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On stochastic MPC formulations with closed-loop guarantees: Analysis and a unifying framework

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Abstract— **We investigate model predictive control (MPC) formulations for linear systems subject to i.i.d. stochastic disturbances with bounded support and chance constraints. Existing stochastic MPC formulations with closed-loop guarantees can be broadly classified in two separate frameworks: i) using robust techniques; ii) feasibility preserving algorithms. We investigate two particular MPC formulations representative for these two frameworks called *robust-stochastic* MPC and *indirect feedback* stochastic MPC. We provide a qualitative analysis, highlighting intrinsic limitations of both approaches in different edge cases. Then, we derive a unifying stochastic MPC framework that naturally includes these two formulations as limit cases. This qualitative analysis is complemented with numerical results, showcasing the advantages and limitations of each method.**

I. Introduction

Model predictive control (MPC) is an optimization-based control strategy that can ensure satisfaction of state and input constraints [1]. In order to account for disturbances, robust or stochastic MPC formulations can be utilized [2].

Stochastic MPC (SMPC): SMPC formulations consider chance constraints, i.e., constraint satisfaction with some user-chosen probability, to avoid the conservatism of worst-case robust formulations [3], [4]. The reformulation of (standard) linear stochastic optimal control problems as a deterministic quadratic program (QP) has been largely solved in the literature, e.g., using a constraint tightening based on analytical reformulations or scenario-based approximation [3], [4]. However, the development of receding horizon SMPC algorithms turns out to be non-trivial. In particular, the stochastic formulation explicitly allows for a certain probability of constraint violation. Hence, the optimization problem can become infeasible during closed-loop operation. In the SMPC literature, two main frameworks have emerged that address this issue and yield recursive feasibility, performance, and chance constraint satisfaction: utilizing *robust* techniques or redesigning the MPC algorithm.

Robust techniques: A natural solution to this problem is *robustly* ensuring recursive feasibility [2], [5]–[10]. In particular, one can use a constraint tightening composed of primarily robust bounds where only the first step is done stochastically, which ensures that the shifted candidate solution is feasible for any possible disturbance [5]–[7]. Due to the primary reliance on robust constraint tightening, we refer to this approach as *robust-stochastic* MPC. A less conservative approach is used in [8]–[10] by directly enforcing robust recursive feasibility using the robust control invariant set.

Feasibility preserving algorithms: In order to address possibly unbounded (e.g., Gaussian) disturbances, a second SMPC framework has emerged over the last decade which modifies standard MPC algorithms to ensure recursive feasibility [11]–[28]. Early work in [11] suggested to only use the new measured state if this retains recursive feasibility, which was also adopted/extended in [12]–[15]. More recently, in [16], it was shown that a modified version of this approach does in fact provide the desired closed-loop guarantees, which was subsequently utilized and extended in [17]–[21]. *Indirect feedback SMPC* [22] provides a simpler way to address feasibility by making the nominal state (and hence the feasible set) completely independent of the realized disturbances. Similar ideas were advocated in [23], and extensions of this idea can be found in [24]–[27], compare also [28] for a comparable approach.

Contribution: In this paper, we contrast these two available SMPC frameworks by carving out their relationships and individual trade-offs.

Since these frameworks were introduced for different problem setups (bounded vs. unbounded support), there exists no[1] analysis regarding their benefits and limitations. We consider two specific SMPC schemes that are representative of these two frameworks: the robust-stochastic MPC (RS-MPC) from [2, Chap. 8.1] and the indirect feedback SMPC (IF-SMPC) from [22]. We provide a qualitative analysis by showing edge cases where either one of the two approaches is guaranteed superior. This analysis is complemented with numerical simulations, which show that the quantitative difference between these formulations can be significant.

As a separate contribution, we derive a novel SMPC framework, called multi-step SMPC (MS-SMPC), which naturally unifies these two SMPC formulations. In particular, the difference between RS-MPC and IF-SMPC is on which state the chance constraints are conditioned: the current measured state or the initial state. The proposed SMPC formulation conditions the chance constraints on a state (up to) M time-steps in the past with some user-chosen M. For $M \in \{1, \infty\}$, this naturally recovers the two existing SMPC formulations.

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[1] In fact, [19], [24], [27] provide some recent comparisons between SMPC schemes. However, due to consideration of unbounded Gaussian noise, these discussions and comparisons do not consider the robust-stochastic MPC schemes [2], [5]–[9].

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Outline: We first present the problem setup in Section II and introduce preliminaries regarding SMPC (Sec. III). Section IV presents the two SMPC formulations, RS-MPC & IF-SMPC, and provides a qualitative analysis regarding benefits and limitations. Then, we derive a unifying framework (Sec. V), provide a numerical comparison (Sec. VI), and end the paper with some conclusions (Sec. VII). The theoretical proof for the unifying SMPC framework and additional details regarding the computation of the constraint tightening can be found online [29, App. A-B].

Notation: The set of integers in an interval $[a, b]$ is denoted by $\mathbb{I}_{[a,b]}$. The modulo operator for $k, M \in \mathbb{I}_{\geq 0}$ is denoted by $\mathrm{mod}(k, M) \in \mathbb{I}_{[0,M-1]}$. For vectors $a, b \in \mathbb{R}^n$, $a \leq b$ denotes an element-wise inequality. We denote a diagonal matrix with diagonal elements $a_i \in \mathbb{R}$, $i \in \mathbb{I}_{[1,n]}$ by $\mathrm{diag}(a_1, \ldots a_n) \in \mathbb{R}^{n\times n}$. A vector of ones is denoted by $\mathbf{1}$. The trace of a square matrix A is denoted by $\mathrm{tr}\,(A)$. The j-th row of a matrix $F \in \mathbb{R}^{r\times n}$ is denoted by $F_{(j)} \in \mathbb{R}^{1\times n}$. For a vector $x \in \mathbb{R}^n$ and a positive semi-definite matrix $Q \in \mathbb{R}^{n\times n}$, we abbreviate $\|x\|_Q^2 := x^\top Q x$. The expected value of a stochastic variable w is denoted by $\mathbb{E}\,[w]$. The probability of an event A is denoted by $\Pr\,[A]$. By $c_{i|k}$, $k \in \mathbb{I}_{\geq 0}$, $i \in \mathbb{I}_{\geq 0}$ we denote a prediction of a variable c at time k, i steps in the future.

II. Problem setup

We consider linear systems of the form

$$x(k+1) = Ax(k) + Bu(k) + Dw(k) \tag{1}$$

with state $x(k) \in \mathbb{R}^n$, input $u(k) \in \mathbb{R}^m$, and additive disturbance $w(k) \in \mathbb{R}^q$. The disturbances are independent and identically distributed (i.i.d.) with zero-mean and variance $\Sigma_\mathrm{w} \in \mathbb{R}^{q\times q}$. In addition, we assume that the disturbances are bounded, i.e., $w(k) \in \mathcal{W}$, $\forall k \in \mathbb{I}_{\geq 0}$ with some known compact set $\mathcal{W}$. The system is constrained by individual half-space chance constraints

$$\Pr\left[H_{\mathrm{x},(j)} x(k) \leq 1\right] \geq p_j,\ j \in \mathbb{I}_{[1,r_\mathrm{x}]} \tag{2}$$

with some probability level $p_j \in (0, 1]$. Furthermore, we have hard polytopic input constraints of the form

$$H_{\mathrm{u},(j)} u(k) \leq 1,\ j \in \mathbb{I}_{[1,r_\mathrm{u}]}. \tag{3}$$

The constraints (2)–(3) can be equivalently formulated into mixed state and input constraints

$$\Pr\left[F_{(j)} x(k) + G_{(j)} u(k) \leq 1\right] \geq p_j,\ j \in \mathbb{I}_{[1,r]} \tag{4}$$

with $r = r_\mathrm{x} + r_\mathrm{u}$ and $p_j = 1$ for $j \in \mathbb{I}_{[r_\mathrm{x}+1, r_\mathrm{x}+r_\mathrm{u}]}$. Furthermore, we have a convex linear-quadratic stage cost $\ell(x, u) := \|x\|_Q^2 + x^\top q + \|u\|_R^2 + u^\top r$ that should be minimized. The control problem can be summarized by the following stochastic optimal control problem

$$\min_{u(\cdot)} \mathbb{E}\left[\sum_{k=0}^{\infty} \ell(x(k), u(k))\right] \tag{5a}$$

$$\text{s.t. } (1),\ (4),\ \forall k \in \mathbb{I}_{\geq 0}. \tag{5b}$$

Remark 1. *(Problem setup) A large part of the stochastic MPC literature deals with the challenges related to unbounded (e.g., Gaussian) disturbances [11], [16], [19], [22]. These approaches typically relax the hard input constraints* (3) *to chance constraints (although non-trivial methods to consider both exist, cf. [26], [3, Table 2]). We consider bounded disturbances to allow for a comparison to the RS-MPC methods in [2], [5]–[10]. Furthermore, we consider hard input constraints due to their prevalence in practical application and the inability of the approaches in [2], [5], [6], [8]–[10] to consider chance input constraints (see Rk. 3 below regarding the treatment of input constraints).*

III. Preliminaries - stochastic optimal control

In the following, we discuss how to reformulate Problem (5) into a QP by using a simple input parametrization and offline constraint tightening, as standard in the SMPC literature [2], [22]. We consider a linear parametrization for the control input[2]

$$u(k) = Kx(k) + c(k) \tag{6}$$

with a free input $c(k) \in \mathbb{R}^m$ and an offline chosen stabilizing state feedback $K \in \mathbb{R}^{m\times n}$. This results in dynamics $x(k+1) = \Phi x(k) + Bc(k) + Dw(k)$ with $\Phi := A + BK$ Schur stable. We define nominal dynamics

$$s(k+1) = \Phi s(k) + Bc(k),\ s(0) = x(0) \tag{7}$$

and an error $e(k) = x(k) - s(k)$, which satisfies

$$e(k+1) = \Phi e(k) + Dw(k),\ e(0) = 0. \tag{8}$$

The following proposition formulates the chance constraints as tightened constraints on the nominal state s.

Proposition 1. *(adapted from [2, Lemma 8.1, Sec. 3.2]) The stochastic constraint* (4) *with* (6), (7), (8) *for* $k \in \mathbb{I}_{\geq 0}$ *is equivalent to*

$$\tilde{F}s(k) + Gc(k) \leq \mathbf{1} - \gamma_k, \tag{9}$$

where $\tilde{F} := F + GK$ *and*

$$\gamma_{k,(j)} = \min_{\gamma} \gamma \tag{10a}$$

$$\text{s.t. } \Pr\left[\tilde{F}_{(j)} e(k) \leq \gamma\right] \geq p_j,\ j \in \mathbb{I}_{[1,r]}. \tag{10b}$$

If $p_j = 1$, (9)–(10) *simplifies to*

$$\tilde{F}s(k) + Gc(k) \leq \mathbf{1} - \sum_{l=0}^{k-1} a_l, \tag{11}$$

$$a_{l,(j)} = \max_{w\in\mathcal{W}} \tilde{F}_{(j)} \Phi^l D w,\ l \in \mathbb{I}_{\geq 0},\ j \in \mathbb{I}_{[1,r]}. \tag{12}$$

Tightening constants γ_k (10) can be computed/approximated offline [9, Sec. V.A], e.g., using the scenario approach, while the constants a_k can be computed using linear programming for polytopic $\mathcal{W}$. We note that the computation of constants γ_k can be equally posed as the computation of probabilistic reachable sets for

[2]More general disturbance affine feedbacks would require a more complex chance constraint reformulation in Prop. 1, see, e.g., [30].

the error, compare [16], [22]. For large horizons $k \geq \bar{k}$, with some $\bar{k} \gg 1$, we can set $\gamma_k = \gamma_{\max}$, with some constant $\gamma_{\max}$ satisfying (10b) for all $k \in \mathbb{I}_{\geq 0}$. Details regarding the computation of γ_k and $\gamma_{\max}$ can be found online [29, App. B].

We consider a finite horizon $N \in \mathbb{I}_{\geq 1}$ and optimize over an input sequence $c_{i|k} \in \mathbb{R}^m$, $i \in \mathbb{I}_{[0,N-1]}$, where we set $c_{i|k} = 0$ for $i \in \mathbb{I}_{\geq N}$. In the considered setting and parametrization, minimizing the expected cost in (5a) yields the same minimizer as minimizing the cost of the mean prediction [22]. Hence, considering an initial state $x(k)$, and a predicted input sequence $c_{i|k} \in \mathbb{R}^m$, $i \in \mathbb{I}_{[0,N-1]}$ over a horizon N, we arrive at the following cost:

$$\mathcal{J}(x(k), c_{\cdot|k}) := \sum_{k=0}^{N-1} \ell(\bar{x}_{i|k}, \bar{u}_{i|k}) + V_{\mathrm{f}}(\bar{x}_{N|k}), \tag{13}$$

with the mean prediction $\bar{x}_{0|k} = x(k)$, $\bar{u}_{i|k} = c_{i|k} + K\bar{x}_{i|k}$, $\bar{x}_{i+1|k} = \Phi\bar{x}_{i|k} + Bc_{i|k}$, $i \in \mathbb{I}_{[0,N-1]}$ [2, Thm. 6.1]. Furthermore, $V_{\mathrm{f}}(x) := \|x\|^2_{P_{\mathrm{f}}} + x^\top p_{\mathrm{f}}$ is the linear-quadratic terminal cost that accounts for the infinite-horizon tail, i.e.,

$$\Phi^\top P_{\mathrm{f}}\Phi + Q + K^\top RK = P_{\mathrm{f}}, \tag{14a}$$

$$\Phi^\top p_{\mathrm{f}} = p_{\mathrm{f}} + q + K^\top r. \tag{14b}$$

Finally, the stochastic optimal control problem can be formulated using the following (finite-dimensional) QP

$$\min_{c_{\cdot|k}} \mathcal{J}(x(k), c_{\cdot|k}) \tag{15a}$$

$$\text{s.t. } s_{0|k} = x(k) \tag{15b}$$

$$s_{i+1|k} = \Phi s_{i|k} + Bc_{i|k},\ i \in \mathbb{I}_{[0,N-1]} \tag{15c}$$

$$\tilde{F}s_{i|k} + Gc_{i|k} \leq \mathbf{1} - \gamma_i,\ i \in \mathbb{I}_{[0,N-1]} \tag{15d}$$

$$s_{N|k} \in \mathcal{X}_{\mathrm{f}}. \tag{15e}$$

The terminal set constraint (15e) enforces (15d) for $i \in \mathbb{I}_{\geq N}$ with $c_{i|k} = 0$, $i \in \mathbb{I}_{\geq N}$ and $\mathcal{X}_{\mathrm{f}} = \{s|\tilde{F}\Phi^i s \leq \mathbf{1} - \gamma_{N+i},\ i \in \mathbb{I}_{\geq 0}\}$, which admits a finite polytopic representation [2, Thm. 8.2]. Solving Problem (15) at time $k = 0$ yields a suboptimal but feasible solution to Problem (5). However, an SMPC scheme based on Problem (15) would not be recursively feasible (cf. [3], [4]).

Definition 1. *(Desired closed-loop properties) The closed-loop system resulting from an SMPC scheme should satisfy the following properties:*

1) *Given initial feasibility, the SMPC formulation is recursively feasible and hence the closed loop is well-defined.*
2) *The chance constraints and input constraints* (2)–(3) *are satisfied for all times $k \in \mathbb{I}_{\geq 0}$.*
3) *The asymptotic average performance is no worse than applying the linear feedback $u = Kx$, i.e.,*

$$\ell_{\mathrm{avg}} := \limsup_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{k=0}^{T-1} \ell(x(k), u(k))\right] \leq \operatorname{tr}\left(P_{\mathrm{f}}\Sigma_{\mathrm{w}}\right). \tag{16}$$

In the following, we assume that ℓ admits a uniform lower bound[3], which holds trivially if ℓ is positive definite. Next, we discuss two SMPC formulations that meet these requirements.

IV. Analysis of SMPC frameworks

In this section, we present the robust-stochastic MPC (RS-MPC) [2] (Sec. IV-A) and the indirect feedback SMPC [22] (IF-SMPC) (Sec. IV-B), which are representative for the two SMPC frameworks commonly considered in the literature. Then, we provide a qualitative analysis, revealing particular shortcomings of either scheme (Sec. IV-C).

A. Robust-stochastic MPC

First, we present RS-MPC [2], which modifies the SMPC formulation (15) by using more conservative constraint tightening constants $\beta_i \geq \gamma_i$ that ensure robust recursive feasibility. In particular, this SMPC scheme is based on the following optimization problem:

$$\min_{c_{\cdot|k}} \quad \mathcal{J}(x(k), c_{\cdot|k}) \tag{17a}$$

$$\text{s.t.} \quad s_{0|k} = x(k) \tag{17b}$$

$$s_{i+1|k} = \Phi s_{i|k} + Bc_{i|k},\ i \in \mathbb{I}_{[0,N-1]} \tag{17c}$$

$$\tilde{F}s_{i|k} + Gc_{i|k} \leq \mathbf{1} - \beta_i,\ i \in \mathbb{I}_{[1,N-1]} \tag{17d}$$

$$s_{N|k} \in \mathcal{X}_{\mathrm{f}}^{\mathrm{RS}} \tag{17e}$$

$$H_{\mathrm{u}}\left(Kx(k) + c_{0|k}\right) \leq \mathbf{1}. \tag{17f}$$

We denote a minimizer by *. In closed loop, we solve Problem (17) at each time $k \in \mathbb{I}_{\geq 0}$ and apply $u(k) = Kx + c^\star_{0|k}$ to the system. The constraint tightening β_i is constructed by treating only the first disturbance stochastically, while the rest is handled robustly:

$$\beta_i = \gamma_1 + \sum_{l=1}^{i-1} a_l, \quad i \in \mathbb{I}_{\geq 1}. \tag{18}$$

Furthermore, stochastic constraints on the state only need to be enforced for $i \geq 1$, since $s_{0|k} = x(k)$ is deterministic. Hence, the hard input constraint for $i = 0$ is implemented separately (17f). A terminal set constraint (17e) enforces the constraints (17d) for $i \geq N$ with $c_{i|k} = 0$, i.e.,

$$\mathcal{X}_{\mathrm{f}}^{\mathrm{RS}} = \{s|\tilde{F}\Phi^i s \leq \mathbf{1} - \beta_{i+N},\ i \in \mathbb{I}_{\geq 0}\}. \tag{19}$$

Compared to Problem (15), the main difference is the fact that the constants γ_i are replaced by more conservative tightenings $\beta_i \geq \gamma_i$, which yields all the desired closed-loop guarantees.

Theorem 1. *(adapted from [2, Thm. 8.1, Cor. 8.1]) The SMPC scheme based on Problem* (17) *ensures the desired closed-loop properties in Definition 1.*

The key insight in this approach is that the tightening β_i turns out to be the least restrictive tightening to *robustly* ensure recursive feasibility with the standard (shifted) candidate solution [2, Theorem 8.1]. In general, the approach tries to

[3]This condition is invoked to obtain the performance bound (16) from an expected cost decrease in $\mathcal{J}$, see [19, Prop. 5], [16, Cor. 2].

robustly ensure satisfaction of the chance constraints, hence the name *robust-stochastic MPC*.

B. Indirect feedback SMPC

An alternative SMPC method with closed-loop guarantees is IF-SMPC [22]. Contrary to the SMPC formulation (15), the nominal state s is not reset to the measured state $x(k)$ at each time k, but follows the nominal dynamics (7). The corresponding optimization problem is provided below:

$$\min_{c_{\cdot|k}} \quad \mathcal{J}(x(k), c_{\cdot|k}) \tag{20a}$$
$$\text{s.t.} \quad s_{0|k} = s(k) = s^{\star}_{1|k-1} \tag{20b}$$
$$s_{i+1|k} = \Phi s_{i|k} + B c_{i|k},\ i \in \mathbb{I}_{[0,N-1]} \tag{20c}$$
$$\tilde{F} s_{i|k} + G c_{i|k} \leq \mathbf{1} - \gamma_{i+k},\ i \in \mathbb{I}_{[0,N-1]} \tag{20d}$$
$$s_{N|k} \in \mathcal{X}_{\mathrm{f}}^{\mathrm{IF}}. \tag{20e}$$

Closed-loop operation is similar to RS-MPC: Problem (20) is solved at each $k \in \mathbb{I}_{\geq 0}$, and we apply $u(k) = Kx + c^{\star}_{0|k}$ to the system. The nominal state $s(k)$ follows the nominal dynamics (7) with $s(0) = x(0)$. We enforce constraints for $i \geq N$ with $c_{i|k} = 0$ using the terminal set

$$\mathcal{X}_{\mathrm{f}}^{\mathrm{IF}} = \{s | \tilde{F}\Phi^i s \leq \mathbf{1} - \gamma_{\max}, i \in \mathbb{I}_{\geq 0}\}, \tag{21}$$

where $\gamma_{\max} \geq \gamma_k$, $k \in \mathbb{I}_{\geq 0}$ (cf. [29, App. B]). As the measured state $x(k)$ enters the optimization problem (20) only through the objective, the scheme is named *indirect feedback SMPC*.

Theorem 2. *(adapted from [22, Thm. 1–2, Cor 1]) The SMPC scheme based on Problem* (20) *ensures the desired closed-loop properties in Definition 1.*

The fact that the nominal state is not reset provides recursive feasibility, as the nominal system develops independently from the error. Closed-loop constraint satisfaction follows directly from enforcing constraints (4) for all $i \in \mathbb{I}_{\geq 0}$ with (20d).

C. Qualitative analysis

In the following, we compare RS-MPC and IF-SMPC using edge cases that reveal particular shortcomings of either scheme.

1) Shortcomings of IF-SMPC: In the following, we show that IF-SMPC is more conservative than RS-MPC if $p \to 1$ by relating to established robust MPC formulations.

First, note that for $p_j = 1$, $j \in \mathbb{I}_{[1,r_{\mathrm{x}}]}$, the constraint tightening constants are equivalent, i.e., $\beta_i = \gamma_i$, compare (10), (12), (18). Furthermore, RS-MPC reduces to a standard robust constraint-tightening MPC [2, Sec. 3.2–3.3], [31].[4] On the other hand, as $k \to \infty$, the tightening in IF-SMPC (20d) increases to $\gamma_{\max} \geq \gamma_{i+k}$, which corresponds to the size of the (minimal) robust positive invariant set for the error. Hence, the IF-SMPC corresponds to the simple/conservative robust tube MPC scheme in [32]. In particular, compared to a standard robust tube MPC (Mayne et al. (2005) [33]), the fixed nominal initial state $s_{0|k} = s^{\star}_{1|k-1}$ (20b) reduces the degrees of freedom. It is well known in the robust MPC literature, that this simpler approach is more conservative than taking into account the new measured state $x(k)$, in particular whenever the realized disturbance is not the worst-case disturbance [23, Sec. 5], [34, Sec. 3]. While the independence of the constraints from the measured state and realized disturbances is a key characteristic of IF-SMPC, this does not allow for a "resetting" of the constraints and thereby makes the approach more conservative as $p \to 1$.

2) Shortcomings of RS-MPC: In the following, we discuss cases where the constraint tightening in RS-MPC is significantly more conservative and where this results in a performance deterioration.

By definition, $\gamma_i \leq \beta_i$, since γ_i is an exact reformulation (Prop. 1) and β_i (18) treats only the first disturbance stochastically and the rest robustly (see also proof [2, Thm. 8.1]). There are several cases where this difference becomes extremely large: (i) Whenever the disturbance only affects the constraints indirectly through the dynamics ($\tilde{F}D = 0$), we have $\gamma_1 = 0$, and thus, RS-MPC enforces (conservative) worst-case constraints, independent of the probability level $p \in (0, 1)$ or the disturbance distribution (see [24, Sec. III.A]). (ii) If w is drawn from a truncated Gaussian with fixed variance, then the conservatism becomes arbitrary large if the support $\mathcal{W}$ is large. (iii) Similarly, the difference becomes significant if $p \ll 1$, or more generally, whenever there is a big difference between chance constraints and a purely robust formulation.

Given the discussion in Section IV-C.1, it is not immediately obvious if $\gamma_i \ll \beta_i$ implies that RS-MPC is more conservative in closed-loop operation. Hence, we next provide a clear case where $\gamma_i < \beta_i$ deteriorates the performance of RS-MPC. Suppose we wish to stabilize a steady-state which is close to a chance constraint. By choosing an LQR feedback $K = K_{\mathrm{LQR}}$, IF-SMPC recovers the optimal LQR performance, assuming the LQR satisfies the chance constraints ($\gamma_i \leq \gamma_{\max} \leq 1$) [24, Lemma 1]. RS-MPC can provide the same guarantees if $\beta_i \leq \beta_{\max} \leq 1$. However, since $\gamma_i < \beta_i$, this condition is in general not satisfied. Then, the standard RS-MPC design is infeasible and instead, a suboptimal design choice is required, e.g., stabilizing a steady-state further away from the constraints or trying to decrease β_i with a suboptimal tube feedback K, resulting in performance deterioration compared to IF-SMPC.

Remark 2. *(Generalizations) The presented analysis focused on two specific SMPC schemes: RS-MPC and IF-SMPC. Nonetheless, the qualitative analysis similarly applies more generally to the two SMPC frameworks: [2], [5]–[10] vs. [11]–[28].*[5] *For example, the conservatism of RS-MPC can be reduced by directly enforcing robust recursive feasibility using a robust control invariant set. However, similar to the discussion in Section IV-C.2, if the worst-case robust bounds are too conservative, the design might be infeasible,*

[4] [2, Sec. 3.2–3.3], [31] additionally enforce (17d) for $i = 0$, which, however, is redundant if $x(0)$ fulfills the state constraints.

[5] The candidate-based re-conditioning in [28, Prop. 1] does not seem to suffer from the limitations of IF-SMPC as $p \to 1$.

especially if the desired steady-state is close to a chance constraint. Considering IF-SMPC, the conservatism of the fixed initial state constraint $s_{0|k} = s^\star_{1|k-1}$ can be reduced by using a less restrictive interpolating initial state constraint between $s^\star_{1|k-1}$ and the measured state $x(k)$ [19]–[21] (cf. also [16]–[18] for previous binary initialization strategies). While this provides some reduction in conservatism, such an interpolating initial state constraint is still significantly more restrictive than the standard initial state constraint in a robust tube MPC scheme [33]. Hence, the schemes [19]–[21] are also more conservative than RS-MPC as $p \to 1$.

Remark 3. *(Input constraints) In IF-SMPC, no distinction is made between state and input constraints. Thus, also probabilistic input constraints can be considered. However, for the considered hard input constraints* (3)*, this treatment can be quite conservative (see discussion Sec. IV-C.1 with $p_j = 1$). By imposing the input constraints for $i = 0$ directly based on the measured state $x(k)$ (see* (17f)*), this conservatism can be reduced. More generally, IF-SMPC could be modified to implement any robust constraints ($p_j = 1$) conditioned on $x(k)$ using the formulas from RS-MPC* (17) *with a different nominal trajectory, while the IF-SMPC formulas are only used for chance constraints ($p_j < 1$), which should reduce conservatism.*

V. Unifying framework - multi-step SMPC

Motivated by the limitations exposed in Section IV-C, we provide a unifying SMPC framework, which contains these two SMPC formulations as extreme cases.

A. Conceptual idea

On a high-level, RS-MPC (Sec. IV-A, [2]) robustly enforces the chance constraints (2) conditioned on the state one time step in the past. Then, recursive feasibility is ensured by accounting for the worst-case disturbance for the rest of the horizon with β_i (18). In contrast, IF-SMPC (Sec. IV-B, [22]) enforces the chance constraints (2) conditioned on the initial state $x(0)$. As a result, constraints are enforced on a nominal state s (independent of the measured state), and a pure stochastic constraint tightening γ_i (10) is used.

As a natural unification, the proposed framework conditions the chance constraints (2) on a state (up to) $M \in \mathbb{I}_{\geq 1}$ steps in the past. Correspondingly, the first M steps are treated stochastically and the rest robustly, i.e.,

$$\tilde{\beta}_i = \gamma_i,\ i \in \mathbb{I}_{[1,M]}, \qquad \tilde{\beta}_{i+1} = \tilde{\beta}_i + a_i,\ i \in \mathbb{I}_{\geq M}. \quad (22)$$

Furthermore, we use a nominal state, which is reset every M steps. This naturally provides a unified framework, corresponding to IF-SMPC and RS-MPC in case $M \in \{\infty, 1\}$, respectively, see discussion in Section V-C.

B. Proposed SMPC algorithm

Since the nominal state is reset every M steps, we define two nominal states

$$s(k+1) = \begin{cases} x(k+1) & \mathrm{mod}(k+1, M) = 0 \\ \Phi s(k) + Bc(k) & \text{else} \end{cases} \quad (23a)$$

$$z(k+1) = \begin{cases} \Phi s(k) + Bc(k) & \mathrm{mod}(k+1, M) = 0 \\ \Phi z(k) + Bc(k) & \text{else} \end{cases} \quad (23b)$$

which are initialized with $s(0) = z(0) = x(0)$. In case $\mathrm{mod}(k+1, M) \neq 0$, these states follow the nominal dynamics. For $\mathrm{mod}(k+1, M) = 0$, s is reset to the measured state and z is the nominal prediction based on the measured state M steps in the past (assuming no disturbances w). In general, $s(k)$ is defined based on $x(k - \mathrm{mod}(k, M))$ and $z(k)$ is defined based on $x(k - M_k)$ with

$$M_k := \begin{cases} M + \mathrm{mod}(k, M) & k \geq M \\ k & k < M \end{cases}. \quad (24)$$

The corresponding SMPC formulation is given by

$$\begin{aligned}
\min_{c_{\cdot|k}}\ & \mathcal{J}(x(k), c_{\cdot|k}) && (25a)\\
\text{s.t. } & s_{0|k} = s(k) && (25b)\\
& s_{i+1|k} = \Phi s_{i|k} + Bc_{i|k},\ i \in \mathbb{I}_{[0,N-1]} && (25c)\\
& \tilde{F}s_{i|k} + Gc_{i|k} \leq \mathbf{1} - \tilde{\beta}_{i+\mathrm{mod}(k,M)},\ i \in \mathbb{I}_{[2M-M_k,N-1]} && (25d)\\
& s_{N|k} \in \mathcal{X}_{\mathrm{f}}^{\mathrm{MS}}(\mathrm{mod}(k, M)) && (25e)\\
& z_{0|k} = z(k) && (25f)\\
& z_{i+1|k} = \Phi z_{i|k} + Bc_{i|k},\ i \in \mathbb{I}_{[0,2M-1-M_k-1]} && (25g)\\
& \tilde{F}z_{i|k} + Gc_{i|k} \leq \mathbf{1} - \tilde{\beta}_{i+M_k},\ i \in \mathbb{I}_{[0,\min\{2M-M_k,N\}-1]} && (25h)\\
& z_{N|k} \in \mathcal{Z}_{\mathrm{f}}^{\mathrm{MS}}. && (25i)
\end{aligned}$$

The constraint (25d) enforces the chance constraints at time $i+k$ conditioned on $s(k)$ and hence on $x(k - \mathrm{mod}(k, M))$, with $\tilde{\beta}_{i+\mathrm{mod}(k,M)}$. The constraint (25h) enforces constraints on $i + k$, conditioned on $z(k)$ and hence on $x(k - M_k)$, with the constant $\tilde{\beta}_{i+M_k}$. Assuming $M_k = M < N$, the constraints on z (25h) are imposed on the first M prediction steps and the constraints on s (25d) on the remaining horizon. The corresponding terminal sets capture the constraints for $i \geq N$ with $c_{i|k} = 0$:[6]

$$\mathcal{X}_{\mathrm{f}}^{\mathrm{MS}}(k) = \{s |\ \tilde{F}\Phi^i s \leq \mathbf{1} - \tilde{\beta}_{N+i+k},\ i \in \mathbb{I}_{\geq 0}\} \quad (26a)$$

$$\mathcal{Z}_{\mathrm{f}}^{\mathrm{MS}} = \{z |\ \tilde{F}\Phi^i z \leq \mathbf{1} - \tilde{\beta}_{N+i+2M-1},\ i \in \mathbb{I}_{\geq 0}\} \quad (26b)$$

with $k \in \mathbb{I}_{[0,M-1]}$. Since the proposed SMPC framework reconditions the chance constraints every M steps, it is similar to a multi-step implementation where the first M inputs are directly applied and the optimization problem is only solved every M steps. For this reason, we call this approach *multi-step* SMPC (MS-SMPC).

[6]If $N \geq 2M \geq 2M - M_k$, the terminal set $\mathcal{Z}_{\mathrm{f}}^{\mathrm{MS}}$ in (25i) can be omitted.

Theorem 3. *The SMPC scheme based on Problem* (25) *ensures the desired closed-loop properties in Definition 1.*

The proof mainly combines the tools and ideas of RS-MPC and IF-SMPC. The details can be found online [29, App. A].

C. Discussion

In the following, we show that the proposed formulation indeed recovers RS-MPC (Sec. IV-A) and IF-SMPC (Sec. IV-B) for $M = 1$ and $M = \infty$, respectively.

For $M = \infty$, the initialization (23), (24) ensures $M_k = k$, and the nominal states $s(k) = z(k)$ are equivalent to the nominal state in IF-SMPC. Furthermore, the constraint tightening (22) is equivalent to IF-SMPC with $\tilde{\beta}_i = \gamma_i$. Finally, looking at Problem (25), note that $\tilde{\beta}_{i+M_k} = \tilde{\beta}_{i+\mathrm{mod}(k,M)} = \gamma_{i+k}$, $s_{\cdot|k} = z_{\cdot|k}$, $\mathcal{Z}_\mathrm{f}^\mathrm{MS} \subseteq \mathcal{X}_\mathrm{f}^\mathrm{MS}(k)$. Hence, the tightened constraints and terminal set constraint in Problem (25) are equivalent to Problem (20).

Next, we consider $M = 1$, which yields $\mathrm{mod}(k, M) = 0$ and $M_k = 1$ (for $k > 0$). The nominal state $s(k) = x(k)$ is equivalent to RS-MPC, while $z(k) = s^\star_{1|k-1}$. Furthermore, we trivially have that $\tilde{\beta}_i = \beta_i$ (see (22),(18)). The constraints on $s_{i|k}$ in Problem (25) are posed for $i \geq 2M - M_k = 1$ and are hence equivalent to Problem (17).[7] The only difference is the constraints on z, which are posed for $i \leq \min\{2M - M_k, N\} - 1 = 0$.[8] The corresponding state constraints ($G_{(j)} = 0$) are redundant and can be removed without changing the solution. However, there is a difference in the handling of the input constraints ($G_{(j)} \neq 0$) for $i = 0$. In Problem (17), the exact input constraints are enforced separately utilizing the (known) current state $x(k)$ in (17f). Problem (25) robustly enforces the input constraint (3) at time k conditioned on $x(k-1)$, which is more conservative. As discussed in Remark 3, this issue can be addressed by explicitly separating the hard input constraints from the chance constraints.

Overall, the proposed MS-SMPC encompasses both RS-MPC and IF-SMPC and hence allows to flexibly trade off limitations exposed in Section IV-C with an appropriate choice of $M \in \mathbb{I}_{\geq 1}$.

VI. NUMERICAL COMPARISON

The following numerical comparison demonstrates the complementary advantages and shortcomings of RS-MPC and IF-SMPC discussed in Section IV-C. In addition, we show that the proposed MS-SMPC (Sec. V) can avoid both pitfalls with an appropriate choice of $M \in \mathbb{I}_{\geq 1}$. First, we introduce the simulation setup (Sec. VI-A). Then, we study two edge cases which demonstrate the limitations of RS-MPC (Sec. VI-B) and IF-SMPC (Sec. VI-C), respectively.

The MPC problems were formulated with YALMIP [35] and solved with quadprog (Sec. VI-B) and MOSEK [36] (Sec. VI-C) in Matlab. The code is available online.[9]

[7]For $k = 0$, the constraints (25d) on s at $i = 2$ are equivalent to the corresponding constraint (25h) on z.

[8]Neglecting the terminal set constraint $z_{N|k} \in \mathcal{Z}_\mathrm{f}^\mathrm{MS}$, which is not needed with $N \geq 2M - M_k = 1$.

[9]gitlab.ethz.ch/ics/MS-SMPC

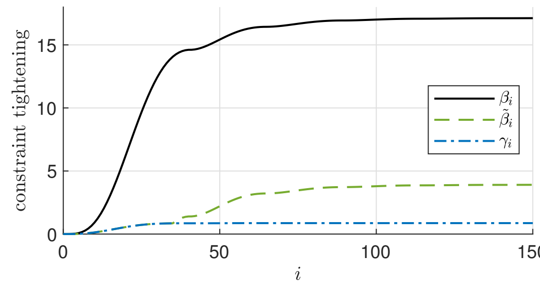

Fig. 1. Comparison of state constraint tightening with tube controller K_LQR.

A. Simulation setup

We consider a 4th-order integrator system from [24] with

$$A = \begin{bmatrix} 1 & T_\mathrm{s} & T_\mathrm{s}^2/2 & T_\mathrm{s}^3/6 \\ 0 & 1 & T_\mathrm{s} & T_\mathrm{s}^2/2 \\ 0 & 0 & 1 & T_\mathrm{s} \\ 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} T_\mathrm{s}^4/24 \\ T_\mathrm{s}^3/6 \\ T_\mathrm{s}^2/2 \\ T_\mathrm{s} \end{bmatrix}, \; D = B$$

and $T_\mathrm{s} = 0.1$. The disturbances $w(k)$ are uniformly distributed over the set $\mathcal{W} = [-4, 4]$ and the input constraint is $u(k) \in [-20, 20]$. This system is interesting as the effect of the disturbance w and the control input u on the first state $x_{(1)}$ is only apparent over longer horizons (see also [24]). Hence, we consider a chance constraint on the first state: $\Pr\left[x_{(1)}(k) \leq 0.1\right] \geq p$, with some probability level p specified later. We use condition (17f) in IF-SMPC and MS-SMPC to implement the current input constraints ($i = 0$) in a non-conservative fashion, similar to RS-MPC (Rk. 3). All simulations are performed for 10^3 realizations over 300 steps, with initial state $x(0) = [0\ 0\ 0\ 0]^\top$. The average cost ℓ_avg (16) and average satisfaction of the chance constraints are approximated using only the interval $k \in \mathbb{I}_{[50,299]}$.

B. Case 1 - performance limitations of RS-SMPC

First, we present an example where IF-SMPC has better performance than RS-MPC. We choose probability level $p = 0.7$, and a stage cost $\ell(x, u) = \|x\|_Q^2 + \|u\|_R^2$ where $Q = \mathrm{diag}\,(Q_{11}, 0, 0, 0)$ and $R = 0.1$, with $Q_{11} = 1.32$. Further, we select prediction horizon $N = 75$ and multi-step horizon $M = 35$.

We consider an optimal LQR state feedback gain K_LQR (tube controller). Figure 1 shows the resulting constraint tightening β_i (RS-MPC), $\tilde{\beta}_i$ (MS-SMPC), and γ_i (IF-SMPC) with K_LQR, indicating large conservatism of the robust-stochastic tightening β_i. Furthermore, we have $\gamma_\mathrm{max} \approx 0.9$, while $\tilde{\beta}_\mathrm{max} \geq \lim_{i\to\infty} \tilde{\beta}_i \approx 4$, and $\beta_\mathrm{max} \geq \lim_{i\to\infty} \beta_i > 15$. Therefore, the tightened constraints are empty, rendering the MS-SMPC and RS-MPC designs infeasible. To allow for application of MS-SMPC and RS-MPC, we choose more aggressive, suboptimal tube controllers K_MS and K_RS with Q_{11} increased to $Q_{\mathrm{MS},11} = 16.6$ and $Q_{\mathrm{RS},11} = 740$, respectively, which results in sufficiently small tightening $\beta_\mathrm{max} \approx 0.9$ and $\tilde{\beta}_\mathrm{max} \approx 0.9$.

Results: The state trajectories and corresponding constraint violations are presented in Figure 2. Table I details

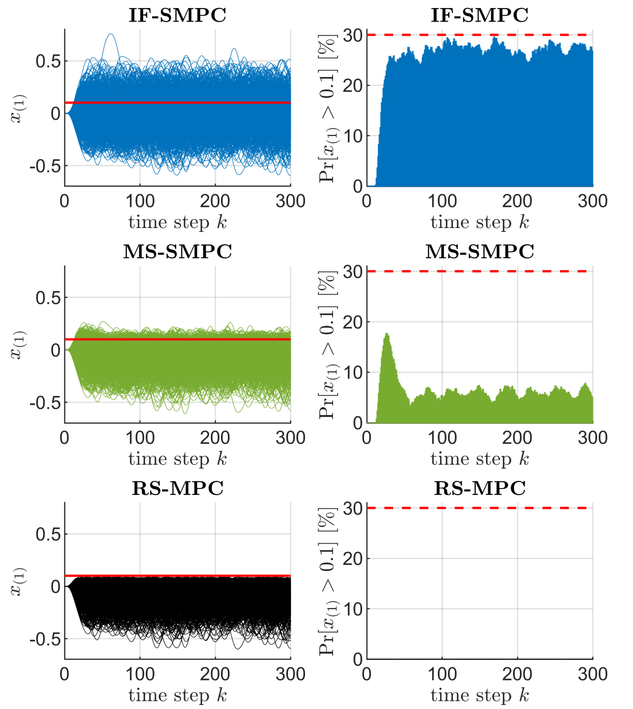

Fig. 2. Case 1: State trajectories (left) and probability of constraint violation (right) of IF-SMPC (blue), MS-SMPC (green, $M = 35$), and RS-MPC (black). The solid red lines depict the state constraints; the dashed red lines indicate the violation level $1 - p$.

performance (normalized w.r.t. LQR performance) and constraint violation probability of all SMPC schemes and their respective tube controllers. Figure 3 shows the performance of MS-SMPC for different multi-step horizons M.[10]

	ℓ_{avg}	Violation probability
K_{LQR}	1.00	27.08%
K_{MS}	1.26	4.07%
K_{RS}	2.29	0.00%
IF-SMPC with K_{LQR}	1.00	27.08%
MS-SMPC with K_{MS}	1.09	5.75%
RS-MPC with K_{RS}	1.20	0.00%

TABLE I

CASE 1: COMPARISON - PERFORMANCE AND VIOLATION PROBABILITY.

We observe a significant performance improvement of IF-SMPC over RS-MPC. Due to $\tilde{F}_{(j)} D \approx 0$ for the state constraint $j = 1$, the constraint on $x_{(1)}$ is treated virtually robustly in the RS-MPC case (Sec. IV-C.2), resulting in practically no constraint violations and higher cost, while the probability of constraint violation for IF-SMPC is close to the specified $1 - p = 30\%$. Further, IF-SMPC matches performance and constraint violations of the LQR. For $M = 35$, MS-SMPC achieves performance and constraint violations in between RS-MPC and IF-SMPC. As expected,

[10]For each M, we compute the least conservative feedback K ensuring $\tilde{\beta}_{\max} \approx 0.9$.

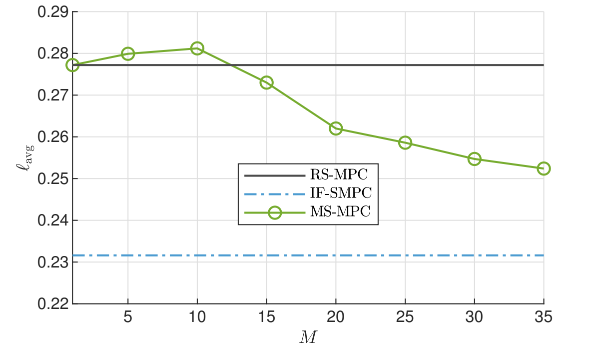

Fig. 3. Case 1: Performance of MS-SMPC for different multi-step horizons M in comparison to RS-MPC and IF-SMPC.

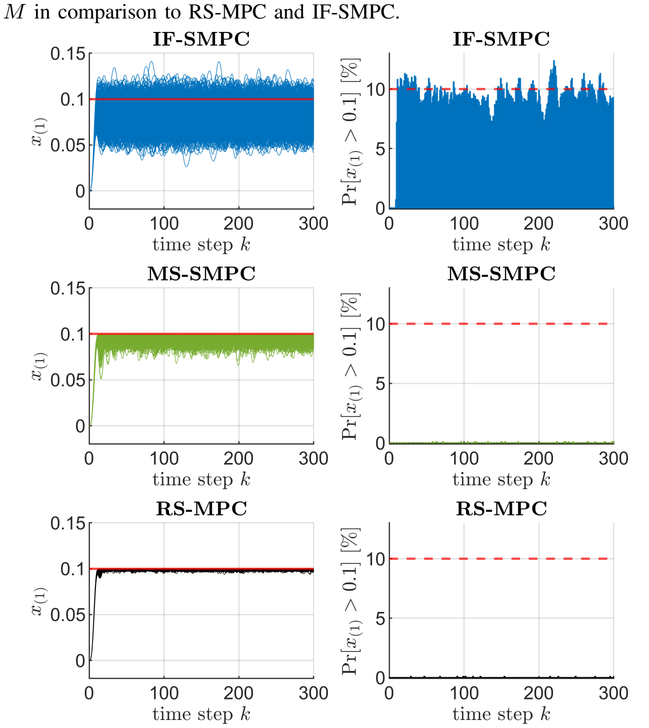

Fig. 4. Case 2: State trajectories (left) and probability of constraint violation (right) of IF-SMPC (blue), MS-SMPC (green, $M = 2$), and RS-MPC (black). The solid red lines depict the state constraints; the dashed red lines indicate the violation level $1 - p$.

Figure 3 indicates that for $M = 1$, MS-SMPC matches the performance of RS-MPC while for $M \to \infty$, we observe convergence to the performance of IF-SMPC (Sec. V-C).

C. Case 2 - performance limitations of IF-SMPC

In the following, we present an example where RS-MPC has better performance than IF-SMPC. To this end, we minimize an economic cost objective with stage cost $\ell(x) = -x_{(1)} + 0.1$. In consequence, $x_{(1)}$ is maximized and the state constraint is always active. We choose probability level $p = 0.9$, tube control gain $K = [-55.45, \ -51.22, \ -23.65, \ -6.54]$, prediction horizon $N = 30$, and multi-step horizon $M = 2$.

Results: State trajectories and constraint violations are shown in Figure 4. Performance and constraint violation probability are compared in Table II. Similarly to Case 1, RS-MPC results in virtually no constraint violations due to the (approximately) robust disturbance treatment by RS-MPC in this example (Secs. IV-C.2, VI-B). Nonetheless, RS-MPC has significantly better performance than IF-SMPC, as the resetting of the nominal state in RS-MPC reduces conservatism considerably. Again, MS-SMPC can achieve performance in between RS-MPC and IF-SMPC.

	$\ell_{\text{avg}} \cdot 10^2$	Violation probability
IF-SMPC	1.71	9.485%
MS-SMPC	0.41	0.008%
RS-MPC	0.06	0.005%

TABLE II

CASE 2: COMPARISON - PERFORMANCE AND VIOLATION PROBABILITY.

In summary, we have seen that there can be significant performance differences between RS-MPC and IF-SMPC and either scheme can be superior, depending on the specific problem. Furthermore, we have seen that MS-SMPC unifies RS-MPC and IF-SMPC and can alleviate the respective limitations with a suitable choice of M.

VII. CONCLUSION

We investigated SMPC schemes with desired closed-loop guarantees. We categorized the corresponding literature in two separate frameworks and we provided a qualitative analysis, highlighting some of the intrinsic features and limitations of these two approaches. We also provided a numerical comparison that support these findings. As a separate contribution, we derived a novel SMPC framework that naturally unifies these two approaches. The proposed unifying SMPC framework forms a natural basis for future investigations.

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