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On the relaxation property of nonlinear circuit elements

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Abstract— **The purpose of this paper is to explore a nonlinear generalization of the LTI theory of *relaxation* systems. LTI relaxation systems have the property that their Hankel operator is the gradient of a quadratic functional. We use this property as a defining property of nonlinear relaxation systems, generalizing the functional from quadratic to convex. Relaxation systems are shown to be special fading memory systems, characterized by strong positivity properties. It is suggested that relaxation systems and their duals define the elements of fading memory systems that admit a physical circuit representation.**

I. Introduction

How to model the memory of a physical device is a central question of control theory. The question goes back to the early modelling of operational amplifiers. It is undergoing a resurgence of interest with the development of neuromorphics, and the realization of neuromorphic circuits from memristive materials, as well as with the development of soft robotics, and the realization of soft actuators from memory alloys [1]–[4].

Memory modelling has known a distinct and somewhat separated development in the two traditions of system theory, namely the input-output theory of systems modelled as operators and the state-space theory of systems modelled as differential equations. In the input-output theory, a key advance came from the concept of fading memory [5]. Fading memory operators provide a nonlinear generalization of the convolution property of LTI systems: the present output of the system is calculated as a weighted combination of past inputs. For linear systems, the weights are the coefficients of the impulse response, that fades away at an exponential rate. In state-space theory, one could argue that the state *is* the memory, in that it summarizes what is needed from the past to determine the future. The physical interpretation of this abstract concept is however provided by dissipativity theory [6], through the concept of storage. State variables of physical circuits are in one-to-one correspondence with the elements that can store energy: in electrical circuits, charge is the storage of electric energy and flux is the storage of magnetic energy. In this context, memory modeling becomes equivalent to energy-based modeling. The memory of a physical system is identified from its storage elements.

The LTI theory of passive systems provides a clear relationship between the fading memory of passive operators and the physical storage of their state-space realization. Any stable passive convolution operator has fading memory *and* the input-output relationship can be realized as an electrical circuit with capacitors and inductors as storage elements. The storage is a quadratic function of the state.

For nonlinear systems, the link between fading memory and storage is quite unclear. Nonlinear state-space models do not necessarily have fading memory. Conversely, it is unclear how to define the storage of a fading memory operator.

The present paper is an attempt to reconcile the separate concepts of fading memory and storage in the physical modelling of nonlinear circuits. Our starting point is the *relaxation* property of the elements of linear circuits. The concept of relaxation elements has a long history going back to Maxwell and his linear relaxation model of viscoelastic materials. When perturbed, a relaxation system returns to equilibrium in a completely monotone manner, that is, *without a hint of oscillation* [6]. In his seminal dissipativity paper, Willems singles out LTI relaxation systems as those passive circuits whose storage is completely determined by the external behavior. Our recent paper [7] has formalized this property by showing that the Hankel operator of a LTI relaxation system is the gradient of a quadratic functional of the past input. Because it is not defined in terms of the state but directly in terms of the past input, we have named this functional the *intrinsic* storage of the system.

To generalize the concept of relaxation system to nonlinear systems, we take the intrinsic storage as the defining property of a relaxation element. While linear relaxation systems derives from *quadratic* storages, the generalization to *convex* storages provides a nonlinear concept of relaxation system. Nonlinear relaxation systems are derived from their convex storage in the same way LTI convolution operators are derived from their quadratic storage.

To build circuits from relaxation elements, we use duality theory. Any relaxation element has a dual defined by the Fenchel dual of its intrinsic storage. From there, circuits can be defined as parallel and series interconnections of relaxation elements. In that sense, relaxation systems become the elements of fading memory systems that can be realized as physical circuits.

The rest of the paper is organized as follows. After some preliminaries, we introduce in Section II a paradigmatic example of memristive element and discuss the limitations of its classical state-space representation. Section IV summa-

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rizes the key properties of LTI relaxation systems. Section V show how to extend the definition of relaxation to nonlinear systems. Section VI briefly discusses the duality theory of relaxation systems. Section VII show how the proposed approach can be employed to determine a relaxation model of the introductory example.

II. A Motivating Example

Our interest in the relaxation property stems from the difficulty to analyze circuits defined from state-space models. We illustrate this bottleneck in the seminal paper [8] that introduced circuit modelling in neuroscience. Hodgkin and Huxley modelled the excitable behavior of a biophysical neuron as the parallel interconnection of a leaky capacitor and two current sources, the so-called potassium and sodium ion channel currents.

They designed an experiment to identify the input-output response of each ion channel current separately, from a series of step input responses of different magnitude. Figure 1 reproduces the experimental responses recorded for the potassium current.

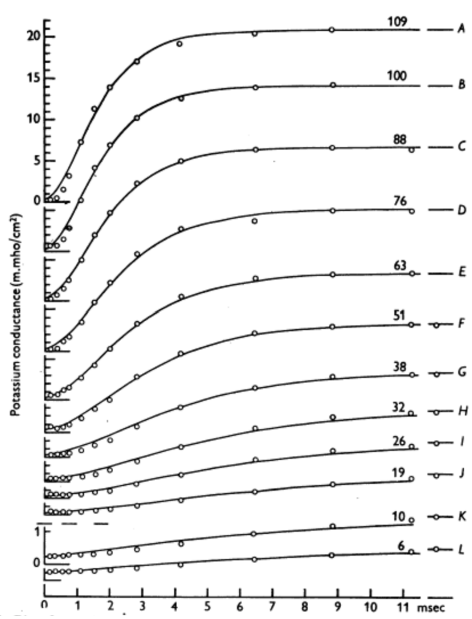

Fig. 1: Graph of data of conductance of potassium ion channel under different polarisation used in Hodgkin-Huxley model [8, Fig. 3].

In order to simulate an action potential, the authors fitted the experimental data to the following state-space model:

$$
\begin{aligned}
y &= 36n^4(u - V_K) \\
\dot{n} &= \alpha_n(u)(1-n) - \beta_n(u)n, \\
\alpha_n(u) &= \frac{0.1 - 0.01(-V_r + u)}{\exp(1 - 0.1(-V_r + u)) - 1} \\
\beta_n(u) &= \frac{0.125}{\exp(0.0125(-V_r + u))}
\end{aligned}
$$

with $V_r = -65.1$ and $V_K = -77$. V_r is the resting potential of the neuron.

Ever since, biophysical neural circuits have been modelled according to that same modelling principle. Both internal currents (ion channels) and external currents (synaptic currents) are modelled as state-space models similar to the potassium current model. Such models have been called *memristive* by Chua and Kang [9]. They are characterized by a dynamical (voltage-gated) conductance. They are trivially passive, since the product of current and voltage is always positive away from the equilibrium potential $V = V_K$. But what is their storage ? Do they have fading memory ? As lumped models of a complex molecular machinery, such elements do not easily fit neither the classical theory of fading memory systems nor the classical theory of energy-based modelling. This lack of modelling framework has become a central bottleneck of analysis and design, both in neuroscience and in neuromorphic engineering [10].

The experimental data of the potassium current suggest a memory element with the relaxation property: each step response models how the system returns monotonically to equilibrium from an initial deviation. We will illustrate in the last section of the paper how the system can be modelled as a relaxation system.

III. Preliminaries

We let $L_2(\mathbb{T})$ denote the space of square-integrable signals $u : \mathbb{T} \to \mathbb{R}$, where $\mathbb{T} \subseteq \mathbb{R}$ represents the time axis. L_2 is a Hilbert space, with inner product given by

$$\langle u, y \rangle = \int_{\mathbb{T}} u(t) y(t) \, \mathrm{d}t.$$

The Hardy space H_2 is the space of complex functions which are bounded and analytic in the right half plane, with inner product

$$\langle \hat{u}, \hat{y} \rangle = \int_{-\infty}^{\infty} \hat{u}(j\omega)^* \hat{y}(j\omega) \, \mathrm{d}\omega,$$

where z^* denotes the complex conjugate of z.

The Laplace transform is a linear bijection from $L_2([0,\infty))$ to H_2. Given $u \in L_2([0,\infty))$, its Laplace transform is given by

$$\hat{u}(s) = (\mathcal{L}u)(s) = \int_0^{\infty} u(t) e^{-st} \, \mathrm{d}t.$$

Given an absolutely integrable impulse response $g \in L_1$, we may define a Hankel operator $\Gamma_g : L_2([0,\infty)) \to L_2([0,\infty))$ by

$$(\Gamma_g u)(t) := \int_0^{\infty} g(t+\tau) u(\tau) \, \mathrm{d}\tau.$$

This may be thought of as a mapping from a past input $\bar{u}$ on the time interval $(-\infty, t]$ to a future output by setting $u(\tau) = \bar{u}(t - \tau)$.

A function $f : \mathbb{R} \to \mathbb{R}$ is *absolutely monotonic* if

$$\frac{\mathrm{d}}{\mathrm{d}t^n} f(t) \geq 0$$

for all t and n. A function $f(t)$ is *completely monotonic* if

$$(-1)^n \frac{\mathrm{d}}{\mathrm{d}t^n} f(t) \geq 0$$

for all t and n.

IV. LTI RELAXATION SYSTEMS

LTI relaxation systems are special convolution operators characterised by a completely monotonic impulse response:

$$(-1)^n \frac{\mathrm{d}}{\mathrm{d}t^n} g(t) \geq 0. \tag{1}$$

It follows from Bernstein's theorem [11] that such impulse responses have the representation

$$g(t) = \int_0^\infty e^{-pt} dN(p) \tag{2}$$

with $N(s)$ a bounded nondecreasing function on $[0, \infty)$. For simplicity, we limit ourselves to finite dimensional LTI relaxation systems with impulse response

$$g(t) = G_0 \delta(t) + \sum_i G_i e^{-p_i t}, \tag{3}$$

where $G_i, p_i \geq 0$ for all i. For convenience, we also limit the exposition to SISO systems, but MIMO relaxation systems are defined similarly, by replacing the scalars p_i and G_i by positive definite matrices. The Laplace transform of (3) is

$$\hat{g}(s) = G_0 + \sum_i \frac{G_i}{s + p_i}. \tag{4}$$

In the sequel, we omit the term G_0, that is, we choose $G_0 = 0$. This is because G_0 models the purely resistive part of the relaxation system. The nonlinear generalization of a linear resistor is a classical topic, to which we briefly return in Section VI. The focus of the present paper is on the dynamical part of the relaxation system. Finally, we exclude from the present paper the limiting case $p = 0$. This is just for the convenience of the exposition, and the limiting case $p = 0$ will be included in the journal version of this conference paper.

For $n = 1$, the Hankel operator of $g(t) = e^{-pt}$ has the expression

$$(\Gamma_g u)(t) = \int_0^\infty u(\tau) e^{-p(t+\tau)} d\tau = \hat{u}(p) e^{-pt}, \; t \geq 0. \tag{5}$$

This formula has the following intepretation: the future output is the impulse response g weighted by the scalar product of the past input with g. In the general case, from the Laplace transform of (5), we get the following time and frequency domain expressions of the Hankel operator.

Lemma 1: Let g be the impulse response of a relaxation system. Then,

$$y_{\Gamma_g}(t) = (\Gamma_g u)(t) = \sum_i G_i \hat{u}(p_i) e^{-p_i t}, \; t \geq 0 \tag{6a}$$

$$\hat{y}_{\Gamma_g}(s) = \mathcal{L}(\Gamma_g u)(s) = \sum_i G_i \frac{\hat{u}(p_i)}{s + p_i}. \tag{6b}$$

In the recent paper [7], we showed that the Hankel operator of an LTI relaxation system is a cyclic monotone operator on $L_2[0, \infty)$. This means that (6a) is the derivative of a closed, convex and proper potential $M : L_2([0, \infty)) \to \mathbb{R}$, defined by

$$M(u) = \frac{1}{2} \langle u, \Gamma_g u \rangle_2 = \frac{1}{2} \int_0^\infty u(t) y_{\Gamma_g}(t) dt. \tag{7}$$

For the simple case of $g(t) = e^{-pt}$, the potential takes the remarkable expression

$$M(u) = \frac{1}{2} \hat{u}(p) \int_0^\infty u(t) e^{-pt} dt = \frac{1}{2} \hat{u}^2(p), \tag{8}$$

which, for general impulse g, leads to

$$M(u) = \frac{1}{2} \sum_i G_i \hat{u}^2(p_i). \tag{9}$$

The main theorem in [7] shows that $M(u)$ defines an intrinsic storage for the relaxation system, given by convolution with the impulse response g. Letting $\bar{u}, \bar{y} \in L_2(\mathbb{R})$, this convolution operator has the expression

$$\bar{y}(t) = \int_{-\infty}^t g(t - \tau) \bar{u}(\tau) \, \mathrm{d}\tau. \tag{10}$$

In order to study properties of the convolution operator using the Hankel operator, we project the past of $\bar{u}$ on $(-\infty, t]$ to a signal $u_t \in L_2([0, \infty))$, given by

$$u_t(\tau) := \bar{u}(t - \tau), \quad \tau \in [0, \infty). \tag{11}$$

Theorem 1: [7, Thm. 6] The functional M satisfies the dissipation inequality

$$\frac{\mathrm{d}M(u_t)}{\mathrm{d}t} \leq \bar{u}(t) \bar{y}(t). \tag{12}$$

⌟

The functional $M(u)$ can be equivalently regarded as the physical storage of the relaxation system and as the memory potential of its Hankel operator. We call M the memory potential, or, equivalently, the intrinsic storage of the convolution operator defined by the completely monotone impulse response g. The memory potential M characterizes at once a fading memory operator and its physical storage.

V. NONLINEAR RELAXATION SYSTEMS

We have seen in the previous section how the quadratic potential $\hat{u}^2(p)$ defines a LTI convolution operator with the relaxation property. To generalize this definition to nonlinear systems, we generalize the quadratic function to any proper convex function $F : \mathbb{R} \to \mathbb{R}_{\geq 0}$ that reaches its minimum at zero. For simplicity, we assume that F is continuously differentiable and we note $f = F'$. Indeed, for the case $g(t) = e^{-pt}$, we have

$$M(u) = F(\hat{u}(p)), \tag{13}$$

where $p \in \mathbb{R}_{\geq 0}$ and $u \in L_2[0, \infty)$.

We then proceed to derive an operator from this memory potential by taking the functional derivative, defined via the first variation:

$$\langle \mathrm{grad}\, M(u), \phi \rangle := \left[\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \left(M(u + \varepsilon \phi) \right) \right]_{\varepsilon = 0}, \tag{14}$$

where $\phi \in L_2[0,\infty)$.

Theorem 2:

$$\operatorname{grad} M(u) = \frac{f(\hat{u}(p))}{s+p}. \tag{15}$$

⌟

Proof: Computing the functional derivative gives

$$\begin{aligned}\langle \operatorname{grad} M(u), \phi\rangle &= \left[\frac{\mathrm{d}}{\mathrm{d}\varepsilon} M(u+\varepsilon\phi)\right]_{\varepsilon=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[F(\hat{u}(p)+\varepsilon\hat{\phi}(p))\right]_{\varepsilon=0} \\ &= \left[f(\hat{u}(p)+\varepsilon\hat{\phi}(p))\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\hat{u}(p)+\varepsilon\hat{\phi}(p))\right] \\ &= f(\hat{u}(p))\hat{\phi}(p). \end{aligned} \tag{16}$$

The result of the theorem then follows from

$$\begin{aligned}\left\langle \frac{f(\hat{u}(p))}{s+p}, \hat{\phi}\right\rangle &= \langle f(\hat{u}(p))e^{-p\,\cdot}, \phi\rangle \\ &= \langle f(\hat{u}(p)), \phi e^{-p\,\cdot}\rangle \\ &= \int_0^\infty f(\hat{u}(p))\phi(t)e^{-pt}dt \\ &= f(\hat{u}(p))\hat{\phi}(p), \end{aligned} \tag{17}$$

where the first identity follow from Parseval's theorem. ■

This defines an operator $\operatorname{grad} M : H_2 \to H_2$. Taking the initial value of this operator gives the memory functional [5], [12] of a time-invariant fading memory system:

$$y(0) = \lim_{s\to\infty} s \operatorname{grad} M(u) = \lim_{s\to\infty} \frac{sf(\hat{u}(p))}{s+p} = f(\hat{u}(p)). \tag{18}$$

In the time domain, the latter reads

$$y(0) = f\left(\int_0^\infty u(\tau)e^{-p\tau}\,\mathrm{d}\tau\right). \tag{19}$$

Finally, the operator can be constructed from the memory functional as follows. Consider the input signal $\bar{u} \in L_2(\mathbb{R})$ and define u_t as in (11), which maps the signal $\bar{u}(t-\cdot) \in L_2(-\infty, t]$ into the signal $u_t(\cdot) \in L_2[0,\infty)$. Then, the operator output $\bar{y} \in L_2(\mathbb{R})$ can be computed by

$$\bar{y}(t) = f\left(\int_0^\infty u_t(\tau)e^{-p\tau}\,\mathrm{d}\tau\right) = f\left(\hat{u}_t(p_i)\right) \tag{20a}$$

$$= f\left(\int_0^\infty \bar{u}(t-\tau)e^{-p\tau}\,\mathrm{d}\tau\right). \tag{20b}$$

At each time, this has the form of a nonlinear readout of the output of a first order lag

$$\dot{\bar{x}}_p(t) = -p\bar{x}_p(t) + \bar{u}(t)\,, \quad \bar{y}(t) = f(\bar{x}_p(t)) \tag{21}$$

with the "initially at rest" condition $\bar{x}_p(-\infty) = 0$.

The derivation above motivates the following definition.

Definition 1: A relaxation system is a fading memory causal time-invariant system given by the input-output relationship

$$\bar{y}(t) = \sum_i f_i(\bar{x}_{p_i}(t)) = \sum_i f_i(\hat{u}_t(p_i)) \tag{22}$$

where $f_i : \mathbb{R} \to \mathbb{R}$ are bounded monotone functions, $\bar{x}_{p_i}(t) = \int_0^\infty \bar{u}(t-\tau)e^{-p_i\tau}d\tau$ and $\hat{u}_t = \mathcal{L}(\bar{u}(t-\cdot))$. For $F_i' = f_i$, the functional

$$M(u) = \sum_i F_i(\hat{u}(p_i)). \tag{23}$$

is called the memory potential of the relaxation system (22).

⌟

We note that the above definition is closely related to the kernel-based fading memory operators defined in the recent paper [12]. Relaxation systems are special kernel-based fading memory systems that derive from a potential. We do not discuss the kernel interpretation of relaxation systems in the present paper but the reader will note that the first order lag $\frac{1}{s+p}$ is indeed (up to a factor) the Szegő reproducing kernel of the Hardy space H_2. Note that systems of the form (21) were originally studied by Popov [13].

The following three theorems show that a nonlinear relaxation system inherits the key properties of an LTI relaxation system.

Theorem 3: Let each $f_i(x)$ be absolutely monotonic on $\mathbb{R}_{\geq 0}$. Then the impulse response of the relaxation system is completely monotonic. ⌟

Proof: The impulse response of the relaxation system is a sum of terms of the form $f_i(e^{-p_i t})$. The function $e^{-p_i t}$ is completely monotonic. The composition of an absolutely monotonic function with a completely monotonic function is completely monotonic [11, Chap. IV, Thm. 2b]. The sum of completely monotonic functions is completely monotonic. ■

Theorem 4: A relaxation system is passive and its memory potential defines an intrinsic storage. ⌟

Proof: Let $(\bar{u},\bar{y}) \in \Sigma$. As above, define $u_t \in L_2[0,\infty)$ such that $u_t(\tau) = \bar{u}(t-\tau)$ for all $\tau \in [0,\infty)$. It follows that $\frac{\mathrm{d}}{\mathrm{d}t}u_t(\tau) = -\frac{\mathrm{d}}{\mathrm{d}\tau}u_t(\tau)$. Then,

$$\frac{\mathrm{d}M(u_t)}{\mathrm{d}t} = \left\langle \operatorname{grad} M(u_t), \frac{\mathrm{d}u_t}{\mathrm{d}t}\right\rangle = -\left\langle \operatorname{grad} M(u_t), \frac{\mathrm{d}u_t}{\mathrm{d}\tau}\right\rangle.$$

Using (17), we then have

$$\begin{aligned}\frac{\mathrm{d}M(u_t)}{\mathrm{d}t} &= -\sum_i f_i(\hat{u}_t(p_i))\left\langle \mathcal{L}^{-1}\left(\frac{1}{s+p_i}\right), \frac{\mathrm{d}u_t}{\mathrm{d}\tau}\right\rangle \\ &= -\sum_i f_i(\hat{u}_t(p_i))\left\langle e^{-p_i\cdot}, \frac{\mathrm{d}u_t}{\mathrm{d}\tau}\right\rangle \\ &= -\sum_i f_i(\hat{u}_t(p_i))\int_0^\infty e^{-p_i\xi}\frac{\mathrm{d}u_t}{\mathrm{d}\tau}(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{24}$$

Integration by parts gives

$$\begin{aligned}\frac{\mathrm{d}M(u_t)}{\mathrm{d}t} &= -\sum_i f_i(\hat{u}_t(p_i))\bigg(\left[e^{-p_i\xi}u_t(\xi)\right]_0^\infty \\ &\quad + \int_0^\infty p_i e^{-p_i\xi}u_t(\xi)\,\mathrm{d}\xi\bigg) \\ &= \sum_i f_i(\hat{u}_t(p_i))u_t(0) - \sum_i p_i f_i(\hat{u}_t(p_i))\hat{u}_t(p_i) \\ &= \bar{y}(t)\bar{u}(t) - \sum_i p_i f_i(\hat{u}_t(p_i))\hat{u}_t(p_i) \\ &\leq \bar{y}(t)\bar{u}(t), \end{aligned} \tag{25}$$

where the final inequality follows from the fact that each f_i is monotone and satisfies $f_i(0) = 0$, so $f_i(\hat{u}(p_i))$ has the same sign as $\hat{u}(p_i)$. ■

Theorem 5: A relaxation system is externally positive: if $u(t) \geq 0$ for all t, $y(t) \geq 0$ for all t. ⌟

Proof: From Definition 1 for each i, the function f_i is monotone. It follows that $f_i(\bar{x}_{p_i}(t)) \geq 0$ whenever $\bar{x}_{p_i}(t) \geq 0$, which is clearly the case if $\bar{u}(t) \geq 0$ for all t. Thus, from (22), this implies that $\bar{y}(y) \geq 0$ for all t if $\bar{u}(t) \geq 0$ for all t. ■

VI. Duality of relaxation systems

A key aspect of LTI circuit theory is that each element has a dual: the dual of a resistor $i = Rv$ is the conductor $v = Gi$, with $G = \frac{1}{R}$. The dual of the capacitor with impedance $\hat{v}(s) = \frac{\hat{i}(s)}{Cs}$ is the inductor $\hat{i}(s) = \frac{\hat{v}(s)}{Ls}$, with $L = \frac{1}{C}$. The authors in [14] make the important observation that this duality property extends to nonlinear resistive elements: if $R(\cdot)$ is a monotone function that satisfies $R(0) = 0$, then the dual of the nonlinear resistor $v = R(i)$ is the nonlinear conductor $i = G(v)$ with G the inverse of R. The duality property comes from applying Fenchel duality to the convex functional $D(i) = \int_0^i R(x)dx$, the so-called resistive content introduced by Millar [15]. The Fenchel dual of $D(i)$ is the resistive co-content defined by

$$D^\star(v) = \sup_i (iv - D(i)). \tag{26}$$

The resistive content can be regarded as the area under the monotone curve $v = R(i)$. The resistive co-content can be regarded as the area under the monotone curve $i = R^{-1}(v)$. Both have the dimension of energy, and they sum to the power area iv.

In a similar way, we can generate a dual of the memory potential $M(i) = F(\hat{i}(p))$, by taking the Fenchel conjugate of F, denoted $F^\star$. For a first-order relaxation system, this gives:

$$F^\star(\hat{v}(p)) = \sup_{\hat{i}(p)\in\mathbb{R}} \left(\hat{v}(p)\hat{i}(p) - F(\hat{i}(p))\right). \tag{27}$$

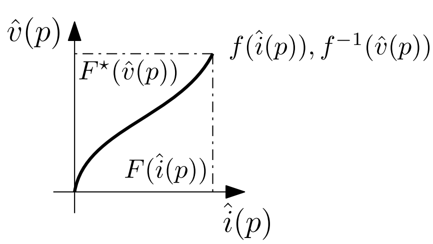

Fig. 2: Graphical interpretation of the memory potential and its conjugate.

Conjugating F and taking the gradient converts a capacitative relaxation system to an inductive relaxation system.

Theorem 6: If the relaxation system generated by $M(i) = F(\hat{i}(p))$ has dynamics

$$\dot{\bar{x}}_p(t) = -p\bar{x}_p(t) + \bar{i}(t)\,, \quad \bar{v}(t) = f(\bar{x}_p(t)), \tag{28}$$

then the relaxation system generated by $M^\star(v) = F^\star(\hat{v}(p))$ has dynamics given by

$$\dot{\bar{x}}_p(t) = -p\bar{x}_p(t) + \bar{v}(t)\,, \quad \bar{i}(t) = f^{-1}(\bar{x}_p(t)). \tag{29}$$

⌟

Proof: The proof mirrors that of Theorem 2, using Fenchel's identity [16, Sec. 2.1]: $\partial F^\star = (\partial F)^{-1} = f^{-1}$. ■

Nonlinear circuits can be defined by interconnecting relaxation elements according to Kirchoff's laws. The parallel interconnection of relaxation systems is a relaxation system defined by the sum of the storages. The series interconnection is defined as the dual of the parallel inteconnection of the dual elements.

VII. A relaxation model of the potassium current

We return to the introductory example of this paper to illustrate how the potassium current model can be approximated by a simple relaxation model.

As a first step, we fit a first-order relaxation model to the step responses of the potassium current. We choose the pole $p_1 = \frac{1}{0.15}$ to fit the slowest time-constant of the experiemental step responses (which is also the one with the smallest step amplitude). We then fit the monotone function f_1 to match the static gain of ten step responses of increasing amplitudes $\{u_i\}_{i=1}^{10}$. A piecewise quadratic fit provides the approximation shown in Figure 3.

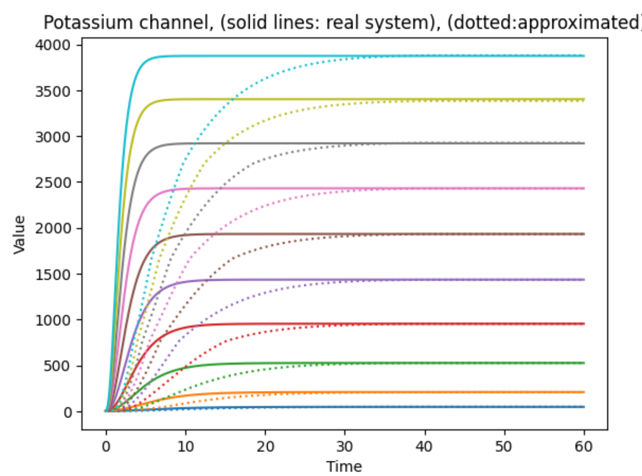

Fig. 3: Approximation of the potassium channel current by a first-order relaxation system.

This first-order relaxation system is sufficient to capture the slow dynamics of the channel. Additional poles can be added to capture the faster responses of larger step responses. With three poles chosen as $p_1 = \frac{1}{0.15}$, $p_2 = \frac{1}{0.3}$ and $p_3 = \frac{1}{0.6}$, we obtain the fit shown in Figure 4. The monotone functions f_i, $i = 1, 2, 3$, are regarded as "activation" functions. Their sum must fit the static gains of the step responses. The activation function of the slow pole is chosen to fit the small amplitude step responses. The activation function of the intermediate pole is chosen to fit the residual error for step responses of intermediate amplitudes. Finally, the activation function of the fastest pole is chosen to correct the residual error at large amplitudes. The figure suggests quite a good fit with a third-order relaxation system. It should be emphasized that the dynamics of ion channels are quite uncertain: they are variable from neuron to neuron and from experiment to experiment. Hence only a qualitative fit of the dynamics is desirable. We also note that the numerical simulation of a relaxation system is very cheap. The operator can be regarded as a temporal analog of the spatial nonlinear convolution operators of computer vision.

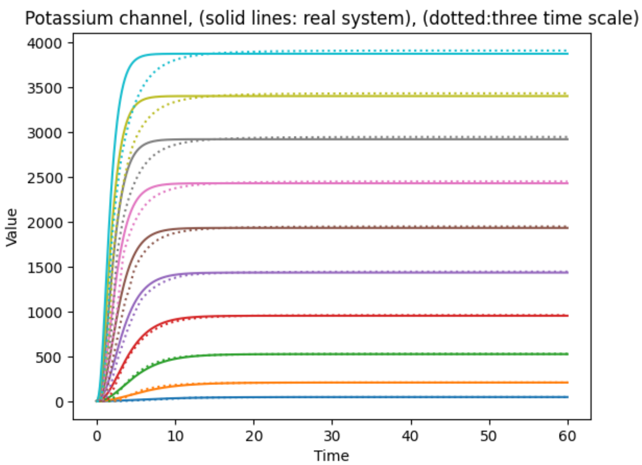

Fig. 4: Approximation of the potassium channel current by a third-order relaxation system

VIII. Conclusion

Relaxation systems have been characterized as time-invariant fading memory operators that derive from a memory potential. The memory potential is a functional of the past input, which can be equivalently regarded as the memory of the operator or its physical storage. Relaxation systems enjoy many properties entailed by the convexity of their potential. They exhibit the relaxation property, they are passive, and externally postitive. Relaxation systems provide a dynamical generalization of nonlinear resistors. In that sense, they can be regarded as alternative models of *memristive* elements, with attractive algorithmic properties.

The recent preprint [17] provides an alternative definition of nonlinear relaxation systems, grounded in the important concept of *reciprocity*. The connection between the two generalizations will be investigated in future research.

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