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Output Regulation of Second-Order Nonlinear Systems subject to Unknown Control Direction Using Extremum Seeking Control

Telema Harry, Martin Guay and Shimin Wang

Abstract— **The Nussbaum gain approach has been the standard technique in solving unknown control direction problems. In this paper, we propose control laws composed of extremum seeking control, an internal model, and a compensator signal to solve the robust practical output regulation problem of a second-order system subject to an unknown control direction. Using the Lie bracket approximation technique, we show that the closed-loop system is bounded and the origin is ϵ–Semi-global Practical Uniform Asymptotic Stable. Finally, we illustrate the effectiveness of the proposed approach with a numerical example of a Van der Pol system.**

I. INTRODUCTION

Output regulation is the standard theoretical framework for solving reference signal tracking and disturbance signal rejection problems. It has elicited much research interest in the control system community in the past five decades with many pioneering results, see *e.g.*, [1], [2], [3] and references therein, because of its wide applications in areas such as aerospace, robotic manipulators, mobile robots and many other engineering applications.

Most output regulation problems are solved under the assumption that the control direction (i.e. sign of control input coefficient) is known *a priori*. However, the control direction is unknown in many engineering applications, especially when all state variables are unavailable for measurements, and large uncertainties exist in the system [4]. Nussbaum gain approach, first introduced in [5], has been a standard technique for solving control problems with unknown control direction [6], [7], [8]. While Nussbaum controllers can be shown to guarantee global stability, they suffer from large initial overshoot when the initial control coefficient sign is guessed incorrectly [9].

In this work, we propose to solve the output regulation problem subject to an unknown control direction using an extremum-seeking controller. Extremum-seeking control is a model-free optimization and control technique with broad applications in many engineering fields see, *e.g.*, [10], [11], [12]. It is a powerful technique that drives a dynamical system to the optimal operating points corresponding to the extremum of an unknown *cost function* without explicit knowledge of the system dynamics [13].

Extremum-seeking control is now a well-established field in the control literature with numerous exciting results. In [9], the authors showed that a Lie bracket approximation-based extremum-seeking control algorithm gives a closed-loop response that is independent of the control direction. An extremum-seeking regulator was proposed to solve a practical output regulator problem in [14]. In [15], a practical output regulation problem was solved for a class of nonlinear with an unknown control direction system using control laws composed of extremum seeking control and internal model principle.

Generally speaking, the output regulation problem can be solved under suitable conditions by regulating an error-dependent objective function to the equilibrium point or the origin. In this work, we shall present a novel technique to solve the output regulation problem of a second-order nonlinear system subject to an unknown control direction. We first design a compensator signal and show that the regulation of this signal to the origin using a combination of extremum seeking control law and internal model solves the problem at hand. Finally, we showed the effectiveness of the proposed approach with a numerical example of a Van der Pol system.

The paper is organized as follows. Problem formulation is given in Section II. In Section III, the internal model and a brief introduction to Lie bracket approximation are presented. The compensator signal, extremum-seeking-based controller design, and the stability analysis of the closed-loop system are presented in Section IV. Some simulation results are presented in Section V and conclusions in Section VI.

A. Notations

For $A_1, \ldots, A_k \in \mathbb{R}^n$, let $\text{col}\,(A_1, \cdots, A_K) = [A_1, \cdots, A_K]^T$. Let $\|\cdot\|$ denote the Euclidian norm of a vector, and $|\cdot|$ denote the absolute value of a scalar. I_m denotes the identity matrix $I \in \mathbb{R}^{n\times n}$. The Lie bracket of two smooth vector fields f and g on a manifold M denoted by $[f, g]$ is defined as $[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$. The Jacobian of a continuously differentiable function $H \in C^1 : \mathbb{R}^n \to \mathbb{R}^m$ is denoted by

$$\frac{\partial H(x)}{\partial x} := \begin{bmatrix} \frac{\partial H_1(x)}{\partial x_1} & \cdots & \frac{\partial H_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial H_m(x)}{\partial x_1} & \cdots & \frac{\partial H_m(x)}{\partial x_n} \end{bmatrix}$$

II. PROBLEM FORMULATION

In this paper, we consider the output regulation of a class of second-order nonlinear systems with unknown control direction of the form:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x_1, x_2, w, m) + g(w, m)u \\ y &= x_1 \end{aligned} \tag{1}$$

where $x := \text{col}(x_1, x_2) \in \mathbb{R}^2$, $y \in \mathbb{R}$, and $u \in \mathbb{R}$ are the state, measured output, and control input respectively. $m \in \mathbb{M} \subset \mathbb{R}^{n_m}$ is a vector of unknown parameter and $w \in \mathbb{R}^q$

is the exogenous signal comprising of disturbance signal to be rejected and the reference signal to be tracked, which is commonly referred to as the exosystem. As the standard practice in output regulation literature, we assume that the exosystem is generated by the following autonomous system:

$$\dot{w} = Sw, \quad y_0 = h(w, m) \tag{2}$$

where $S \in \mathbb{R}^{q\times q}$ is an unknown constant matrix, and $y_0 \in \mathbb{R}$ is the exogenous signal output measurement. We assume that the functions $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^q \times \mathbb{R}^{n_m}$, $g : \mathbb{R}^q \times \mathbb{R}^{n_m}$, $h : \mathbb{R}^q \times \mathbb{R}^{n_m}$ are continuous in time and sufficiently smooth satisfying the equality $f(0, 0, 0, m) = 0$ and $h(0, m) = 0$. The sign of $g(w, m)$ is called the control direction. The error signal to be regulated is given by:

$$e = y - y_0 = h_e(x_1, w, m). \tag{3}$$

We propose a dynamic output feedback controller of the form:

$$u = \phi(\zeta, \eta), \quad \zeta = \psi(x, \eta, e), \quad \dot{\eta} = l(x, \eta) \tag{4}$$

where η is the internal model, ζ is a compensator signal to be defined, $\phi(\cdot)$ is a continuously differentiable function, $\psi(\cdot)$ and $l(\cdot)$ are smooth functions vanishing at the origin. We formally define the output regulation problem composed of systems (1), (2) and (3) as follows:

Problem 1. *Given the nonlinear system composed of* (1) *and* (2)*, with unknown control direction, design a control law of the form* (4)*, such that, for any initial condition* $x(0)$*,* $\eta(0)$*, and* $\zeta(0)$*, the trajectories of the closed-loop system composed of* (1) *and* (4) *are bounded for all* $t \geq 0$ *for any* $w \in \mathbb{W} \subset \mathbb{R}^q$*, the regulated error signal* $e(t)$ *achieves practical asymptotic stability:* $\lim_{t\to\infty} |e(t)| \leq \varepsilon$ *where* ε *is a small positive number.*

Remark 1. *Reference [16] studied the cooperative output regulation of the same class of system when the control direction of each agent is known. Reference [8] considered the problem without prior knowledge of the control direction. The problem was solved using Nussbaum-type gain and adaptive control methodology. We will adopt a model-free technique based on extremum-seeking control for a single agent. Our approach does not rely on the system dynamics, giving our approach the ability to handle systems with model uncertainties.*

III. PRELIMINARIES

A. Internal Model

A popular technique for solving output regulation problems is known as the internal model principle which can be intuitively described as: "Any good regulator must create a model of the dynamic structure of the environment in the closed-loop system" [17]. Isidori and Byrnes [1] showed that the solvability of the so-called regulator equations in nonlinear systems is a necessary condition for solving output regulation problems. The regulator equation for nonlinear systems is given as:

$$\begin{aligned} \frac{\partial \mathbf{x}(w, m)}{\partial w} Sw &= f(\mathbf{x}(w, m), \mathbf{u}(w, m), w, m) \\ 0 &= h_e(\mathbf{x}(w, m), \mathbf{u}(w, m), w, m) \end{aligned} \tag{5}$$

where $\mathbf{x}(w, m)$ and $\mathbf{u}(w, m)$ are smooth functions vanishing at the origin i.e. $\mathbf{x}(0, 0) = 0$ and $\mathbf{u}(0, 0) = 0$. The regulator equation solution associated with the system (1) and (3) can be obtained by inspection as follows:

$$\mathbf{x}_1(w, m) = h(w, m), \quad \mathbf{x}_2(w, m) = \frac{\partial h(w, m)}{\partial w} Sw$$
$$\mathbf{u}(w, m) = g^{-1}(w, m) \left(\frac{\partial \mathbf{x}_2(w, m)}{\partial w} Sw - f(\mathbf{x}_1, \mathbf{x}_2, w, m) \right)$$

where $\mathbf{x}(w, m) = \text{col}(\mathbf{x}_1(w, m), \mathbf{x}_2(w, m))$ and $\mathbf{u}(w, m)$ are referred to as the steady-state states and steady-state plant input respectively. Next, we state some standard assumptions.

Assumption 1. *All the eigenvalues of* S *are semi-simple with zero real parts. i.e. the unknown exogenous signal is neutrally stable*

Assumption 2. $|g(w, m)| \neq 0$ *for all* $w \in \mathbb{R}^q$*,* $m \in \mathbb{R}^{n_m}$

Assumption 3. *The functions* $\mathbf{x}_2(w, m)$*,* $\mathbf{u}(w, m)$ *are polynomials in* w *with polynomials depending on* m *for all* $m \in \mathbb{M}$.

Under Assumption 1, the dynamics of the exosystem (2) are such that $w(t)$ evolves on a compact invariant set $\mathbb{W} \subset \mathbb{R}^q$ as $t \to \infty$. Assumption 2 guarantees the existence of the regulator equation (5) solution. Under Assumption 1 – 3, there exist n integers and a monic polynomial

$$P(\lambda) = \lambda^n - \wp_1 - \wp_2\lambda^1 - \cdots - \wp_n\lambda^{n-1}$$

with imaginary roots such that for all trajectories of $w \in \mathbb{W}$ and all $m \in \mathbb{M}$, the following equations are satisfied:

$$\begin{aligned} \frac{d^n \mathbf{x}_2(w, m)}{dt^n} &= \wp_{10}\mathbf{x}_2(w, m) + \wp_{20}\frac{d\mathbf{x}_2(w, m)}{dt} \\ &\quad + \cdots + \wp_{n_0}\frac{d^{n-1}\mathbf{x}_2(w, m)}{dt^{n-1}} \\ \frac{d^n \mathbf{u}(w, m)}{dt^n} &= \wp_1\mathbf{u}(w, m) + \wp_2\frac{d\mathbf{u}(w, m)}{dt} \\ &\quad + \cdots + \wp_n\frac{d^{n-1}\mathbf{u}(w, m)}{dt^{n-1}}. \end{aligned}$$

The positive scalar coefficients $\wp_n$ are independent of the exogenous signal state w and the parameter m but depend only on the matrix S [16], [18].

Motivated by [16], we define the following internal model:

$$\begin{aligned} \dot{\eta}_1 &= M_0\eta_1 + Q_0 x_2 \\ \dot{\eta}_2 &= M_1\eta_2 + Q_1 u \end{aligned} \tag{6}$$

where for $i = 0, 1$, $M_i \in \mathbb{R}^{n_i\times n_i}$ is any Hurwitz matrix and $Q_i \in \mathbb{R}^{n_i\times 1}$, such that the pair (M_i, Q_i) are controllable. The system composed of the plant (1) and system (6) is commonly referred to as the augmented system [2], and it can be transformed into the so-called stabilization problem via coordinate transformation, such that for all $\text{col}(w,m) \in \mathbb{R}^q \times \mathbb{R}^{n_m}$, the origin of the augmented system is an equilibrium point. For $i = 0, 1$ we set:

$$\Phi_i := \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \wp_1 & \wp_2 & \cdots & \wp_n \end{bmatrix}, \quad \Gamma_i := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T.$$

$$\theta_0(w,m) := T_0 \text{col}\Big(\mathbf{x}_2(w,m), \cdots, \frac{d^{(n_0-1)}\mathbf{x}_2(w,m)}{d^{n_0-1}t}\Big)$$
$$\theta_1(w,m) := T_1 \text{col}\Big(\mathbf{u}(w,m), \cdots, \frac{d^{n_1-1}\mathbf{u}(w,m)}{d^{n_1-1}t}\Big)$$

and $\Psi_i = \Gamma_i T_i^{-1}$. It can be easily verified that the pair (Γ_i, Φ_i) is observable. Since Φ_i and M_i have distinct eigenvalues of zero real parts, the matrix T is the unique nonsingular matrix solution of the Sylvester equation [19]:

$$T_i \Phi_i - M_i T_i = Q_i \Gamma_i \tag{7}$$

Next, we perform coordinate and input transformation of the augmented system by defining new states and inputs:

$$\begin{aligned} \bar{x}_1 &= x_1 - \mathbf{x}_1(w,m), \quad \bar{x}_2 = x_2 - \Psi_0 \eta_1 \\ \bar{u} &= u - \Psi_1 \eta_2, \quad \tilde{\eta}_1 = \eta_1 - \theta_0(w,m) - Q_0 \bar{x}_1 \\ \tilde{\eta}_2 &= \eta_2 - \theta_1(w,m) - g^{-1}(w,m) Q_1 \bar{x}_2 \end{aligned} \tag{8}$$

We can re-write the augmented system in terms of the new coordinate transformation in system (8) Operation of system (8) on the augmented system results in:

$$\begin{aligned} \dot{\tilde{\eta}}_1 &= M_0 \tilde{\eta}_1 + M_0 Q_0 \bar{x}_1 \\ \dot{\bar{x}}_1 &= \Psi_0(\tilde{\eta}_1 + Q_0 \bar{x}_1) + \bar{x}_2 \\ \dot{\tilde{\eta}}_2 &= M_1 \tilde{\eta}_2 + \bar{f}_1(\bar{x}_1, \bar{x}_2, \tilde{\eta}_1, w, m) \\ \dot{\bar{x}}_2 &= \bar{f}_2(\bar{x}_1, \bar{x}_2, \tilde{\eta}_1, \tilde{\eta}_2, w, m) + g(w,m)\bar{u} \end{aligned} \tag{9}$$

where, $\bar{f}_1(\bar{x}_1, \bar{x}_2, \tilde{\eta}_1, w, m) = -\frac{\partial g^{-1}(w,m)}{\partial w} S w Q_1 \bar{x}_2 + g^{-1}(w,m) M_1 Q_1 \bar{x}_2 - g^{-1}(w,m) Q_1 \bar{f}(\bar{x}_1, \bar{x}_2, \tilde{\eta}_1, w, m)$, $\bar{f}_2(\bar{x}_1, \bar{x}_2, \tilde{\eta}_1, \tilde{\eta}_2, w, m) = \bar{f}(\bar{x}_1, \bar{x}_2, \tilde{\eta}_1, w, m) + g(w,m)\Psi_1(\tilde{\eta}_2 + g^{-1}(w,m) Q_1 \bar{x}_2)$. with $\bar{f}(\bar{x}_1, \bar{x}_2, \tilde{\eta}_1, w, m) = \bar{f}\Big(\bar{x}_1 + \mathbf{x}_1(w,m), \bar{x}_2 + \Psi_0(\tilde{\eta}_1 + Q_0 \bar{x}_1 + \theta_0(w,m)), w, m\Big) + g(w,m)\Psi_1 \theta_1(w,m) - \frac{\partial \mathbf{x_2}(w,m)}{\partial v} S w - \Psi_0 \dot{\tilde{\eta}}_1 - \Psi_0 Q_0 \dot{\bar{x}}_1$

One important characteristic of the augmented system (9) is that the origin of the system is zero for all $w \in \mathbb{R}^q$ and $m \in \mathbb{R}^{n_m}$ and the error signal e is equal to zero at the origin.

B. Lie Bracket Approximations

The proposed controller is based on a Lie bracket averaging extremum seeking control design. In this section, we provide a brief introduction to the Lie bracket averaging technique for a class of perturbed input-affine nonlinear systems of the following form:

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)\sqrt{\omega} u_i(\omega t) \tag{10}$$

where $x \in \mathbb{R}^n$, $x(0) \in \mathbb{R}^n$, $\omega > 0$, $t \in [0, \infty)$. We also assume that the functions $f(x)$ and $g(x)$ are twice continuously differentiable and the signals $u_i(\omega t)$ are periodic with period T, such that $\int_0^T u_i(\omega\tau)\mathrm{d}\tau = 0$ and uniformly bounded for all $t \geq 0$. It is shown in [20], [21] that the corresponding lie bracket approximation of system (10) is given by:

$$\dot{\bar{x}}^a = b_0(t, \bar{x}^a) + \frac{1}{T} \sum_{\substack{i=1 \\ j=i+1}} [b_i, b_j](t, \bar{x}^a) v_{j,i}(t) \tag{11}$$

where

$$v_{j,i} = \frac{1}{T}\int_0^T u_j(\theta) \int_0^\theta u_i(\tau), \mathrm{d}\tau \mathrm{d}\theta$$

The Lie bracket averaging technique can be used to design robust extremum seeking control laws, ([22], [23]) for different extremum seeking controller designs. One important property of the Lie bracket average technique is that the trajectories of the average system (11) converges to the trajectory of the nominal system (10) as the dither frequency $\omega \to \infty$. This can be formally stated using results from [24] as follows. Consider a nonlinear system:

$$\dot{x} = f(t, x) \tag{12}$$

whose trajectory is given by $x(t) = \varphi(t, t_0, x_0)$ for all $t \geq 0$, $x(t_0) = x_0$ and the vector field $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ depends on the dither frequency $\omega \in (0, \infty)$. Let the corresponding average system dynamics be given as:

$$\dot{x}^\varepsilon = f^\varepsilon(t, x^\varepsilon) \tag{13}$$

with a solution $x^\varepsilon(t) = \varphi^\varepsilon(t, t_0, x_0^\varepsilon)$ for all $t \geq 0$ and $x^\varepsilon(t_0) = x_0^\varepsilon$. Then, the convergence property is defined as follows:

Definition 1. *(Converging Trajectories Property [23]). Systems* (12) *and* (13) *are said to satisfy the convergence trajectories property if for every $T \in (0, \infty)$ and compact set $K \subset \mathbb{R}^n$ satisfying $\{(t, t_0, x_0) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n : t \in [t_0, t_0 + T], x_0 \in K\} \subset Dom\psi$, for every $d \in (0, \infty)$ there exists ε^* such that for all $t_0 \in \mathbb{R}$, for all $x_0 \in K$ and for all $\varepsilon \in (0, \varepsilon^*)$,*

$$\|\varphi^\varepsilon(t, t_0, x_0^\varepsilon) - \varphi(t, t_0, x_0)\| < d, \quad \forall t \in [t_0, t_0 + T] \tag{14}$$

Definition 2. *[23] ϵ–Seminglobal Practical Uniform Asymptotic Stability (ϵ–SPUAS) [23]: The equilibrium point of* (10) *is said to be (ϵ–SPUAS) if it satisfies the following three conditions uniform stability, uniform boundedness, and global uniform attractivity.*

In what follows, we state the following results from [24]

Lemma 1. *[23], If systems* (12) *and* (13) *satisfies the convergence trajectories property and if the origin is a global uniform asymptotic stable (GUAS) equilibrium point of* (12)*, then the origin of* (13) *is ϵ–SPUAS.*

We now state some standard assumptions required for our analysis in the next section.

Assumption 4. *The objective function $H(\zeta)$ is such that*

1) Its gradient vanishes only at the minimizer ζ^*, that is:

$$\left.\frac{\partial H}{\partial \zeta}\right|_{\zeta=\zeta^*} = 0.$$

2) At the minimizer, the Hessian is assumed to be strictly positive i.e.

$$\frac{\partial^2 H}{\partial \zeta^2} > \beta I, \quad \forall\, \zeta \in \mathbb{R}^n$$

where $\beta \in \mathbb{R}$ is a strictly positive constant.

Assumption 5. *At the minimizer ζ^* the tracking error $e \le \varepsilon$. where ε is a small positive number in the neighbourhood of the origin*

Assumption 6. *Given any compact set $\Theta \subset \mathbb{R}^{n_w} \times \mathbb{M}$, there exist a continuously differentiable positive definite function $V\colon \mathbb{R}^n \to \mathbb{R}$ and some class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$ and $\overline{\alpha}(\cdot)$, and a positive constant δ satisfying*

$$\underline{\alpha}(\|z\|) \le V(z) \le \overline{\alpha}(\|z\|)$$
$$\dot{V}(z,\zeta) \le -\alpha(\|z\|) + \delta\gamma(\|\zeta\|)$$

Assumption 4 is a standard assumption in the extremum-seeking control literature. See [25]. It guarantees that the unknown optimum of the cost function is obtained. Under Assumption 5, the tracking error $e(t)$ tends to the origin at the same rate with $\zeta \to \zeta^*$. Under Assumtion 6, the subsystem $\dot{\bar{x}} = \bar{f}(\bar{x}_1, \tilde{\eta}_1, \tilde{\eta}_2, \zeta)$ is ISS with respect to the state $\bar{x}$ and input ζ.

IV. Main Result

A. Compensator Signal Design

The compensator signal $\zeta \in \mathbb{R}$ is such that at the unknown optimum ζ^*, Problem 1 is solved. The control objective then becomes the design of an output feedback controller that can steer ζ to ζ^* asymptotically, which in turn indicates driving $e(t)$ to the origin asymptotically. We propose a compensator signal of the form:

$$\zeta = k_p \bar{x}_1 + \bar{x}_2 \tag{15}$$

with k_p a sufficiently large positive design parameter. $\bar{x}_1$ and $\bar{x}_2$ are same as (8).

B. Extremum Seeking Controller Design

We propose the extremum-seeking controller based on the Lie bracket averaging technique in the form:

$$u = \frac{k}{\alpha}\sqrt{\omega}\cos\Big(\alpha H(\zeta) + \omega t\Big)\rho(\zeta) + \Psi_1\eta_2 \tag{16a}$$
$$\dot{\eta}_1 = M_0\eta_1 + Q_0 x_2 \tag{16b}$$
$$\dot{\eta}_2 = M_1\eta_2 + Q_1 u \tag{16c}$$

where $H(\cdot) : \mathbb{R} \to \mathbb{R}$ is an unknown twice continuously differentiable function and it is known as the cost function.

Theorem 1. *Under Assumptions 1 – 4 and Assumption 6, there exists a positive real number k_p and a smooth function $\rho(\cdot)$, such that the global stabilization problem of system* (9) *is solved by the output feedback control law composed of* (15) *and* (16).

Proof: We re-write augmented system (9) in terms of the compensator signal (15) as follows

$$\dot{\tilde{\eta}}_1 = M_0\tilde{\eta}_1 + M_0 Q_0 \bar{x}_1 \tag{17a}$$
$$\dot{\bar{x}}_1 = \Psi_0(\tilde{\eta}_1 + Q_0\bar{x}_1) - k_p\bar{x}_1 + \zeta \tag{17b}$$
$$\dot{\tilde{\eta}}_2 = M_1\tilde{\eta}_2 + \bar{f}_{11}(\bar{x}_1, \zeta, \tilde{\eta}_1, w, m) \tag{17c}$$
$$\begin{aligned}\dot{\zeta} = {}& k_p\Big(\Psi_0(\tilde{\eta}_1 + Q_0\bar{x}_1) - k_p\bar{x}_1 + \zeta\Big) \\ &+ \bar{f}_{22}(\bar{x}_1, \zeta, \tilde{\eta}_1, \tilde{\eta}_2, w, m) + g(w,m)\bar{u}\end{aligned} \tag{17d}$$

where

$$\begin{aligned}&\bar{f}_{11}(\bar{x}_1, \tilde{\eta}_1, \zeta, w, m) = -\frac{\partial g^{-1}(w,m)}{\partial w} SwQ_1(\zeta - k_p\bar{x}_1) + \\ &\quad g^{-1}(w,m)(M_1Q_1(\zeta - k_p\bar{x}_1) - Q_1\bar{f}(\bar{x}_1, \tilde{\eta}_1, \zeta, w, m)), \\ &\bar{f}_{22}(\bar{x}_1, \tilde{\eta}_1, \tilde{\eta}_2, \zeta, w, m) = \bar{f}(\bar{x}_1, \tilde{\eta}_1, \zeta, w, m) \\ &\quad + g(w,m)\Psi_1\Big(\tilde{\eta}_2 + g^{-1}(w,m)Q_1(\zeta - k_p\bar{x}_1)\Big).\end{aligned}$$

We can easily verify that $\bar{f}_{11}(0,0,0,w,m) = 0$, $\bar{f}_{22}(0,0,0,0,w,m) = 0$ for all $\mathrm{col}(w,m) \in \mathbb{R}^{n_w} \times \mathbb{R}^{n_m}$. Thus, the origin is the equilibrium point of the augmented system (17) at which the regulated output e is identically equal to zero. The closed-loop system, composed of the output feedback control law (16) and the augmented system (17) can be written as

$$\dot{\tilde{\eta}}_1 = M_0\tilde{\eta}_1 + M_0 Q_0 \bar{x}_1 \tag{18a}$$
$$\dot{\bar{x}}_1 = \Psi_0(\tilde{\eta}_1 + Q_0\bar{x}_1) - k_p\bar{x}_1 + \zeta \tag{18b}$$
$$\dot{\tilde{\eta}}_2 = M_1\tilde{\eta}_2 + \bar{f}_{11}(\bar{x}_1, \zeta, \tilde{\eta}_1, w, m) \tag{18c}$$
$$\begin{aligned}\dot{\zeta} = {}& k_p\Big(\Psi_0(\tilde{\eta}_1 + Q_0\bar{x}_1) - k_p\bar{x}_1 + \zeta\Big) \\ &+ \bar{f}_{22}(\bar{x}_1, \zeta, \tilde{\eta}_1, \tilde{\eta}_2, w, m) \\ &+ g(w,m)\frac{k}{\alpha}\sqrt{\omega}\cos\Big(\alpha H(\zeta) + \omega t\Big)\rho(\zeta).\end{aligned} \tag{18d}$$

The corresponding average system is obtained after some straightforward computation as:

$$\dot{\tilde{\eta}}_1^a = M_0\tilde{\eta}_1^a + M_0 Q_0 \bar{x}_1^a \tag{19a}$$
$$\dot{\bar{x}}_1^a = \Psi_0(\tilde{\eta}_1^a + Q_0\bar{x}_1^a) - k_p\bar{x}_1^a + \zeta^a \tag{19b}$$
$$\dot{\tilde{\eta}}_2^a = M_1\tilde{\eta}_2^a + \bar{f}_{11}(\bar{x}_1^a, \tilde{\eta}_1^a, \zeta^a, w, m) \tag{19c}$$
$$\begin{aligned}\dot{\zeta}^a = {}& k_p\Psi_0\tilde{\eta}_1^a + (Q_0\Psi_0 - k_p)k_p\bar{x}_1^a + k_p\zeta^a \\ &+ \Psi_1\eta_2^a + \bar{f}_{22}(\tilde{\eta}_1^a, \bar{x}_1^a, w, m) \\ &- \frac{1}{2}k_p g^2(w,m)\rho^2(\zeta)\frac{\partial H}{\partial \zeta}.\end{aligned} \tag{19d}$$

Remark 2. *The average system clearly shows that the extremum-seeking algorithm produces the gradient of the cost function $H(\zeta^a)$ with respect to ζ^a multiplied by $g(w,m)$ square, which is always positive irrespective of the sign.*

We set $X_0^a := \mathrm{col}(\tilde{\eta}_1^a, \bar{x}_1^a)$ and consider the X_0^a–subsystem. Under Assumption 1, the exogenous signal $w(t)$ is bounded for all initial condition $\mathrm{col}(w_0, m) \in \mathbb{R}^{n_w} \times \mathbb{M}$. Since M_0 is Hurwitz, there exists a positive definite matrix $P_0 \in \mathbb{R}^{n\times n}$ satisfying

$P_0M_0 + M_0^TP_0 \leq -I_0a_0$. Where a_0 is a positive number $a_0 \in \mathbb{R}$. Next, we pose a Lyapunov function $V_0 = (\tilde{\eta}_1^a)^T P_0\tilde{\eta}_1^a + (\bar{x}_1^a)^T \bar{x}_1^a$. The time derivative of $V_0(X_0^a)$ along the trajectory of the X_0^a–subsystem gives:

$$\begin{aligned}\dot{V}_0 =& (\tilde{\eta}_1^a)^T P_0\Big(M_0\tilde{\eta}_1^a + M_0Q_0\bar{x}_1^a\Big)\\ &+ \Big(M_0\tilde{\eta}_1^a + M_0Q_0\bar{x}_1^a\Big)^T P_0\tilde{\eta}_1^a\\ &+ 2\bar{x}_1^a\Big(\Psi_0(\tilde{\eta}_1^a + Q_0\bar{x}_1^a) - k_p\bar{x}_1^a + \zeta^a\Big)\\ \leq& -(a_0-2)\|\tilde{\eta}_1^a\|^2 - \Big(k_p - \|P_0M_0Q_0\|^2\\ &- \|\Psi_0\|^2 - 2\|\Psi_0Q_0\| - 1\Big)\|\bar{x}_1^a\|^2 + \|\zeta^a\|^2.\end{aligned}$$

Letting $a_0 \geq 2$ and $k_p \geq \|P_0M_0Q_0\|^2 + \|\Psi_0\|^2 + 2\|\Psi_0Q_0\| + 1$ gives:

$$\dot{V}_0 \leq -A_1\|X_0^a\|^2 + \|\zeta^a\|^2 \tag{20}$$

with $A_1 = \min(a_0, k_p)$. According to the changing of supply rate technique [26] given any smooth function $\Lambda_0(X_0^a) > 0$, there exist a continuously differentiable function $U_0(X_0^a)$ and some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_0(\cdot)$ and $\overline{\alpha}_0(\cdot)$ satisfying

$$\underline{\alpha}_0(\|X_0^a\|) \leq U_0(X_0^a) \leq \overline{\alpha}_0(\|X_0^a\|)$$

such that for all $\text{col}(w,m) \in \mathbb{W}\times\mathbb{M}$

$$\dot{U}_0(X_0^a) \leq -\Lambda_0(X_0^a)\|X_0^a\|^2 + \varphi(\zeta^a)(\zeta^a)^2$$

where $\varphi(\cdot)$ is some smooth function satisfying $\varphi(\cdot) \geq 1$.

Next we consider the $\text{col}(X_0, \tilde{\eta}_2, \zeta)$–subsystem. Since M_1 is Hurwitz, then, there exists a positive definite matrix $P_1 \in \mathbb{R}^{n\times n}$ satisfying $P_1M_1 + M_1^TP_1 \leq -I_1a_1$ where a_1 is a positive real number. Next, we pose the following Lyapunov function:

$$V(X_0^a, \tilde{\eta}_2^a, \zeta^a) = U_0(X_0) + (\tilde{\eta}_2^a)^TP_1\tilde{\eta}_2^a + (\zeta^a)^2$$

Then for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_2(\cdot)$ and $\overline{\alpha}_2(\cdot)$, $V(X_0^a, \tilde{\eta}_2^a, \zeta^a)$ satisfies

$$\underline{\alpha}_2(\|(X_0^a, \tilde{\eta}_2^a, \zeta^a)\|) \leq V(X_0^a, \tilde{\eta}_2^a, \zeta^a) \leq \overline{\alpha}_2(\|(X_0^a, \tilde{\eta}_2^a, \zeta^a)\|).$$

The time derivative along the trajectory of $\text{col}(X_0^a, \tilde{\eta}_2^a, \zeta^a)$ is given by:

$$\begin{aligned}\dot{V} =& \dot{U}_0 - a_1(\tilde{\eta}_2^a)^T\tilde{\eta}_2^a + 2\tilde{\eta}_2^aP_1\hat{f}_{11}(\cdot) + 2k_p(\zeta^a)^2\\ &+ 2\zeta^a\Big(k_p\Psi_0\tilde{\eta}_1^a + (Q_0\Psi_0 - k_p)k_p\bar{x}_1^a\Big) + 2\Psi_1\eta_2^a\zeta^a\\ &+ 2\zeta^a\hat{f}_{22}(\cdot) - \zeta k^2g^2(w,m)\rho^2(\zeta^a)\frac{\partial H}{\partial\zeta^a}\end{aligned}$$

It can be verified that when $H(\zeta^a) = \frac{1}{2}(\zeta^a)^2$

$$\begin{aligned}\dot{V} \leq& -\Lambda_0(X_0^a)\|X_0^a\|^2 - a_1\|\tilde{\eta}_2^a\|^2\\ &- k_p^2g^2(w,m)\rho^2(\zeta^a)(\zeta^a)^2\end{aligned} \tag{21}$$

Therefore, by Lyapunov stability theorem, the solution of the average system (19) exists and is bounded for all $t \geq 0$. In particular, $\lim_{t\to\infty}\text{col}(X_0^a(t), \tilde{\eta}_2^a(t), \zeta^a(t)) = 0$. Therefore, system (19) is globally uniformly asymptotically stable. Hence, by Lemma 1, there exists some compact set, such that for any initial condition belonging to the compact, the closed-loop response of the original system (18) is ϵ–SPUAS. This completes the proof. □

V. Simulation Example

We present a numerical example to illustrate the effectiveness of our design. In what follows, we solve the Van der Pol oscillators problem in [16] when the control direction is unknown.

$$\begin{aligned}&\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \mu(m)x_2(1-x_1^2) + g(m)u\\ &y = x_1\end{aligned} \tag{22}$$

and the exogenous signal is given by: $\dot{w} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}w$ where the uncertain parameter $\mu(m) = 2 + m^2$, and where the gain $g(m) = -(1+m^2)$ is unknown to the controller. Instead, the controller is assumed to have access to direct measurement of ζ. We can easily obtain the solution to the regulator equation (5) for the plant and exosystem by inspection as: $\mathbf{x_1}(w,m) = w_1$, $\mathbf{x_2}(w,m) = w_2$ and $\mathbf{u}(w,m) = -g(m)^{-1}\mu(m)w_2(1-w_1^2)$. We can also show that $\frac{d^2\mathbf{x_2}(w,m)}{dt^2} = -\mathbf{x_2}(w,m)$, $\frac{d^4\mathbf{u}(w,m)}{dt^4} = -9\mathbf{u}(w,m) - 10\frac{d^2\mathbf{u}(w,m)}{dt^2}$.

We Let $M_0 = \begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}$, $Q_1 = \begin{bmatrix}0\\ 1\end{bmatrix}$ and

$$M_0 = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -10 & -18 & -15 & -6\end{bmatrix}, \quad Q_1 = \begin{bmatrix}0\\ 0\\ 0\\ 1\end{bmatrix}$$

Then, solving the Sylvester equation (7) gives $\Psi_0 = \begin{bmatrix}0 & 1\end{bmatrix}$ and $\Psi_1 = \begin{bmatrix}1 & 18 & 5 & 6\end{bmatrix}$. The controller is designed with $\rho(\zeta) = (\zeta+1)^2$, $H(\zeta) = \zeta^2$ and tuning parameters: $k = 2.5$, $\alpha = 1$, $k_p = 500$. The simulation is performed assuming $m = 1.5$, $w(0) = [2, -3]^T$, $x(0) = [2, -2]^T$, and $\eta(0) = 0$. Figure 1 and Figure 2 shows the tracking error and compensator signals respectively over different dither frequencies (ω). Both signal converges to the origin asymptotically. Figure 3 shows the trajectory of the Van der Pol oscillator and the exogenous signal.

VI. Conclusion

In this paper, we studied the robust practical output regulation problem of a second-order nonlinear system subject to an unknown control direction using control laws composed of an extremum seeking control, an internal model, and a compensator signal. We showed that the output regulation problem can be solved by regulating a suitable compensator signal to the origin. The algorithm does not rely on the system dynamics and the compensator signal model which gives it the advantage of being able to handle systems with model uncertainties. Finally, we showed that the closed-loop system is bounded and the origin is ϵ–Semi-global Practical Uniform Asymptotic Stable

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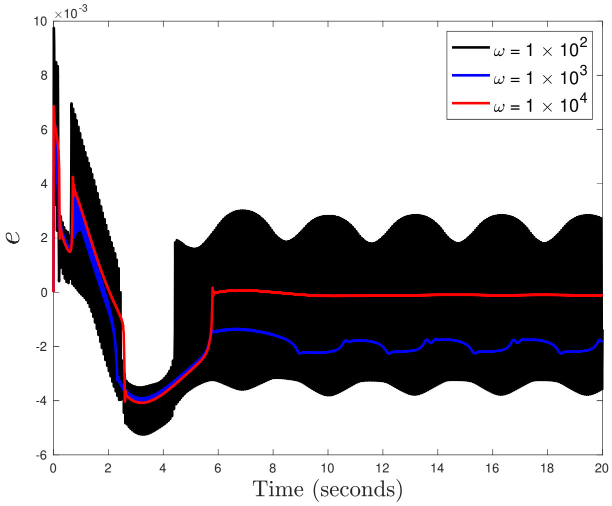

Fig. 1. Tracking error trajectory (e)

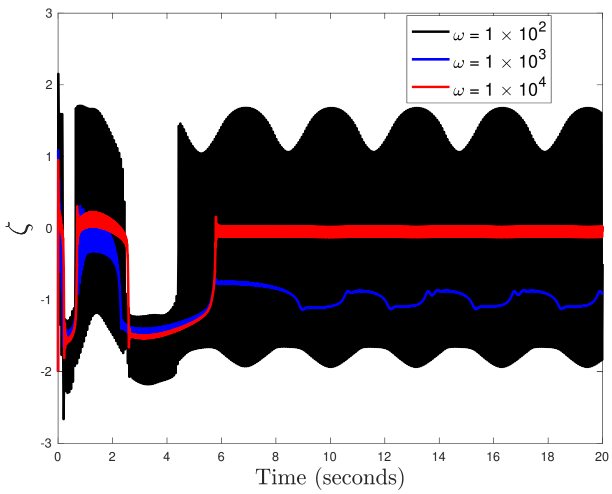

Fig. 2. Compensator signal (ζ) tracjectory

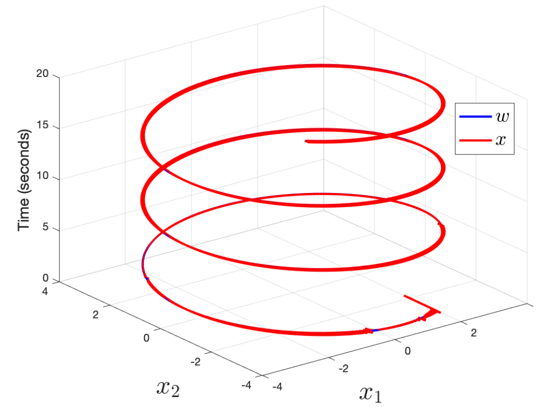

Fig. 3. 3-D plot of controlled Van der Pol oscillator and exogenous signal ($\omega = 1 \times 10^5$)

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