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Jointly Observable State Estimation for Linear Systems over Periodic Time-varying Networks

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Abstract— **This paper deals with the distributed state estimation problem for jointly observable multi-agent systems operated over network with time-varying topologies. The approach is shown to extend the design of distributed observers to unstable observable systems on communication networks with intermittent connection losses. This includes the connected static networks as well as every-time connected switching networks as special cases. Sufficient conditions for the existence of distributed observers for general linear observable systems over periodic communication networks are presented. A toy numerical example and a practical application are provided to illustrate the effectiveness of the theoretical results.**

I. Introduction

The objective of the distributed state estimation problem for multi-agent systems is the design of distributed observers for each agent (follower system, distributed sensor) that can reconstruct the state of an observable system (leader system, large-scale dynamical systems) using only local measurements and additional information obtained from its closest neighbours [1], [2], [3]. Existing approaches that have been proposed to address the distributed state estimation problem are classified into four categories that reflect the observability assumptions of the system to be observed and the communication network. They include: 1) disjointly observable systems[4], 2) semi-jointly observable systems[1],[5], 3) locally jointly observable systems [3], and, 4) jointly observable systems [2],[6], [7]. Among these assumptions, the jointly observable assumption is the mildest possible restriction on the system as it allows the reconstruction of the system's state using local interactions among agents, where each agent in the network may fail to be observable.

The distributed state estimation problem for jointly observable multi-agent systems has been solved under various communication conditions where both the linear system case [2], [6], [7], [8] and the nonlinear system case [9] have been considered. Specifically, a simple Luenberger-type local observer was proposed in [7] for a static communication network where a Kalman observable canonical decomposition was used to address different architectures of the distributed observer as in [6], [10], [11] Static communication networks focus on idealized conditions for distributed control and estimation. In practice, communication networks can change dynamically due to various factors such as time-varying environment, communication distance, failures of links and network congestion. Some efforts have been made to investigate the distributed state estimation problem for continuous-time systems over time-varying graphs [12], [13], [14] and [15]. Particularly, [12] addressed the problem of every-time strongly connected switching networks, while [13], [14] investigated the case of jointly connected switching networks. This is the weakest possible assumption on switching networks. It can accommodate networks that can be intermittently disconnected. Nevertheless, the problems considered [13], [14] and [15] require either the system matrix of the system to be neutrally stable, i.e all the eigenvalues of the system matrix are semi-simple with zero real parts, or all the observe pairs consist of the system matrix and the output matrix to be marginally stabilizable, respectively.

In this study, we propose the design of distributed observer for a general linear time-invariant system to estimate the state of the system to be observed subject to jointly observable systems operated over time-varying graphs. In the proposed approach, all the agents in the networks cooperate to reconstruct the state of the system and share the estimated state with their neighbours through local communication. The results allow the observed system to be unstable and operated over a communication network with intermittent connections that can be disconnected and reconnected at every sampling time instant. In this study, sufficient conditions are presented for the existence of distributed observers for general linear systems over periodic communication networks.

The rest of this paper is organized as follows. In Section II, we introduce some standard assumptions and lemmas. Section III is devoted to the design and analysis of the distributed observer. This is followed by two simulation examples in Section IV and brief conclusions in Section V.

Notation: Let $\|\cdot\|$ denote both the Euclidean norm of a vector and the Euclidean-induced matrix norm (spectral norm) of a matrix. $\mathbb{R}$ is the set of real numbers. $\mathbb{N}$ denotes all natural numbers. I_n is the $n\times n$ identity matrix. For $A\in\mathbb{R}^{m\times n}$, $\text{Ker}(A)=\{x\in\mathbb{R}^n|Ax=0\}$ and $\text{Im}(A)=\{y\in\mathbb{R}^m|y=Ax \text{ for some } x\in\mathbb{R}^n\}$ denote the kernel and range of A, respectively. For a subspace $\mathcal{V}\subset\mathbb{R}^n$, the orthogonal complement of $\mathcal{V}$ is denoted by $\mathcal{V}^\perp=\{x\in\mathbb{R}^n|x^Tv=0,\forall v\in\mathcal{V}\}$. The operation $\otimes$ denotes the Kronecker product of matrices. For $b_i\in\mathbb{R}^{n_i\times p}$, $i=1,\dots,m$, $\text{col}(b_1,\dots,b_m)\triangleq\begin{bmatrix}b_1^T & \cdots & b_m^T\end{bmatrix}^T$. For $a_i\in\mathbb{R}^{p\times n_i}$, $i=1,\dots,m$, $\text{row}(a_1,\dots,a_m)\triangleq\begin{bmatrix}a_1 & \cdots & a_m\end{bmatrix}$. For $X_1\in\mathbb{R}^{n_1\times m_1},\dots,X_k\in\mathbb{R}^{n_k\times m_k}$,

$$\text{diag}(X_1,\dots,X_k)\triangleq\begin{bmatrix}X_1 & & \\ & \ddots & \\ & & X_k\end{bmatrix}.$$

For any $P\in\mathbb{R}^{n\times n}$, let λ_p and λ_P denote the minimum and maximum eigenvalues of P, respectively.

II. Problem Formulation and Assumptions

We consider a linear time-invariant system of the form:

$$\dot{x}(t) = Ax(t), \tag{1a}$$

$$y_i(t) = C_i x(t), \qquad i = 1, \cdots, N \tag{1b}$$

where $x(t) \in \mathbb{R}^n$ is the vector of state variables to be estimated, $y_i \in \mathbb{R}^{p_i}$ denotes the output of system (1) detected by agent (sensor or follower) i, and, A and C_i are known matrices of suitable dimension, for $i = 1, \cdots, N$.

As in [13], [14], the multi-agent system is composed of N agents. The network topology of the multi-agent system is described by a switching digraph $\mathcal{G}_{\sigma(t)} = \left(\bar{\mathcal{V}}, \mathcal{E}_{\sigma(t)}\right)$ where $\sigma(t)$ is a switching signal having a dwelling time $\tau > 0$, $\mathcal{V} = \{1, \cdots, N\}$ and $(i,j) \in \mathcal{E}_{\sigma(t)}$ if and only if $a_{ji}(t) > 0$ at time instant t. We use $\bar{\mathcal{N}}_i(t)$ to denote the neighbour set of note i at time instant t. More details of the graph theory can be found in [16].

The objective of this paper is to design distributed observers over time-varying networks that estimate the state variables of a jointly observable system (1) where the estimated state $\hat{x}_i(t)$ of each ith local agent's observer converges to the state $x(t)$, i.e.,

$$\lim_{t\to\infty} (\hat{x}_i(t) - x(t)) = 0, \qquad i = 1, \cdots, N.$$

Before proceeding with the main results of this study, we state the following assumptions:

Assumption 1: There exists a subsequence $\{i_k\}$ of $\{k : k = 0, 1, \cdots\}$ with $t_{i_{k+1}} - t_{i_k} < \upsilon$ for some positive υ such that the union graph $\bigcup_{j=i_k}^{i_{k+1}-1} \mathcal{G}_{\sigma(t_j)}$ is a strongly connected graph.

Assumption 2: The switching signal $\sigma(t)$ is periodic.

Remark 1: Under Assumption 2, we can assume that the switching signal is as follows:

$$\sigma(t) = \begin{cases} 1, & \text{If } sT \le t < (s+\omega_1)T; \\ 2, & \text{If } (s+\omega_1)T \le t < (s + \sum_{\rho=1}^{2} \omega_\rho)T; \\ \vdots & \qquad\qquad \vdots \\ p, & \text{If } (s + \sum_{\rho=1}^{p-1} \omega_\rho)T \le t < (s+1)T; \end{cases} \tag{2}$$

where T is a positive constant, $s = 0, 1, 2, \cdots$, and ω_ρ, $\rho = 1, \cdots, p$ are positive constants satisfying $\sum_{\rho=1}^{p} \omega_p = 1$. Under Assumptions 1 and 2, for any $k = 0, 1, 2, \cdots$, the union graph $\bigcup_{j=i_k}^{i_{k+1}-1} \mathcal{G}_{\sigma(t_j)} = \bigcup_{\rho=1}^{p} \mathcal{G}_\rho \equiv \mathcal{G}$. We let $\mathcal{L}$ denote the Laplacian matrix of graph $\mathcal{G}$.

Assumption 3: The system defined in (1) is jointly observable.

Remark 2: For $i \in \mathcal{V}$, we assume that the observability index of (A, C_i) is v_i, such that rank$(\mathcal{O}_i) = v_i$, where $\mathcal{O}_i \in \mathbb{R}^{np_i \times n}$ is the observability matrix and defined as following $\mathcal{O}_i = \text{col}\left(C_i, C_iA, \cdots, C_iA^{n-1}\right)$. For $i \in \mathcal{V}$, the observable subspace and unobservable subspace of (A, C_i) are defined as Im$(\mathcal{O}_i^T) \subset \mathbb{R}^n$ and Ker$(\mathcal{O}_i) \subset \mathbb{R}^n$, respectively, and satisfy Ker$(\mathcal{O}_i)^\perp = $ Im$(\mathcal{O}_i^T)$.

For $i \in \mathcal{V}$, let $V_i = \text{row}(V_{ui}, V_{oi}) \in \mathbb{R}^{n\times n}$ be an orthogonal matrix such that $V_iV_i^T = I_n$. The matrix $V_{ui} \in \mathbb{R}^{n\times(n-v_i)}$ is such that all columns of V_{ui} are from an orthogonal basis of the Ker$(\mathcal{O}_i)$ satisfying Im$(V_{ui}) = $ Ker$(\mathcal{O}_i)$. In addition, the matrix $V_{oi} \in \mathbb{R}^{n\times v_i}$ be a matrix such that all columns of V_{oi} are from an orthogonal basis of the Im$(\mathcal{O}_i^T)$ satisfying Im$(V_{oi}) = $ Im$(\mathcal{O}_i^T)$.

For $i \in \mathcal{V}$, the matrices A and C_i of the system in (1) yield the Kalman observability decomposition defined as follows:

$$V_i^T A V_i = \begin{bmatrix} A_{ui} & A_{ri} \\ 0_{v_i\times(n-v_i)} & A_{oi} \end{bmatrix}, \tag{3a}$$

$$C_iV_i = \begin{bmatrix} 0_{p_i\times(n-v_i)} & C_{oi} \end{bmatrix}, \tag{3b}$$

where the pair (A_{oi}, C_{oi}) is observable, $A_{oi} \in \mathbb{R}^{v_i\times v_i}$, $A_{ri} \in \mathbb{R}^{(n-v_i)\times v_i}$, $A_{ui} \in \mathbb{R}^{(n-v_i)\times(n-v_i)}$ and $C_{oi} \in \mathbb{R}^{p_i\times v_i}$ admit the matrix decompositions: $A_{ui} = V_{ui}^T A V_{ui}$, $A_{ri} = V_{ui}^T A V_{oi}$, $A_{oi} = V_{oi}^T A V_{oi}$ and $C_{oi} = C_i V_{oi}$.

Remark 3: Let $C_o = \text{diag}(C_{o1}, \cdots, C_{oN})$, $A_r = \text{diag}(A_{r1}, \cdots, A_{rN})$, $V_o = \text{diag}(V_{o1}, \cdots, V_{oN})$, $V_u = \text{diag}(V_{u1}, \cdots, V_{uN})$, $A_o = \text{diag}(A_{o1}, \cdots, A_{oN})$ and $A_u = \text{diag}(A_{u1}, \cdots, A_{uN})$.

We now review some lemmas proposed in [17], [7], [16].

Lemma 1: [17] Consider the linear switched system:

$$\dot{y} = A_{\sigma(t)} y \tag{4}$$

where $y \in \mathbb{R}^n$ is the state, $\sigma : [0, \infty) \mapsto \mathcal{P} = \{1, 2, \cdots, p\}$ is the switching signal satisfying Assumption 2, and $A_\rho \in \mathbb{R}^{n\times n}$, $\rho = 1\cdots, p$. If the matrix $\sum_{\rho=1}^{p} \omega_p A_p$ is Hurwitz with ω_ρ, $\rho = 1, \cdots, p$, obtained from (2), then there exists a positive constant $\bar{T}_0$ such that for $0 < T < \bar{T}_0$, the origin of system (4) is exponentially stable.

Lemma 2: [16] Suppose that the communication network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is strongly connected. Let $\theta = \text{col}(\theta_1, \cdots, \theta_N)$ be the left eigenvector of the Laplacian matrix $\mathcal{L}$ associated with the eigenvalue 0, i.e., $\mathcal{L}^T\theta = 0$. Then, $\Theta = \text{diag}(\theta_1, \cdots, \theta_N) > 0$ and $\hat{\mathcal{L}} = \Theta\mathcal{L} + \mathcal{L}^T\Theta \ge 0$.

Lemma 3: [7] Suppose that the communication network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is strongly connected. Then, the following statements are equivalent:

1) The system (1) is jointly observable;
2) The matrix $V_u^T\big(\hat{\mathcal{L}} \otimes I_n\big)V_u$ is positive definite;
3) The matrix $V_u^T\left(\mathcal{L} \otimes I_n\right)V_u$ is nonsingular.

Under Assumption 1 and 3, the matrix $V_u^T(\hat{\mathcal{L}} \otimes I_n)V_u$ is positive definite matrix from Lemma 3. Let λ_l and λ_L denote the minimum and maximum eigenvalues of $V_u^T(\hat{\mathcal{L}} \otimes I_n)V_u$, respectively. Let $V_u = \text{diag}(V_{u1}, \cdots, V_{uN})$ and $A_u = \text{diag}(A_{u1}, \cdots, A_{uN})$. Then, we have the following lemma.

Lemma 4: Under Assumption 3, consider the following linear time-invariant system

$$\dot{z} = \big[A_u - \gamma V_u^T\left(\mathcal{L} \otimes I_n\right)V_u\big]z, \tag{5}$$

where $z \in \mathbb{R}^{\sum_{i=1}^{N}(n-\nu_i)}$. Suppose that the communication network $\mathcal{G}$ is strongly connected. Then, the system (5) satisfies the following properties:

1) System (5) is globally asymptotically stable for sufficiently large enough γ.

2) The matrix $A_u - \gamma V_u^T (\mathcal{L} \otimes I_n) V_u$ is Hurwitz for sufficiently large enough γ.

Proof: Define the following Lyapunov function candidate for system (5)

$$V(z) = \sum_{i=1}^{N} \theta_i z_i^T z_i. \tag{6}$$

Then,

$$\theta_m \|z\|^2 \leq V(z) \leq \theta_M \|z\|^2, \tag{7}$$

where $\theta_m = \min\{\theta_1, \cdots, \theta_N\}$ and $\theta_M = \max\{\theta_1, \cdots, \theta_N\}$. The time derivative of $V(z)$ along (5) can be evaluated as

$$\begin{aligned} \dot{V} =& 2\sum_{i=1}^{N} \theta_i z_i^T A_{ui} z_i - \gamma z^T \big[V_u^T (\hat{\mathcal{L}} \otimes I_n) V_u\big] z \\ \leq& 2\theta_M \|A\| \|z\|^2 - \gamma \lambda_l \|z\|^2. \end{aligned}$$

Besides, it is easily to verified that $\|A_{ui}\| \leq \|A\|$, for $i = 1, \cdots, N$. From (5) and (7), we have

$$\dot{V} \leq \frac{2\theta_M^2 \|A\| - \gamma \lambda_l \theta_m}{\theta_M \theta_m} V.$$

Hence, for any $\gamma > \frac{2\theta_M^2 \|A\|}{\lambda_l \theta_m}$, system (5) is globally asymptotically stable (exponentially stable). From the Theorem 4.5 in [18], the matrix $A_u - \gamma V_u^T (\mathcal{L} \otimes I_n) V_u$ is Hurwitz for sufficiently large enough γ. ■

Lemma 5: Under Assumptions 1, 2 and 3, consider the following linear system

$$\dot{z} = \big[A_u - \gamma V_u^T \left(\mathcal{L}_{\sigma(t)} \otimes I_n\right) V_u\big] z, \tag{8}$$

where $z = \text{col}(z_1, \cdots, z_N)$ with $z_i \in \mathbb{R}^{n-\nu_i}$, there exist some constants $\bar{\gamma}_0$ and $\bar{T}_0$, such that, for all $\gamma \geq \bar{\gamma}_0$ and $0 < T < \bar{T}_0$, the system (8) is exponentially stable at the origin,

Proof: Under Assumptions 1 and 2, for any $k = 0, 1, 2, \cdots$, the union graph $\bigcup_{j=i_k}^{i_{k+1}-1} \mathcal{G}_{\sigma(t_j)} = \bigcup_{\rho=1}^{p} \mathcal{G}_\rho = \mathcal{G}$ is a strongly connected directed graph. Clearly, $\mathcal{L} = \sum_{\rho=1}^{p} \omega_\rho \mathcal{L}_\rho$. Then, for $\sum_{\rho=1}^{p} \omega_\rho = 1$, we have

$$\sum_{\rho=1}^{p} \omega_\rho \big[A_u - \gamma V_u^T (\mathcal{L}_\rho \otimes I_n) V_u\big] = A_u - \gamma V_u^T (\mathcal{L} \otimes I_n) V_u.$$

Thus, from Lemma 4, there exists $\bar{\gamma}_0$ such that $A_u - \gamma V_u^T (\mathcal{L} \otimes I_n) V_u$ is is Hurwitz for any $\gamma \geq \bar{\gamma}_0$. Then, from Lemma 1, there exists a positive constant $\bar{T}_0$ such that for any $0 < T < \bar{T}_0$, system (8) is exponentially stable at the origin. ■

III. Main Results

A. Distributed Observer over Time-Varying Graphs

We now introduce the following linear dynamic observer as follows:

$$\dot{\hat{x}}_i = A\hat{x}_i + L_i(C_i\hat{x}_i - y_i) + \gamma M_i \sum_{j \in \mathcal{N}_i(t)} (\hat{x}_j - \hat{x}_i), \tag{9}$$

where, for $i = 1 \cdots, N$, $\hat{x}_i \in \mathbb{R}^n$ is the estimation of x,

$$L_i = V_i \begin{bmatrix} 0 \\ L_{oi} \end{bmatrix}, \quad M_i = V_i \begin{bmatrix} I_{n-v_i} & 0 \\ 0 & 0 \end{bmatrix} V_i^T,$$

γ is sufficiently large enough positive number to be determined and $L_{oi} \in \mathbb{R}^{v_i \times p_i}$ is chosen such that $(A_{oi} + L_{oi} C_{oi})$ is Hurwitz.

For $i = 1, \cdots, N$, let $\tilde{x}_i = \hat{x}_i - x$ be the estimation error of the ith observer. Then, we have

$$\begin{aligned} \dot{\tilde{x}}_i =& A\tilde{x}_i + L_i C_i \tilde{x}_i + \gamma M_i \sum_{j \in \mathcal{N}_i(t)} (\tilde{x}_j - \tilde{x}_i) \\ =& (A + L_i C_i)\tilde{x}_i - \gamma M_i \sum_{j=1}^{N} l_{ij}(t) \tilde{x}_j, \end{aligned} \tag{10}$$

where $l_{ij}(t)$ is the (i,j)-th entry of the Laplacian matrix $\mathcal{L}_{\sigma(t)}$ at time moment t. Let $\tilde{x}_{oi} = V_{oi}^T \tilde{x}_i$ and $\tilde{x}_{ui} = V_{ui}^T \tilde{x}_i$, for $i = 1, \cdots, N$. Then, we have the following system from (3) and (10),

$$\begin{aligned} \dot{\tilde{x}}_{ui} =& A_{ui}\tilde{x}_{ui} + A_{ri}\tilde{x}_{oi} \\ & - \gamma V_{ui}^T \sum_{j=1}^{N} l_{ij}(t) \big[V_{uj}\tilde{x}_{uj} + V_{oj}\tilde{x}_{oj}\big], \end{aligned} \tag{11a}$$

$$\dot{\tilde{x}}_{oi} = (A_{oi} + L_{oi} C_{oi}) \tilde{x}_{oi}. \tag{11b}$$

Let $\tilde{x}_u = \text{col}(\tilde{x}_{u1}, \cdots, \tilde{x}_{uN})$, $\tilde{x}_o = \text{col}(\tilde{x}_{o1}, \cdots, \tilde{x}_{oN})$, $V_o = \text{diag}(V_{o1}, \cdots, V_{oN})$, and $A_r = \text{diag}(A_{r1}, \cdots, A_{rN})$. Then, the system (11a) can be put into the following compact form,

$$\begin{aligned} \dot{\tilde{x}}_u =& A_u \tilde{x}_u + A_r \tilde{x}_o - \gamma V_u^T \left(\mathcal{L}_{\sigma(t)} \otimes I_n\right) \left[V_u \tilde{x}_u + V_o \tilde{x}_o\right] \\ =& M(t)\tilde{x}_u + N(t)\tilde{x}_o \end{aligned} \tag{12}$$

where $M(t) = A_u - \gamma V_u^T \left(\mathcal{L}_{\sigma(t)} \otimes I_n\right) V_u$ and $N(t) = A_r - \gamma V_u^T \left(\mathcal{L}_{\sigma(t)} \otimes I_n\right) V_o$.

Then, we have the following results.

Lemma 6: Under Assumption 3, the system (1) and the linear switched system (9) are such that:

$$\lim_{t \to \infty} (\hat{x}_i(t) - x(t)) = 0, \; i = 1, \cdots, N,$$

for any $x(0) \in \mathbb{R}^n$ and $\hat{x}_i(0) \in \mathbb{R}^n$, provided that the following system

$$\dot{\tilde{x}}_u = M(t)\tilde{x}_u, \tag{13}$$

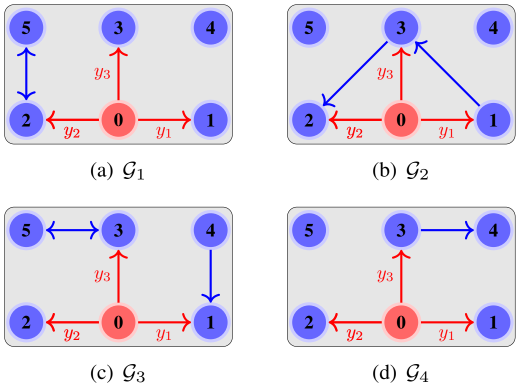
is exponentially stable.

Theorem 1: Consider system (1) and the linear switched system (9). Under Assumptions 1, 2 and 3, there exists positive $\bar{\gamma}_0 > 0$ and $\bar{T}_0$ such that for any $\gamma \geq \bar{\gamma}_0$ and $0 < T < \bar{T}_0$, $\lim_{t \to \infty}(\hat{x}_i(t) - x(t)) = 0$ exponentially for any $x(0) \in \mathbb{R}^n$ and $\hat{x}_i(0) \in \mathbb{R}^n$, $i = 1, \cdots, N$.

Proof: In order to analyze the system (12), we first assume that $\tilde{x}_o(t) = 0$ for all $t \geq 0$, the system (12) will reduce to the system

$$\dot{\tilde{x}}_u = \left(A_u - \gamma V_u^T \left(\mathcal{L}_{\sigma(t)} \otimes I_n\right) V_u\right) \tilde{x}_u \tag{14}$$

System (14) is in the form of (8). Under Assumptions 1, 2 and 3, from Lemma 5, there exists positive $\bar{\gamma}_0 > 0$ and $\bar{T}_0$ such that for any $\gamma \geq \bar{\gamma}_0$ and $0 < T < \bar{T}_0$, (14) is exponentially stable at origin. Therefore, by using Lemma 6, we have $\lim_{t \to \infty}(\hat{x}_i(t) - x(t)) = 0$, exponentially for any $x(0) \in \mathbb{R}^n$ and $\hat{x}_i(0) \in \mathbb{R}^n$, $i = 1, \cdots, N$. ■



(a) $\mathcal{G}_1$ (b) $\mathcal{G}_2$

(c) $\mathcal{G}_3$ (d) $\mathcal{G}_4$

Fig. 1. Communication topology $\mathcal{G}_{\sigma(t)}$

IV. Numerical Example

A. Example 1: A toy example

In this example, we consider linear distributed systems composed of one leader system and five followers as shown in Figure 1. The dynamic of the leader is in the form (1) with

$$A = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},\ B = 0,\ C = \left[\begin{array}{ccc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \begin{array}{c} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{array}.$$

Next, we assume that the switching network topology $\bar{\mathcal{G}}_{\sigma(t)}$ is dictated by the following switching signal:

$$\sigma(t) = \begin{cases} 1 & \text{If } sT \le t < \left(s + \frac{1}{4}\right) T \\ 2 & \text{If } \left(s + \frac{1}{4}\right) T \le t < \left(s + \frac{1}{2}\right) T \\ 3 & \text{If } \left(s + \frac{1}{2}\right) T \le t < \left(s + \frac{3}{4}\right) T \\ 4 & \text{If } \left(s + \frac{3}{4}\right) T \le t < (s + 1) T \end{cases}$$

where $s = 0, 1, 2, \cdots$. The four digraphs $\bar{\mathcal{G}}_k$, $k = 1,\ 2,\ 3,\ 4$, are illustrated in Figure 1, where the node 0 is associated with leader (or the exosystem), and the other nodes are associated with the five followers. One can remark that none of the pairs (A, C_i) are observable, while (A, C) is observable. Based on the decomposition (3), we choose the following matrices:

$$V_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix},\ V_2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix},$$
$$V_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ V_4 = I_3,\ V_5 = I_3.$$

The partition y_1, y_2 and y_3 of the output can be measured by followers 1, 2 and 3 as shown in Figure 1. This topology satisfies Assumption 1 with

$$\mathcal{L} = \sum_{i=1}^{4} \mathcal{L}_\rho = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 & -1 \\ -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix}.$$

Let $\theta = 10^{-3} \times \mathrm{col}(1, 1, 1, 1, 2)$. Hence, $\lambda_L = 1.05 \times 10^{-2}$ and $\lambda_l = 1.3 \times 10^{-3}$. Thus, we can design a control law composed of the equation (9) with the following parameters: $\gamma = 45$, $L_{o1} = \mathrm{col}(-4 - 2)$, $L_{o2} = \mathrm{col}(-4 - 2)$, $L_{o3} = -2.5$, $L_4 = \mathrm{col}(0, 0, 0)$, $L_5 = \mathrm{col}(0, 0, 0)$, $M_4 = I_3$ and $M_5 = I_3$. The simulation is conducted with the following initial conditions: $x(0) = \mathrm{col}(1, 2, 3)$ and $\hat{x}_i(0) = \mathrm{col}(0, 0, 0)$.

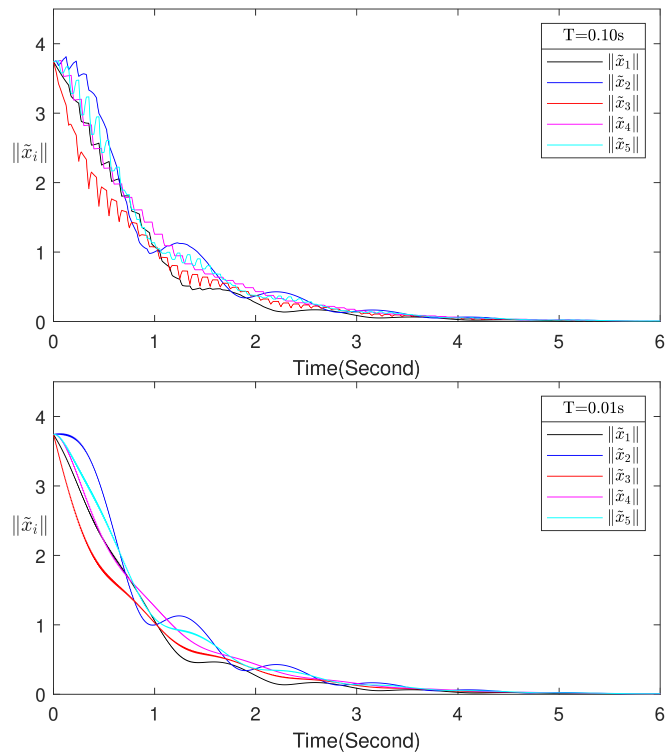


Fig. 2. Estimation errors of all nodes with different T, $i = 1, \ldots, 5$.

Figure 2 shows the estimation errors of all nodes for different values of T. The results demonstrate that the estimation error approaches the estimation error of the averaged estimate as the frequency of switching is increased. Figure 3 shows the estimation performance of each node's sub-state for $T = 0.1$.

B. Example 1: Applications to Quarter-car active automotive suspension system

In this next example, we consider a modified example from [19] to illustrate the proposed design of the local observers for distributed estimation over time-varying networks. Consider the quarter-car active automotive suspension system described in Figure 4. The schematic describes the automotive system for each individual wheel. Each wheel

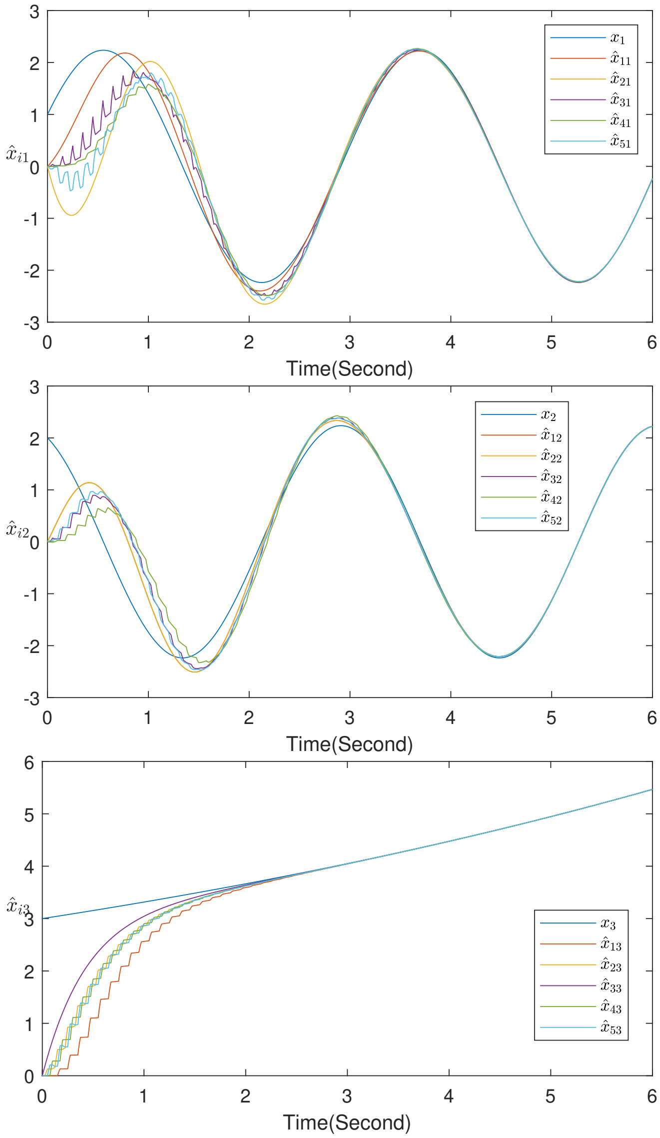


Fig. 3. Estimation performance of all nodes with $T = 0.1$, $i = 1, \ldots, 5$.

assembly is assumed to be monitored by two separate sensors. The system consists of a spring k_s, a damper b_s and an active force actuator F_a. The sprung mass m_s represents the quarter-car equivalent of the vehicle body mass. The unsprung mass m_u represents the equivalent mass due to the axle and tire. The spring k_t represents the vertical stiffness of the tire. The variables z_s, z_u and z_r are the vertical displacements from static equilibrium of the sprung mass, unsprung mass and the road, respectively.

The dynamics of this system are governed by the following system

$$\dot{x}_c = A_c x_c + B_u F_a + B_d \dot{z}_r \tag{15}$$

where F_a is the active force of the actuator, $\dot{z}_r$ is an input describing how the road profile enters into the system, the state vector $x_c = \mathrm{col}(x_{c1}, x_{c2}, x_{c3}, x_{c4})$ include the suspension deflection $x_{c1} = z_s - z_u$, the absolute velocity $x_{c2} = \dot{z}_s$ of the sprung mass m_s, the tire deflection $x_{c3} = z_u - z_r$ and

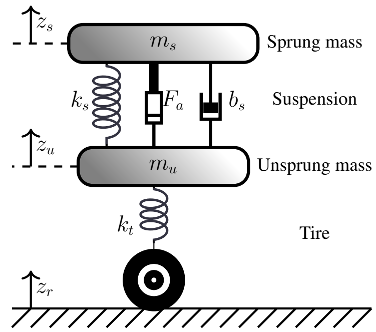


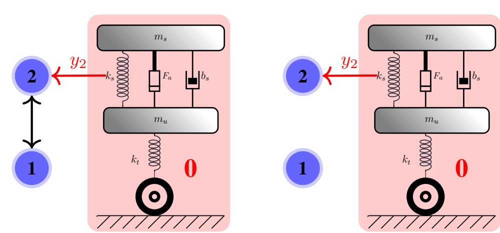
Fig. 4. Quarter-car active automotive suspension system

the absolute velocity $x_{c4} = \dot{z}_u$ of the unsprung mass m_u. The matrices A_c, B_u and B_d are given by:

$$A_c = \begin{bmatrix} 0 & 1 & 0 & -1 \\ \frac{-k_s}{m_s} & \frac{-b_s}{m_s} & 0 & \frac{b_s}{m_s} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{b_s}{m_u} & \frac{-k_t}{m_u} & \frac{b_s+b_t}{-m_u} \end{bmatrix},$$

$$B_u = \mathrm{col}\big(0, \frac{1}{m_s}, 0, \frac{-1}{m_s}\big), B_d = \mathrm{col}\big(0, 0, -1, \frac{b_t}{m_u}\big).$$

where $m_s = 240Kg$, $m_u = 36Kg$, $b_s = 980N.sec/m$, $k_s = 1.6 \times 10^4 N/m$, $k_t = 1.6 \times 10^5 N/m$ and $b_t = 0$.



(a) $\mathcal{G}_1$ (b) $\mathcal{G}_2$

Fig. 5. Communication topology and measurements $\mathcal{G}_{\sigma(t)}$

The time-varying networks are shown in Figure. 5. The network structure is driven by the switching signal:

$$\sigma(t) = \begin{cases} 1 & \text{If } sT \le t < \left(s + \frac{1}{2}\right)T \\ 2 & \text{If } \left(s + \frac{1}{2}\right)T \le t < (s+1)T \end{cases}$$

where $s = 0, 1, 2, \cdots$. We assume that the road profile can be viewed as a disturbance described by the following equation

$$\dot{v} = Sv, \tag{16a}$$

$$\dot{z}_r = Fv, \tag{16b}$$

where $v \in \mathbb{R}^4$ is the state of the exosystem., $F = \left[\begin{smallmatrix} 2 & -1 & -1 & 2 \end{smallmatrix}\right]$ and $S = \mathrm{diag}(2a, a)$ are the output matrix and system matrix of the exosystem with $a = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. We define $x = \mathrm{col}(x_c, v)$ and rewrite (15) and (16) into the following compact form:

$$\dot{x} = Ax + BF_a \tag{17}$$

where $A = \left[\begin{smallmatrix} A_c & F \\ 0_{4\times4} & S \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} B_u \\ 0_4 \end{smallmatrix}\right]$. It is also assumed that the active force actuator is $F_a = 0$. The variable $y_2 = C_2 x$ is measured by agent 2 with $C_2^T = \text{col}(1, 0, 0, 0, 0_4)$. Agent 1 does not measure anything from the quarter-car active automotive suspension system. For this configuration, we get that $rank(\mathcal{O}_1) = 0$ and $rank(\mathcal{O}_2) = 8$. Furthermore, we confirm that (A, C) is observable with $C = \text{col}(C_1, C_2)$. Using the decomposition (3), we compute the transformed matrices and choose $\gamma = 25$ and the vector L_{o2} such that the matrix $(A_{o2} + L_{o2}C_{o2})$ at $\{-2; -4; -6; -8; -3; -9; -5; -7\}$. The initial conditions for the simulation are generated randomly. Figure.6 shows the estimation errors of all agents. It can be

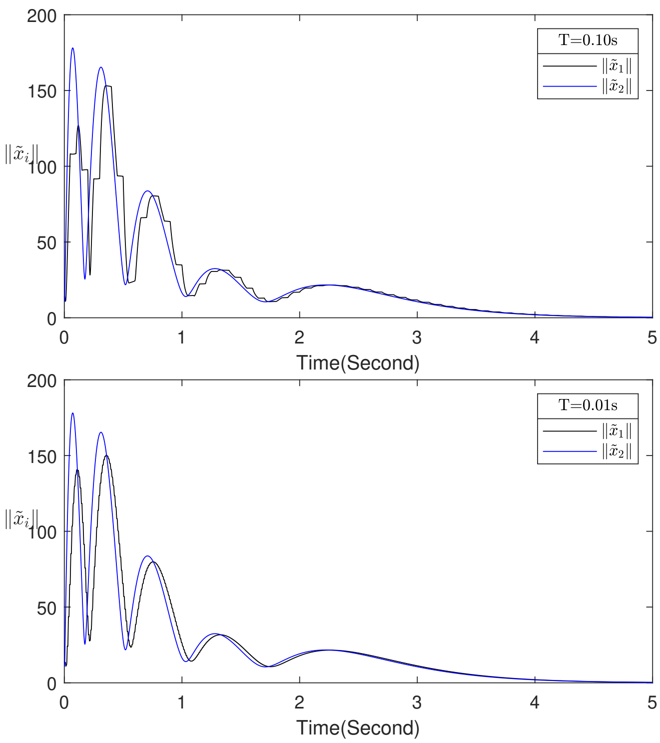


Fig. 6. Estimation errors of all followers with different T.

observed that agent 2 can reconstruct the state of (15). It also demonstrates that the distributed observer with fast-switching communication enables the estimation for agent 1.

V. Conclusion

The distributed observer for a general linear time-invariant system is proposed to estimate the state of the to-be-observed system over the jointly observable assumption and time-varying graphs. The results allow the system to be in a general form and the communication network to be disconnected at every instant. This can naturally include connected static networks or every-time connected switching networks as special cases. Some sufficient conditions for the existence of distributed observers for general linear systems over periodic communication networks have been presented. It is shown that all the nodes in the networks can cooperate to reconstruct the state of the system and share the estimated state with their neighbours through local communication in presence of time-varying communication architectures.

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