref, and u_req do not affect the regulated error $\theta_\mathrm{sh} - E y_\mathrm{ref}$ in steady state even under uncertainty.

A. Regulation conditions

Our first goal is to understand what requirements to the feedback controller R conditions (18) impose. To this end, it can be shown that all stabilizing controllers can be characterized as

$$\begin{aligned} R(s) &= \begin{bmatrix} R_\theta(s) & R_\omega(s) \end{bmatrix} \\ &= \left(1 + b\frac{Q_1(s) + sQ_2(s)}{\chi_\mathrm{cl}(s)}\right)^{-1} Q(s) - \frac{1}{b}\begin{bmatrix} \chi_0 & \chi_1 - a \end{bmatrix}, \end{aligned}$$

K_m [N m/A]	R_a [Ω]	J [kg m²]	f [N m s/rad]	τ_max [N m]
0.126	2.08	0.008	0.005	0.235

TABLE I: Numerical values of motor and load parameters

where $Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \in H_\infty$ but otherwise arbitrary (the Youla parameter [10, Sec. 3.7]), $a = (f + K_\mathrm{m}^2/R_\mathrm{a})/J$, $b = K_\mathrm{m}/(R_\mathrm{a}J)$, and $\chi_\mathrm{cl}(s) = s^2 + \chi_1 s + \chi_0$ is an arbitrary Hurwitz polynomial (the closed-loop characteristic polynomial under $Q_1 = Q_2 = 0$). All stable ES and ET_d are then

$$\begin{aligned} E\begin{bmatrix} S(s) & T_\mathrm{d}(s) \end{bmatrix} &= \frac{1}{\chi_\mathrm{cl}(s)}\left(1 + b\frac{Q_1(s) + sQ_2(s)}{\chi_\mathrm{cl}(s)}\right) \\ &\quad \times \begin{bmatrix} s(s+\chi_1) & a - \chi_1 & b \end{bmatrix} - \frac{bQ_2(s)}{\chi_\mathrm{cl}(s)}\begin{bmatrix} s & -1 & 0 \end{bmatrix}. \end{aligned}$$

It is readily seen that condition (18) holds then iff

$$Q(\mathrm{j}\omega_i) = -\begin{bmatrix} \chi_\mathrm{cl}(\mathrm{j}\omega_i)/b & 0 \end{bmatrix} \neq 0$$

and this condition implies that the transfer function of the angle channel of the controller, $R_\theta(s)$, must have at least one pole at $s = \mathrm{j}\omega_i$. This justifies the use of the controller of form (1) with an internal model $M(s)$ having poles at each $s = \mathrm{j}\omega_i$.

Remark 2: It may happen that only the second condition of (18) is required. For example, it is not unreasonable to assume that only $\omega_i = 0$ is of interest in setpoint tracking problems. In such situations we may be concerned only with $ET_\mathrm{d}(\mathrm{j}\omega_i)$ if $\omega_i = 0$. If this is the case, then the condition on Q is relaxed to

$$Q_1(\mathrm{j}\omega_i) + \mathrm{j}\omega_i Q_2(\mathrm{j}\omega_i) = -\frac{\chi_\mathrm{cl}(\mathrm{j}\omega_i)}{b} \neq 0,$$

which does not entail $Q_2(\mathrm{j}\omega_i) = 0$. Moreover, if $Q_1(\mathrm{j}\omega_i) = 0$ is chosen, then we may end up with a controller solving the regulator problem without an internal model in the regulated channel (rather in the complementary velocity one). Still, the case of $Q_2(\mathrm{j}\omega_i) = 0$ is not ruled out and (1) is a legitimate choice. ▽

B. Design

Assume that condition (18) has to be satisfied for three frequencies,

$$\omega_0 = 0, \quad \omega_1 = \frac{1}{2}\pi, \quad \text{and} \quad \omega_2 = \frac{8}{3}\pi.$$

To ensure (18) in this case we consider the model

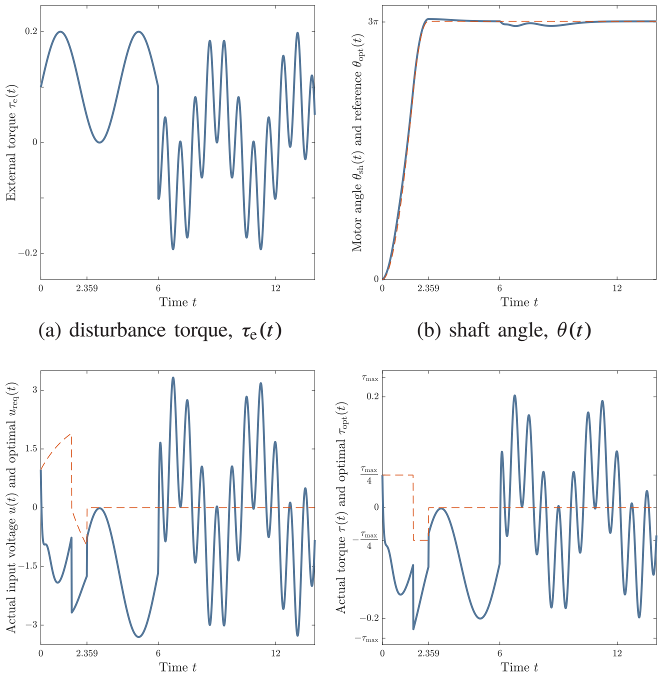
$$M(s) = \frac{(s + a_\mathrm{m})^5}{s(s^2 + \omega_1^2)(s^2 + \omega_2^2)}$$

for some $a_\mathrm{m} > 0$, which satisfies $\boldsymbol{\mathcal{A}}_1$ and $\boldsymbol{\mathcal{A}}_3$. We then choose $\bar{P}$, Υ_1, and Υ_2 according to Proposition 2. With the motor numerical data as in Table I, for which

$$P(s) = \begin{bmatrix} 1 \\ s \end{bmatrix} \frac{7.5672}{s(s + 1.578)} \quad \text{and} \quad k_\tau = 16.5187,$$

and $a_\mathrm{m} = 4$, we end up with

$$\bar{P}(s) = \begin{bmatrix} 1 \\ s - 20 \end{bmatrix} \frac{7.5672}{s^2 - 18.42s + 281.1}$$



(a) disturbance torque, $\tau_e(t)$ (b) shaft angle, $\theta(t)$

(c) input voltage, $u(t)$ (d) motor torque, $\tau(t)$

Fig. 2: Simulations

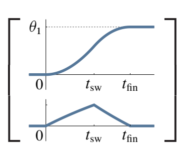
and the internal model compensators

$$\Upsilon_1(s) = \frac{422.84(s^2 + 4.88s + 6.28)(s^2 + 5.84s + 14.33)}{(s+4)^5},$$
$$\Upsilon_2(s) = -\frac{312.65(s^2 + 3.56s + 3.83)(s^2 + 4.63s + 17.09)}{(s+4)^5}.$$

The poles of $\bar{P}(s)$ are quite different from those of $P(s)$, they are actually in the open right-half plane. Still, this itself is not a problem and the design of $\bar{R}$ can be carried out as the standard static state feedback. Specifically, we place both closed-loop poles at $s = -2$ by $\bar{R}(s) = -\begin{bmatrix} 22.6436 & 2.9631 \end{bmatrix}$ and then implement the overall fifth-order controller R as in (17).

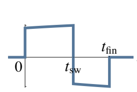
The reference signal y_{ref} is chosen to be the time-optimal shaft trajectory to attain a required steady-state θ_1 under a limited torque τ generated by the motor. For the maximum torque τ_{max} in Table I we choose the constraint to be $\tau_{max}/4$ (to have enough margins to compensate the external torque as well) and design the optimal torque for the load dynamics $J\ddot{\theta}_{sh} + f\dot{\theta}_{sh} = \tau$ under $\theta_{sh}(0) = \dot{\theta}_{sh}(0) = 0$, see [11, Ch. 7], although details are not essential here. Having calculated the optimal $\theta_{sh} = \theta_{opt}$, the reference signal

$$y_{ref}(t) = \begin{bmatrix} \theta_{opt}(t) \\ \omega_{opt}(t) \end{bmatrix} =$$



where $\omega_{opt} = \dot{\theta}_{opt}$, the required voltage $u_{req} = (1/P_\theta)\theta_{opt}$ is of the form

$$u_{req}(t) = \frac{R_a}{K_m}\tau_{opt}(t) + K_m\,\omega_{opt}(t) =$$



and τ_{opt} is bang-bang in the range $[-\tau_{max}/3, \tau_{max}/3]$.

C. Simulations

Simulated responses of the 2DOF controller in Fig. 1 for y_{ref} and u_{req} as above and $\theta_1 = 3\pi$ are presented in Fig. 2. The disturbance

$$\tau_e(t) = 0.1\begin{cases} 1 + \sin(\omega_1 t) & \text{if } 0 < t < 6 \\ \sin(\omega_1 t) - \cos(\omega_2 t) & \text{if } t > 6 \end{cases}$$

see Fig. 2(a). The resulting shaft angle is then as in Fig. 2(b), which also presents y_{ref} in the dashed line. The presence of the internal model M in the angle channel ensures that the disturbance is asymptotically rejected, as expected. The control signal is depicted in Fig. 2(c), where the dashed line corresponds to u_{req}. The resulted torque generated by the motor is shown in Fig. 2(d) and it is within the bounds of $\pm\tau_{max}$ (but this naturally depends on the actual disturbance). The bang-bang torque for which the reference trajectory was calculated is presented by the dashed line in Fig. 2(d).

V. Concluding Remarks

The paper has proposed a novel procedure of designing internal model controllers capable of reducing the stabilization problem of high-dimensional augmented systems, containing the plant and the internal model, to that of an internal model-free counterpart of the plant. A key in the procedure is the use of internal model compensation (IMC) elements, which are stable systems enabling the reduction. An explicit state-space construction of IMC has been derived.

A perspective future research direction is to analyze the implementation of IMC elements that could result in an affine dependence of the closed-loop dynamics on defining static parameters of the internal model. We expect that such an implementation could be instrumental in adding adaptation mechanisms, similarly to the state-feedback study in [12].

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