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Small gain theorems for infinite networks of discontinuous dynamical systems and applications

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Abstract— **We prove two small gain theorems for uniform asymptotic, uniform finite-time and uniform fixed-time stability of infinite networks of interconnected nonlinear systems with discontinuous right-hand sides. Their applicability to decentralized control problems is demonstrated on an example of infinite networks of mechanical systems described by lower-triangular form systems with discontinuous dynamics and power integrators.**

I. INTRODUCTION

In the last years, stability and control of networks with infinite set of nodes [15], [7], [9], [24], [20], [21] becomes a hot topic, although some classical papers devoted to infinite networks [6], [23] raised related problems as well. One motivation comes from biology, power networks and other applications whenever one deals with very complex biological networks [1],[31], or networks with polymeric or spatially invariant structure [3],[8],[33]. It is also challenging to study dynamic multi-agent systems with unknown number of agents [16], as well as stability properties independent of the number of the agents [4]. In some cases it is also convenient to replace control systems of partial differential equations (PDE) with *partial difference* equations, which leads to the same networks of ordinary differential equations (ODE) with unlimited number of nodes [2].

Paper [9] introduced the concept of ℓ_∞-input-to-state stability (ℓ_∞-ISS) of infinite networks, which is a natural generalization of the usual notion of ISS in the case of networks with a countable set of nodes. This paper also proposed new sufficient conditions for ℓ_∞-ISS of such networks and demonstrate their efficiency in design of decentralized stabilizers for infinite networks. However, this work proposes a rather conservative "small-gain theorem", in which it is assumed that all the gains for each node are strictly less than the identity. Of course, it is natural to ask about whether any extension of the classical small-gain conditions for *finite* networks [18], [10], [19], [11] can be obtained for the case of *infinite* networks. The answer was given in [24], [20], [21] and in other recent papers by these and other authors. These works provide more general and less conservative small gain conditions for input-to-state stability of infinite networks in comparison with [9] on the one hand, and to generalize the well-known classical small gain theorems for finite networks [18], [10], [19], [11] on the other.

However works [24], [20], [21] deal with the case when the entire network is well-posed, which means that the solution to every Cauchy problem for the entire network not only exists but also is unique. This assumption does not hold, for instance, if we deal with decentralized *finite-time* stabilization. In addition, even an asymptotic stabilization can lead to continuous but not Lipschitz continuous feedbacks, if the nodes have uncontrollable linearzation as in [9].

Following this line and also being motivated by [14], we are interested in possible extensions of the small-gain theorems from [9], [24], [20], [21] and other related papers to the case of infinite networks composed of interconnected nonlinear systems with *discontinuous* dynamics and in their distributed stabilization. Also, being motivated by [22], [5], [27], we are interested not only in ISS or asymptotic stability of infinite networks, but also in sufficient conditions for their finite-time and fixed-time stability. To our best knowledge this question is still open.

In the first part of this paper, we prove two small gain theorems devoted to uniform asymptotic, finite-time and fixed-time stability conditions for infinite networks of discontinuous dynamical systems and the second part is devoted to their applications to decentralized finite-time stabilization of infinite networks of lower triangular form systems with power integrators and with discontinuous dynamics.

II. PRELIMINARIES

We define ℓ_∞ as the Banach space of sequences of real numbers of the form $Z = \{z_i\}_{i\in\mathbb{Z}}$ s.t. $\sup\limits_{i\in\mathbb{Z}}|z_i| < +\infty$ with the norm defined by $||Z||_{\ell_\infty} := \sup\limits_{i\in\mathbb{Z}}|z_i|$. Given any $A \subset \mathbb{R}^N$, let $\overline{A}$, $\mathrm{int}A$ and $\mathrm{co}A$ denote the closure, the interior and the convex hull of A respectively. Given a metric space $(\mathcal{M}, \mathrm{d})$, let $AC([a,b];\mathcal{M})$ denote the class of absolutely continuous maps $\mathbb{R} \supset [a,b] \ni t \mapsto \mathcal{X}(t) \in \mathcal{M}$. In addition to the standard definitions of the comparison functions of classes $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$, we follow [17] and say that $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a generalized $\mathcal{K}$-function, or a $\mathcal{GK}$-function, if it is continuous, with $\alpha(0) = 0$ and satisfies $\alpha(s) = \max\{0,\ \bar{\alpha}(s) - \bar{\alpha}(s_0)\}$, where $\bar{\alpha}(\cdot)$ is a $\mathcal{K}$-function and $s_0 \geq 0$ is a given parameter. A continuous function $\beta : [0,+\infty[\times[0,+\infty[\to [0,+\infty[$ is said to be a generalized $\mathcal{KL}$-function, or a $\mathcal{GKL}$-function if for each $t \geq 0$ the function $\beta(\cdot,t)$ is a $\mathcal{GK}$-function and for each $s \geq 0$ the function $\beta(s,\cdot)$ is decreasing with $\beta(s,t) \to 0$ as $t \to +\infty$ with some $T(s) \leq +\infty$ and $t \mapsto \beta(s,t)$ is decreasing.

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III. Main Definitions

The following chain of definitions is motivated by [9], by [17], [27], [28], and by [14]. Throughout the paper, we deal with infinite networks of the following form

$$\dot{x}_i(t) = f_i(x_i(t), \{x_j(t)\}_{j\in J(i)}), \qquad i \in \mathbb{Z}, \tag{1}$$

where $x_i = [x_{i,1}, \ldots, x_{i,n_i}]^\top \in \mathbb{R}^{n_i}$ is the state vector of the i-th subsystem for each $i \in \mathbb{Z}$. We assume that, for each $i \in \mathbb{Z}$, the set $J(i) \subset \mathbb{Z}$ of the "neighbors" of the i-th subsystem is some *finite* set of the corresponding indices from $\mathbb{Z}$, and for each $i \in \mathbb{Z}$, we have $i \notin J(i)$. We always assume that the state vectors $x = \{x_i\}_{i\in\mathbb{Z}}$ of the entire network (1) are elements of ℓ_∞ and their ℓ_∞-norms are defined by

$$\|x\|_{\ell_\infty} := \sup_{i\in\mathbb{Z}} |x_i|_\infty = \sup_{i\in\mathbb{Z}} \max_{j=\overline{1,n_i}} |x_{i,j}| < \infty. \tag{2}$$

Also, for any $[t_0, T] \subset \mathbb{R}$, any $i \in \mathbb{Z}$, and any $x_i(\cdot) \in C([t_0, T]; \mathbb{R}^{n_i})$, the Chebyshev norm of $x_i(\cdot)$ is defined by $\|x_i(\cdot)\|_{C([t_0,T];\mathbb{R}^{n_i})} := \max_{t\in[t_0,T]} |x_i(t)|_\infty$.

Motivated by [14], we also suppose that each f_i is *piecewise continuous*, which means the following. For each $i \in \mathbb{Z}$, there is a *non-degenerate and locally finite partitioning* $\{\Omega_{i,k}\}_{k\in\mathbb{N}}$ of $\mathbb{R}^{n_i} \times \mathbb{R}^{\sum_{j\in J(i)} n_j}$, which means the following properties: all $\Omega_{i,k}$ are closed subsets of the corresponding $\mathbb{R}^{n_i} \times \mathbb{R}^{\sum_{j\in J(i)} n_j}$, and for each $i \in \mathbb{Z}$, first, $\mathrm{int}\Omega_{i,k_1} \cap \mathrm{int}\Omega_{i,k_2} = \emptyset$, whenever $k_1 \neq k_2$, second $\cup_{k=1}^{+\infty} \Omega_{i,k} = \mathbb{R}^{n_i} \times \mathbb{R}^{\sum_{j\in J(i)} n_j}$, third, $\overline{\mathrm{int}\Omega_{i,k}} = \Omega_{i,k}$, and, fourth, for each compact set $K \subset \mathbb{R}^{n_i} \times \mathbb{R}^{\sum_{j\in J(i)} n_j}$, there is only *finite* set of indices $\mathrm{J}_i(K) \subset \mathbb{N}$ such that $K \cap \Omega_{i,k} \neq \emptyset$, whenever $k \in \mathrm{J}_i(K)$. For each $i \in \mathbb{Z}$, we assume the existence of a sequence of continuous $f_{i,k} \in C(\Omega_{i,k}; \mathbb{R}^{n_i})$, $k \in \mathbb{N}$, such that for every $(x_i, \{x_j\}_{j\in J(i)}) \in \mathbb{R}^{n_i} \times \mathbb{R}^{\sum_{j\in J(i)} n_j}$ there exists $k \in \mathrm{J}_i(\{(x_i, \{x_j\}_{j\in J(i)})\})$ such that

$$f_i(x_i, \{x_j\}_{j\in J(i)}) = f_{i,k}(x_i, \{x_j\}_{j\in J(i)}) \tag{3}$$

We assume that the origin is an equilibrium point, i.e., $f_{i,k}(0, 0, 0) = 0$ for all $i \in \mathbb{Z}$, $k \in \mathrm{J}_i(\{(0, 0, 0)\})$.

We also assume that the dynamics of (1) is locally ℓ_∞-bounded, which means that for each $R \geq 0$ we have:

$$\sup_{i\in\mathbb{Z}} \max_{\substack{|x_i|_\infty \leq R, \\ |x_j|_\infty \leq R,\, j\in J(i) \\ k\in \mathrm{J}_i(\{(x_i,\{x_j\}_{j\in J(i)})\})}} |f_{i,k}(x_i, \{x_j\}_{j\in J(i)})|_\infty < \infty. \tag{4}$$

In addition, we assume that

$$\sup_{i\in\mathbb{Z}} n_i < +\infty. \tag{5}$$

As in [14], for each $i \in \mathbb{Z}$, we define the following set-valued map F_i of $\mathbb{R}^{n_i} \times \mathbb{R}^{\sum_{j\in J(i)} n_j}$ to $2^{\mathbb{R}^{n_i}}$.

$$\begin{aligned} &F_i(x_i, \{x_j\}_{j\in J(i)}) := \\ &\mathrm{co}\{f_{i,k}(x_i, \{x_j\}_{j\in J(i)}) | k \in \mathrm{J}_i(\{(x_i, \{x_j\}_{j\in J(i)})\})\}. \end{aligned} \tag{6}$$

Remark 1: Note that, for each $i \in \mathbb{Z}$, the set-valued map $(x_i, \{x_j\}_{j\in J(i)}) \mapsto F_i(x_i, \{x_j\}_{j\in J(i)})$ is upper semicontinuous and each set $F_i(x_i, \{x_j\}_{j\in J(i)})$ is convex, closed and bounded for each $(x_i, \{x_j\}_{j\in J(i)}) \in \mathbb{R}^{n_i} \times \mathbb{R}^{\sum_{j\in J(i)} n_j}$.

Motivated by [14], we define extended Filippov solutions to (1) as follows.

Definition 1: Let $\mathcal{T} \subset \mathbb{R}$ be a nonempty interval, which can be open $\mathcal{T} =]a, b[$, closed $\mathcal{T} = [a, b]$, or half-open $\mathcal{T} = [a, b[$, $\mathcal{T} =]a, b]$. By definition we say that $\mathcal{T} \ni t \mapsto x(t) = \{x_i(t)\}_{i\in\mathbb{Z}} \in \ell_\infty$ is *a trajectory of (1) on* $\mathcal{T}$, if and only if for each $[a', b'] \subset \mathcal{T}$ and each $i \in \mathbb{Z}$ the map $t \mapsto x_i(t)$ is of class $AC([a', b']; \mathbb{R}^{n_i})$, and

$$\forall i \in \mathbb{Z} \quad \dot{x}_i(t) \in F_i(x_i(t), \{x_j(t)\}_{j\in J(i)}) \text{ a.e. on } t \in \mathcal{T}. \tag{7}$$

Let us emphasize that, by this definition, if $\mathcal{T} \ni t \mapsto x(t) = \{x_i(t)\}_{i\in\mathbb{Z}}$ is a trajectory of (1), then $x(t) \in \ell_\infty$ for all $t \in \mathcal{T}$.

Definition 2: As in [9], given any nonempty (open, half-open, or closed) interval $\mathcal{T} \subset \mathbb{R}$, any $t_0 \in \mathcal{T}$, and any $x^0 = \{x_i^0\}_{i=-\infty}^{+\infty}$ in ℓ_∞, let $\Xi(t_0, x^0, \mathcal{T})$ denote the set of all trajectories of (1) on $\mathcal{T}$ (i.e., extended Filippov solutions to (1) in the sense of Definition 1) such that $x(t_0) = x^0$.

Motivated by [17], [27], [28], [9] it is natural to give the following definitions

Definition 3: System (1) is said to be ℓ_∞-uniformly asymptotically stable (ℓ_∞-UAS) if and only if there exists $\beta \in \mathcal{KL}$ such that for each $t_0 \in \mathbb{R}$, and each $x^0 = \{x_i^0\}_{i\in\mathbb{Z}}$ in ℓ_∞, we have $\Xi(t_0, x^0, [t_0, +\infty[) \neq \emptyset$ and each trajectory $t \mapsto x(t)$ from $\Xi(t_0, x^0, [t_0, +\infty[)$ satisfies

$$\forall t \geq t_0 \qquad \|x(t)\|_{\ell_\infty} \leq \beta(\|x^0\|_{\ell_\infty}, t - t_0). \tag{8}$$

Definition 4: System (1) is said to be ℓ_∞-finite-time uniformly stable (ℓ_∞-finite-time US), if there is $\beta \in \mathcal{GKL}$ such that $\beta(r, s) = 0$ for each $s \geq T(r)$ with some $r \mapsto T(r)$ of class $C([0, +\infty[; [0, +\infty[)$ and such that $T(0) = 0$, and, for each $t_0 \in \mathcal{T}$, and each $x^0 = \{x_i^0\}_{i=1}^\infty$ in ℓ_∞, we have $\Xi(t_0, x^0, [t_0, +\infty[) \neq \emptyset$ and each trajectory $t \mapsto x(t)$ from $\Xi(t_0, x^0, [t_0, +\infty[)$ satisfies

$$\forall t \geq t_0 \qquad \|x(t)\|_{\ell_\infty} \leq \beta(\|x^0\|_{\ell_\infty}, t - t_0). \tag{9}$$

The function $r \mapsto T(r)$ is called the settling time for (1).

Definition 5: System (1) is said to be ℓ_∞-fixed-time uniformly stable (ℓ_∞-fixed-time US), if it is ℓ_∞-finite-time uniformly stable in the sense of the previous Definition 4 and its settling time, i.e, the function $r \mapsto T(r)$ from the previous Definition 4 is uniformly bounded, i.e., $\sup_{r\in[0,+\infty[} T(r) < +\infty$.

IV. Small Gain Theorems for Infinite Networks of Discontinuous Systems

In general, throughout this section, we assume that system (1) satisfies the following conditions

(i) There are positive definite and uniformly radially unbounded ISS Lyapunov functions $V_i(\cdot)$ of class $C^1(\mathbb{R}^{n_i}; [0, +\infty[)$, $i \in \mathbb{Z}$ with the corresponding $\bar{\alpha}_{\min}(\cdot) \in \mathcal{K}_\infty$ such that $\bar{\alpha}_{\min}(|x_i|_\infty) \leq V_i(x_i)$ for all $x_i \in \mathbb{R}^{n_i}$, $i \in \mathbb{Z}$, and there are the corresponding positive definite decay rates $\alpha_i(\cdot) \in C([0, +\infty[; [0, +\infty[)$, $i \in \mathbb{Z}$, and the corresponding Lyapunov gains $\gamma_{i,j}(\cdot)$ of class $\mathcal{K}\cup\{0\}$, $i\in\mathbb{Z}$, $j\in J(i)$ such that for all $x_i\in\mathbb{R}^{n_i}$, $i\in\mathbb{Z}$ the following ISS Lyapunov inequalities hold

$$\begin{aligned}&V_i(x_i)\geq \max_{j\in J(i)}\gamma_{i,j}(V_j(x_j))\} \Rightarrow\\ &\nabla V_i(x_i)f_{i,k}(\{x_j\}_{j\in J(i)})\leq -\alpha_i(V_i(x_i))\\ &\text{ for all }\quad k\in \mathrm{J}_i(\{(x_i,\{x_j\}_{j\in J(i)})\}).\end{aligned}\tag{10}$$

(ii) For each $R>0$ we have:

$$\begin{aligned}&\sup_{i\in\mathbb{Z}}\max_{|x_i|\leq R} V_i(x_i)<+\infty,\\ &\sup_{i\in\mathbb{Z}}\max_{|x_i|\leq R}\left|\frac{\partial V_i(x_i)}{\partial x_i}\right|<+\infty\end{aligned}\tag{11}$$

Note that (11) also implies the existence of $\bar{\alpha}_{\max}(\cdot)\in\mathcal{K}_\infty$ such that $V_i(x_i)\leq\bar{\alpha}_{\max}(|x_i|_\infty)$ for all $x_i\in\mathbb{R}^{n_i}$,

Motivated by Section 7 of [20], our first small-gain theorem addresses the case of the so-called "spatially invariant" networks and we assume that the following conditions hold

(C1) There exists a "period" $N\in\mathbb{N}$ such that for each $k\in\mathbb{Z}$ the following identities hold: $J(i)+kN=J(i+kN)$ and $\gamma_{i+kN,j+kN}(\cdot)=\gamma_{i,j}(\cdot)$ and $\alpha_{i+kN}(\cdot)=\alpha_i(\cdot)$ for all i,j from $\mathbb{Z}$.

Given $i\in\mathbb{Z}$, let $I(i)$ denote $I(i):=\{j=i+kN\mid k\in\mathbb{Z}\}$. Also, we put by definition $\gamma_{i,j}(\cdot):=0$, whenever $j\notin J(i)$. Our second Condition (C2) is as follows:

(C2) All the gains $\gamma_{i,j}(\cdot)$ from (10) satisfy the condition

$$\forall r>0\quad \left(\gamma_{i_{q+1},i_q}\circ\gamma_{i_q,i_{q-1}}\circ\ldots\circ\gamma_{i_3,i_2}\circ\gamma_{i_2,i_1}\right)(r)<r\tag{12}$$

for every $q\in\{1,\ldots,N\}$ and for all i_p, $p\in\{1,\ldots,q+1\}$ such that $I(i_p)\neq I(i_j)$ whenever $1\leq p<j\leq q$ and such that $I(i_1)=I(i_{q+1})$.

Remark 2: Note that, as a corollary of [24], [20], [21], we obtain that, if conditions (i),(ii),(C1),(C2) hold, then the following statements hold:

(I) There exists a map $[0,\infty[\ni t\mapsto\varrho(t)=\{\varrho_i(t)\}_{i=-\infty}^{+\infty}\in\ell_\infty$ such that the following properties of Definition 5.1 from [11] hold:

(I.1) Each $\varrho_i(\cdot)$ is of class $\mathcal{K}_\infty$ and $\varrho_i^{-1}(\cdot)$ satisfies the local Lipschitz condition on $]0,+\infty[$ for each $i\in\mathbb{Z}$.

(I.2) For each compact set $K\subset]0,+\infty[$ there are $c_K>0$, $C_K>0$ such that $0<c_K\leq\left(\varrho_i^{-1}\right)'(r)<C_K$ for each $i\in\mathbb{Z}$ and for each point $r\in K$ of differentiability of $\varrho_i^{-1}(\cdot)$.

(I.3) For each $i\in\mathbb{Z}$ and each $r>0$ we have $\max\limits_{j\in J(i)}\{\gamma_{i,j}(\varrho_j(r))\}<\varrho_i(r)$.

(II) $\varrho_{i+kN}(\cdot)=\varrho_i(\cdot)$ for all $i\in\mathbb{Z}$, $k\in\mathbb{Z}$.

However, if one wants to design this map $[0,\infty[\ni t\mapsto\varrho(t)=\{\varrho_i(t)\}_{i=-\infty}^{+\infty}\in\ell_\infty$ constructively, then the method from [19] can be applied as follows. For each $i\in\mathbb{Z}$, and each $j\in\mathbb{Z}\backslash(\{i\}\cup J(i))$ we define $\hat{\gamma}_{i,j}(\cdot):=\gamma_{i,j}(\cdot)=0$ and for each $j\in J(i)$, we fix any $\hat{\gamma}_{i,j}(\cdot)>\gamma_{i,j}(\cdot)$ from class $\mathcal{K}_\infty$ such that for each $k\in\mathbb{Z}$ we have and $\hat{\gamma}_{i+kN,j+kN}(\cdot)=\hat{\gamma}_{i,j}(\cdot)$ $i\in\mathbb{Z}$, $j\in\mathbb{Z}\backslash\{i\}$, and such that all $\hat{\gamma}_{i,j}(\cdot)$ and $\hat{\gamma}_{i,j}^{-1}(\cdot)$ satisfy the local Lipschitz condition on $]0,+\infty[$ (in fact, they can be chosen from class C^1 or C^∞ on $]0,+\infty[$) and such that

$$\forall r>0\quad \left(\hat{\gamma}_{i_1,i_2}\circ\hat{\gamma}_{i_2,i_3}\circ\ldots\circ\hat{\gamma}_{i_{l+1},i_1}\right)(r)<r\tag{13}$$

for all $i_p\neq i_{p+1}$, $p=1,\ldots,l$, $i_{l+1}\neq i_1$, $1\leq l\leq N-1$. Then, following [19], define for all $i\in\mathbb{Z}$:

$$\begin{aligned}&\forall r>0\quad \varrho_i(r):=\\ &\sup_{q\in\mathbb{N}}\max_{i_l\neq i_{l+1}}\max_{l\leq p\leq q}\left(\hat{\gamma}_{i,i_1}\circ\hat{\gamma}_{i_1,i_2}\circ\ldots\circ\hat{\gamma}_{i_p,i_{p+1}}\right)(r)\\ &=\max_{q\in\mathbb{N}}\max_{i_l\neq i_{l+1}}\max_{l\leq p\leq q}\left(\hat{\gamma}_{i,i_1}\circ\hat{\gamma}_{i_1,i_2}\circ\ldots\circ\hat{\gamma}_{i_p,i_{p+1}}\right)(r)\end{aligned}\tag{14}$$

(Note that, by (13) and by (C1),(C2), the $\sup\limits_{q\in\mathbb{N}}$ in (14) is equal to the corresponding $\sup\limits_{q\leq N}=\max\limits_{q\leq N}$). From (14) it follows that $\max\limits_{j\in J(i)}\hat{\gamma}_{i,j}(\varrho_j(r))\leq\varrho_i(r)$ for all $r>0$ and for all $i\in\mathbb{Z}$. Then

$$\forall r>0\qquad \max_{j\in J(i)}\gamma_{i,j}(\varrho_j(r))<\varrho_i(r),\quad i\in\mathbb{Z}.\tag{15}$$

and conditions (I.1)-(I.3) and (II) are satisfied.

Our first theorem is as follows.

Theorem 1: Assume that conditions (i),(ii),(C1) and (C2) hold. Then the following statements hold:

(III) For each $t_0\in\mathbb{R}$, and each $x^0=\{x_i^0\}_{i\in\mathbb{Z}}$ in ℓ_∞, we have $\Xi(t_0,x^0,[t_0,+\infty[)\neq\emptyset$ and (1) is ℓ_∞-UAS in the sense of Definition 3. Furthermore, there is a positive definite decay rate $\alpha^\star(\cdot)\in C([0,+\infty[;[0,+\infty[)$ such that the Lyapunov function given by

$$V(x)=\sup_{i\in\mathbb{Z}}\left\{\varrho_i^{-1}(V_i(x_i))\right\}\tag{16}$$

and every $t\mapsto x(t)$ from $\Xi(t_0,x^0,[t_0,+\infty[)$ satisfy

$$\frac{d}{dt}V(x(t))\leq-\alpha^\star(V(x(t)))\text{ a.e. on }[t_0,+\infty[.\tag{17}$$

(IV) If all $\gamma_{i,j}(\cdot)$ are linear for all i,j from $\mathbb{Z}$, and $\alpha_i(r)\sim r^{1-\theta_1}$ as $r\to+0$ with some $\theta_1\in]0,1[$, then (1) is ℓ_∞-finite-time US in the sense of Definition 4.

(V) If all $\gamma_{i,j}(\cdot)$ are linear for all i,j from $\mathbb{Z}$, and $\alpha_i(r)=\max\{\bar{K}_{1,i}r^{1-\theta_1},\bar{K}_{2,i}r^{1+\theta_2}\}$ for all $r\geq0$ with some $\theta_1\in]0,1[$, $\theta_2>0$, and some $\bar{K}_{1,i}>0$, $\bar{K}_{2,i}>0$ such that $\bar{K}_{1,i}=\bar{K}_{1,i+kN}$, $\bar{K}_{2,i}=\bar{K}_{2,i+kN}$ for all $i\in\mathbb{Z}$, $k\in\mathbb{Z}$, then (1) is ℓ_∞-fixed-time US in the sense of Definition 5.

Our second version of the small gain theorems for networks (1) does not require the assumption of spatial invariance and assumptions (C1),(C2) are removed.

Theorem 2: Assume that conditions (i),(ii), hold and $\gamma_{i,j}(\cdot)=\gamma(\cdot)$ and $\alpha_i(\cdot)=\alpha(\cdot)$ for all $i\in\mathbb{Z}$, $j\in\mathbb{Z}\setminus\{i\}$, where $\gamma(\cdot)$ is an element of $\mathcal{K}\cup\{0\}$ such that $\forall r>0$ $\gamma(r)<r$. Then the following statements hold:

(VI) For each $t_0\in\mathbb{R}$, and each $x^0=\{x_i^0\}_{i\in\mathbb{Z}}$ in ℓ_∞, we have $\Xi(t_0,x^0,[t_0,+\infty[)\neq\emptyset$ and (1) is ℓ_∞-UAS in the sense of Definition 3. Furthermore, there is a positive definite decay rate $\alpha^\star(\cdot)\in C([0,+\infty[;[0,+\infty[)$ such that the Lyapunov function given by

$$V(x)=\sup_{i\in\mathbb{Z}}\left\{V_i(x_i)\right\}\tag{18}$$

and every $t \mapsto x(t)$ from $\Xi(t_0, x^0, [t_0, +\infty[)$ satisfy

$$\frac{d}{dt}V(x(t)) \leq -\alpha^\star(V(x(t))) \text{ a.e. on } [t_0, +\infty[. \quad (19)$$

(VII) If $\alpha(r) \sim r^{1-\theta_1}$ as $r \to +0$ with some $\theta_1 \in]0,1[$, then (1) is ℓ_∞-finite-time US in the sense of Definition 4.

(VIII) If $\alpha(r) = \max\{\bar{K}_1 r^{1-\theta_1}, \bar{K}_2 r^{1+\theta_2}\}$ for all $r \geq 0$ with some $\theta_1 \in]0,1[$, $\theta_2 > 0$ and some $\bar{K}_1 > 0$, $\bar{K}_2 > 0$, then (1) is ℓ_∞-fixed-time US in the sense of Definition 5.

V. Proofs of Theorem 1 and Theorem 2

We will prove statements (III)-(V) of Theorem 1 only, whereas the proof of statements (VI)-(VIII) of Theorem 2 will be the same with the identical Ω-path $\varrho_i(t) := t$.

Define the auxiliary $\alpha(\cdot) \in C([0,+\infty[;[0,+\infty[)$, $\alpha_{\min}(\cdot) \in \mathcal{K}_\infty$, $\alpha_{\max}(\cdot) \in \mathcal{K}_\infty$ as follows:

$$\begin{aligned}
&\forall r > 0 \quad \alpha(r) := \tfrac{1}{2} \inf_{i\in\mathbb{Z}} \min_{\frac{r}{2}\leq\varrho\leq 2r} \{(\alpha_i \circ \varrho_i)(\varrho)\} \\
&= \tfrac{1}{2} \min_{i\in\mathbb{Z}} \min_{\frac{r}{2}\leq\varrho\leq 2r} \{(\alpha_i \circ \varrho_i)(\varrho)\}, \\
&\forall r > 0 \quad \alpha_{\max}(r) := \sup_{i\in\mathbb{Z},\, j\in\mathbb{Z}} \left(\varrho_i^{-1} \circ \bar{\alpha}_{\max}\right)(r) \\
&= \max_{i\in\mathbb{Z},\, j\in\mathbb{Z}} \left(\varrho_i^{-1} \circ \bar{\alpha}_{\max}\right)(r) \\
&\forall r > 0 \quad \alpha_{\min}(r) := \inf_{i\in\mathbb{Z},\, j\in\mathbb{Z}} \left(\varrho_i^{-1} \circ \bar{\alpha}_{\min}\right)(r) \\
&= \min_{i\in\mathbb{Z},\, j\in\mathbb{Z}} \left(\varrho_i^{-1} \circ \bar{\alpha}_{\min}\right)(r).
\end{aligned} \quad (20)$$

Fix any $t_0 \in \mathbb{R}$ and any $x^0 = \{x_i^0\}_{i\in\mathbb{Z}} \in \ell_\infty$. The Proof of Theorem 1 is composed of the following Steps. In the first Step we prove the existence of some $\theta > 0$ and the existence of at least one trajectory, of (1), defined on $[t_0, t_0+\theta]$. Then, in second Step, we prove Condition (III) with (17) on this specific small interval $[t_0, t_0+\theta]$; and then we extend this trajectory with (17) inductively to $[t_0+\theta, t_0+2\theta]$, $[t_0+2\theta, t_0+3\theta]$, ..., i.e., to $[t_0, +\infty[$, *and prove that there is no any Zeno effect after such an extension.*

Step 1. Let us first define some fixed subinterval of $[t_0, +\infty[$ on which there is at least one trajectory of (1). For this, define

$$\begin{aligned}
&R_0 := \| x^0 \|_{\ell_\infty}; \qquad V^0 := \sup_{i\in\mathbb{Z}} \{\varrho_i^{-1}(V_i(x_i^0))\}; \\
&\mathcal{E}_{i,\text{ext}} := \{x_i \in \mathbb{R}^{n_i} \mid |x_i|_\infty \leq 2R_0 + \alpha_{\min}^{-1}\left(\varrho_i(2(V^0+1))\right) \\
&+1\} \quad \text{for all } i \in \mathbb{Z}; \\
&\mathcal{E}_{i,\text{in}} := \{x_i \in \mathbb{R}^{n_i} \mid \varrho_i^{-1}(V_i(x_i)) \leq V^0 + 1\} \\
&\text{for all } i \in \mathbb{Z};
\end{aligned} \quad (21)$$

$$\begin{aligned}
&\mathcal{M}^* := \sup_{i\in\mathbb{Z}} \Big\{ \max_{x_i\in\mathcal{E}_{i,\text{ext}}} \left|\tfrac{\partial V_i(x_i)}{\partial x_i}\right|_1 + 1 \Big\}, \\
&M_0 := \sup_{i\in\mathbb{Z}} \Big\{ \max_{\substack{x_i \in \mathcal{E}_{i,\text{ext}}; \\ x_j \in \mathcal{E}_{j,\text{ext}},\, j\in J(i); \\ k \in \mathrm{J}_i(\{(x_i, \{x_j\}_{j\in J(i)})\})}} \Big(1 \\
&+ |f_{i,k}(x_i, \{x_j\}_{j\in J(i)})|_\infty\Big) \Big\}.
\end{aligned} \quad (22)$$

Then fix any $\bar{\varrho} \in]0, R_0+1[$ such that for all $i \in \mathbb{Z}$, we have

$$\begin{aligned}
&\forall x_i' \in \mathcal{E}_{i,\text{ext}} \ \forall x_i'' \in \mathcal{E}_{i,\text{ext}} \quad |x_i' - x_i''|_\infty < \bar{\varrho} \Rightarrow \\
&|\varrho_i^{-1}(V_i(x_i')) - \varrho_i^{-1}(V_i(x_i''))| < \tfrac{V^0}{4}.
\end{aligned} \quad (23)$$

Then we define $\theta > 0$ by $\theta := \frac{\bar{\varrho}}{4M_0 + 4\mathcal{M}^* + 1}$. In contrast to [9] (Proof of Theorem 1, Step 1), we follow the same pattern as [12], Lemma 1 and Theorem 1 on pp. 75-78, and consider the Euler approximations for (7) on $[t_0, t_0+\theta]$. Then, as in [9] (Proof of Theorem 1, Step 1), we combine the Arzela-Ascoli theorem with Cantor's diagonal argument w.r.t. indices i, m, where i is the number of the component of the state vector of our infinite-dimensional system and m is the number of the Euler approximation, and find a subsequence $1 \leq m_1 < m_2 < \ldots < m_q < m_{q+1} < \ldots$ such that for each fixed $i \in \mathbb{Z}$ there is $x_i^*(\cdot)$ of class $C([t_0, t_0+\theta]; \mathbb{R}^{n_i})$ such that $\| x_i^{(m_q)}(\cdot) - x_i^*(\cdot)\|_{C([t_0,t_0+\theta];\mathbb{R}^{n_i})} \to 0$ as $q \to \infty$. Thus, we define $x^*(t) = \{x_i^*(t)\}_{i=1}^{\infty} \in \ell_\infty$ for every $t \in [t_0, t_0+\theta]$ and, arguing as in [12], Lemma 1 and Theorem 1 on pp. 75-78 and Lemmas 9,13 on pp. 62-64, we prove that and each $x_i^*(\cdot)$ is of class $AC([t_0, t_0+\theta]; \mathbb{R}^{n_i})$ and that $x^*(\cdot)$ satisfies (7) for all $i \in \mathbb{Z}$ as desired.

Step 2. In this Step 2, we extend the trajectory $t \mapsto x^*(t) = \{x_i^*(t)\}_{i\in\mathbb{Z}}$ constructed in Step 1 to the entire $[t_0, +\infty[$ and prove (III) for this extended trajectory $t \mapsto x^*(t) = \{x_i^*(t)\}_{i\in\mathbb{Z}}$.

For each $\Delta_1 > 0$ and each $\Delta_2 \geq \Delta_1$, define

$$K_i(\Delta_1, \Delta_2) := \left[\varrho_i\left(\frac{\Delta_1}{2}\right),\ \varrho_i(4\Delta_2)\right], \qquad i \in \mathbb{Z}, \quad (24)$$

and, using property (I.2), for every $i \in \mathbb{Z}$ find $c^{(i)}_{K_i(\Delta_1,\Delta_2)} > 0$ and $C^{(i)}_{K_i(\Delta_1,\Delta_2)} > 0$ such that

$$\begin{aligned}
&C^{(i)}_{K_i(\Delta_1,\Delta_2)} \geq \left(\varrho_i^{-1}\right)'(r) \geq c^{(i)}_{K_i(\Delta_1,\Delta_2)} \\
&\text{a.e. on } r \in K_i(\Delta_1, \Delta_2), \ \ i \in \mathbb{Z}.
\end{aligned} \quad (25)$$

Since $\varrho_{i+kN}(\cdot) = \varrho_i(\cdot)$ for all $i \in \mathbb{Z}$, $k \in \mathbb{Z}$ by property (II), we also obtain: $K_{i+kN}(\Delta_1, \Delta_2) = K_i(\Delta_1, \Delta_2)$, and $c^{(i)}_{K_i(\Delta_1,\Delta_2)} = c^{(i+kN)}_{K_{i+kN}(\Delta_1,\Delta_2)}$, and $C^{(i)}_{K_i(\Delta_1,\Delta_2)} = C^{(i+kN)}_{K_{i+kN}(\Delta_1,\Delta_2)}$ for all $i \in \mathbb{Z}$, $k \in \mathbb{Z}$. Then, for every fixed interval $[\Delta_1, \Delta_2] \subset]0, +\infty[$, we define

$$\begin{aligned}
c(\Delta_1, \Delta_2) &:= \inf_{i\in\mathbb{Z}} c^{(i)}_{K_i(\Delta_1,\Delta_2)} = \min_{i\in\mathbb{Z}} c^{(i)}_{K_i(\Delta_1,\Delta_2)}, \\
C(\Delta_1, \Delta_2) &:= \sup_{i\in\mathbb{Z}} C^{(i)}_{K_i(\Delta_1,\Delta_2)} = \max_{i\in\mathbb{Z}} C^{(i)}_{K_i(\Delta_1,\Delta_2)}.
\end{aligned} \quad (26)$$

To complete the proof of (III) and to obtain the existence of the decay rate $\alpha^*(\cdot)$, it suffices to prove the following lemma (which holds for this constructed trajectory $t \mapsto x^*(t)$ of (1) as well as for any other defined on $[t_0, t_0+\theta]$).

Lemma 1: For every $\Delta_2 \geq \Delta_1 > 0$ and almost everywhere on $t \in [t_0, t_0+\theta]$ we have:

$$\begin{aligned}
&\Delta_1 \leq V(x^*(t)) \leq 2\Delta_2 \Rightarrow \\
&\tfrac{d}{dt}[V(x^*(t))] \leq -c(\Delta_1, \Delta_2)\alpha(V(x^*(t))).
\end{aligned} \quad (27)$$

Without loss of generality assume that $V^0 > 0$. The case $V^0 = 0$ and all the others will be considered in the end of this Step 2. Define

$$\begin{aligned}
\varepsilon := \tfrac{1}{4} \min \Big\{ V^0, \min_{V\leq 2V^0} \min_{i\in\mathbb{Z}} \Big(V \\
- \max_{j\in\mathbb{Z}\setminus\{i\}} \left\{ \left(\varrho_i^{-1} \circ \gamma_{i,j} \circ \varrho_j\right)(V) \right\} \Big) \Big\}.
\end{aligned} \quad (28)$$

From (22),(24),(25), it follows that

$$\begin{array}{l}\forall i\in\mathbb{Z}\qquad\forall t'\in[t_0,t_0+\theta]\qquad\forall t''\in[t_0,t_0+\theta]\\ |\varrho_i^{-1}(V_i(x_i^*(t')))-\varrho_i^{-1}(V_i(x_i^*(t'')))|\\ \le C(\frac{V^0}{4},2V^0)\mathcal{M}^*L^\star|t'-t''|.\end{array}\tag{29}$$

Find any $\tau\in]0,\frac{\theta}{2}]$ such that

$$\begin{array}{l}\forall t\in[t_0,t_0+\theta-\tau]\qquad\forall s\in[0,\tau]\qquad\forall i\in\mathbb{Z}\\ |\varrho_i^{-1}(V_i(x_i^*(t+s)))-\varrho_i^{-1}(V_i(x_i^*(t)))|\le\frac{\varepsilon}{4}.\end{array}\tag{30}$$

Then, in particular,

$$\begin{array}{l}\forall i{\in}\mathbb{Z}\ \ \forall t\in[t_0,t_0+\tau]\ \ |\varrho_i^{-1}(V_i(x_i^*(t)))\\ -\varrho_i^{-1}(V_i(x_i^*(t_0)))|\le\frac{\varepsilon}{4}.\end{array}\tag{31}$$

$$\forall t\in[t_0,t_0+\tau]\qquad\qquad V(x^*(t))\le V^0+\frac{\varepsilon}{4}.\tag{32}$$

For each $t{\in}[t_0,t_0{+}\tau]$ and each $\delta\ \in]0,\varepsilon]$, by $I(t,\delta){\subset}\mathbb{Z}$ denote the following set of indices

$$I(t,\delta):=\{j\in\mathbb{Z}\mid\varrho_j^{-1}(V_j(x_j^*(t)))\ge V^0-\delta\}.\tag{33}$$

Then we prove the following lemmas.

Lemma 2: For each $i{\in}\mathbb{Z}$ and each $\Delta_2{\ge}\Delta_1{>}0$ we have:

$$\begin{array}{l}\max\left\{V^0-\frac{3\varepsilon}{4},\ \Delta_1\right\}\le\varrho_i^{-1}(V_i(x_i^*(t)))\le2\Delta_2\Rightarrow\\ \frac{d}{dt}[\varrho_i^{-1}(V_i(x_i^*(t)))]\le-c(\Delta_1,\Delta_2)\alpha_i(V_i(x_i^*(t)))\\ \quad\text{a. e. on }\ t\in[t_0,t_0{+}\tau].\end{array}\tag{34}$$

Proof of Lemma 2 follows from (31), (32) combined with (i), (I) and with (25)-(27). From Lemma 2 we obtain the following lemma.

Lemma 3: (L.1) For each $i\in\mathbb{Z}$ we have:

$$\varrho_i^{-1}(V_i(x_i^*(t)))\le\max\{\varrho_i^{-1}(V_i(x_i^*(t_0))),V^0-\frac{3\varepsilon}{4}\}.$$

(L.2) For each $t_1\in[t_0,t_0{+}\tau]$, each $i\in I(t_1,\frac{\varepsilon}{4})$, each $\Delta_1\in\ ]0,\ V^0{-}\frac{3\varepsilon}{4}]$, and each $\Delta_2\ge2V^0$, we obtain:

$$\begin{array}{l}\frac{d}{dt}[\varrho_i^{-1}(V_i(x_i^*(t)))]\le-c(\Delta_1,\Delta_2)\alpha_i(V_i(x_i^*(t)))\\ \quad\text{a. e. on }\ t\in[t_1,t_0{+}\tau].\end{array}$$

(L.3) For each $j\ \in\ \mathbb{Z}\setminus I(\frac{\varepsilon}{2})$ and each $i\ \in\ I(t,\frac{\varepsilon}{4})$ we have: $\varrho_j^{-1}((V_j(x_j^*(t)))\ \le\ \varrho_i^{-1}(V_i(x_i^*(t)))$ for all $t{\in}[t_0,t_0{+}\tau]$.

Proof of Item (L.1) immediately follows from Lemma 2. Proof of Item (L.2) follows from (31),(33) and from Lemma 2. Proof of Item (L.3) follows from (31),(33) and from Item (L.1) of Lemma 3.

Since $I(t,\frac{\varepsilon}{4})\subset I(t,\frac{\varepsilon}{2})$, Item (L.3) of Lemma 3 yields:

$$\begin{array}{l}\forall t{\in}[t_0,t_0{+}\tau]\quad V(x^*(t))=\sup\limits_{i\in\mathbb{Z}}\varrho_i^{-1}(V_i(x_i^*(t)))\\ =\sup\limits_{i\in I(\frac{\varepsilon}{2})}\varrho_i^{-1}(V_i(x_i^*(t))).\end{array}\tag{35}$$

Let us prove Lemma 1. Take any $\Delta_2\ \ge\ \Delta_1\ >\ 0$ and assume that $\Delta_1\ \ge\ V^0{-}\frac{3\varepsilon}{4}$ without loss of generality. Eq. (29) implies that $t\ \mapsto\ V(x^*(t))$ satisfies the same Lipschitz inequality (29) as for $t\mapsto\varrho_i^{-1}(V_i(X_i^*(t)))$ with the same Lipschitz constant. Hence, there is $\frac{dV(x^*(t))}{dt}$ a.e. on $[t_0,t_0+\theta]$, and we only need to prove (27) at every $t\in[t_0,t_0+\theta]$ such that there is $\frac{d}{dt}[V(x^*(t))]$. Assume the converse, then there is $t^*\in[t_0,t_0+\tau[$ such that

$$\begin{array}{l}\Delta_1{\le}V(x^*(t^*)){\le}2\Delta_2\quad\text{and}\\ \left.\frac{dV(x^*(t))}{dt}\right|_{t=t^*}\ge-c(\Delta_1,\Delta_2)\alpha(V(x^*(t^*)))\end{array}\tag{36}$$

Then there is $h_0\in]0,t_0+\tau-t^*[$ such that

$$\begin{array}{l}\forall h\in]0,h_0]\qquad V(x^*(t^*+h))\ge V(x^*(t^*))\\ -c(\Delta_1,\Delta_2)\alpha(V(x^*(t^*)))h\end{array}\tag{37}$$

Using the definition of $V(x)$, fix any subsequence of indeces $\{i_k\}_{k=1}^{+\infty}\subset\mathbb{Z}$ such that $\varrho_{i_k}^{-1}(V_{i_k}(x_{i_k}^*(t^*{+}h_0)))\to V(x^*(t^*+h_0))$ as $k\to+\infty$. By (20),(37), there is $k_0\in\mathbb{N}$ such that

$$\begin{array}{l}\forall k\in\mathbb{Z}_{\ge k_0}\quad\varrho_{i_k}^{-1}(V_{i_k}(x_{i_k}^*(t^*+h_0))\ge V(x^*(t^*))\\ \quad-\frac{3c(\Delta_1,\Delta_2)}{2}\alpha(V(x^*(t^*)))h_0\ge\varrho_{i_k}^{-1}(V_{i_k}(x_{i_k}^*(t^*))\\ \quad-\frac{3c(\Delta_1,\Delta_2)}{2}\alpha(V(x^*(t^*)))h_0\ge\varrho_{i_k}^{-1}(V_{i_k}(x_{i_k}^*(t^*))\\ -\frac{3c(\Delta_1,\Delta_2)}{2}h_0\times\min\limits_{\frac{V(x^*(t^*))}{2}\le\varrho\le2V(x^*(t^*))}\{(\alpha_i\circ\varrho_i)(\varrho)\}\\ \ge\varrho_{i_k}^{-1}(V_{i_k}(x_{i_k}^*(t^*))-\frac{3}{4}c(\Delta_1,\Delta_2)\alpha_i(V_{i_k}(x_{i_k}^*(t^*))h_0.\end{array}\tag{38}$$

By the choice of $\tau\in]0,\frac{\theta}{2}]$, by (30), and by Item (L.2) of Lemma 3, there is $k_1\in\mathbb{N}$ such that

$$\begin{array}{l}\forall k\in\mathbb{Z}_{\ge k_1}\quad\varrho_{i_k}^{-1}(V_{i_k}(x_{i_k}^*(t^*+h_0))\le\varrho_{i_k}^{-1}(V_{i_k}(x_{i_k}^*(t^*))\\ -c(\Delta_1,\Delta_2)\alpha_i(V_{i_k}(x_{i_k}^*(t^*))h_0.\end{array}\tag{39}$$

Then, for each $k\in\mathbb{Z}_{\ge\max\{k_0,k_1\}}$, both (38) and (39) hold true and contradict each other. Arguing as above we extend (27) to the entire $[t_0,t_0+\theta]$ under the assumption that $V^0>0$. The proof of Lemma 1 is complete.

In the case, when $V^0=0$, we also define $\bar{\varrho}\in]0,R_0+1[$ and $\theta:=\frac{\varrho}{4M_0+1}$ as above in Step 1 by (21)-(23), repeat the Euler's iteratoins from Step 1 and define $t\mapsto x^*(t)=\{x_i^*(\cdot)\}_{i\in\mathbb{Z}}$ on $[0,t_0{+}\theta]$ as above in Step 1. Then we redefine: $t_0:=t_1$ and repeat the argument of Step 2 for on this new $[t_0,t_0+\theta]:=[t_1,t_1+\theta]$. Then we reduce the proof to the above paragraphs. Then we extend inductively all the above to $[t_0,t_0+2\theta]$, $[t_0,t_0+3\theta]$, $\ldots$, and complete the proof of Item (III).

To prove Item (IV), we note that, if all $\gamma_{i,j}(\cdot)$ are linear, then $\varrho_i(r)=\varrho_ir$ are also linear with some constants $\varrho_i>0$ such that $\varrho_{i+kN}=\varrho_i$ for all $i\in\mathbb{Z}$, $k\in\mathbb{Z}$. Then $V(x)=\sup\limits_{i\in\mathbb{Z}}\left\{\frac{V_i(x_i)}{\varrho_i}\right\}$. In addition, the assumptions of (IV) imply the existence of $\tilde{\alpha}_i>0$, $i\in\mathbb{Z}$, with $\tilde{\alpha}_{i+kN}=\tilde{\alpha}_i$ for all $i\in\mathbb{Z}$, $k\in\mathbb{Z}$ and the existence of some $r_0>0$ such that $\alpha_i(r)\ge\tilde{\alpha}_ir$ for all $i\in\mathbb{Z}$ $r\in]0,r_0[$. Then, arguing as in the Proof of Lemma 1 from [25] (see the passage from (L_1)-(L_1) to (28) in [25]), and taking into account the "N-periodicity" of all coefficients we prove that every trajectory $t\mapsto x(t)$, of (1) satisfies the same differential inequality as (25) in [25]. Then, using a separation of variables and integrating this inequality as in Theorem 1 from [22] or Theorem 4.2 from [5], we obtain that the same inequality with the same estimate for the uniform finite settling time for system (1) as (29),(30) in [25].

Item (V) is also proved by integrating the corresponding Lyapunov inequalities for the case $\alpha_i(r)\ =$

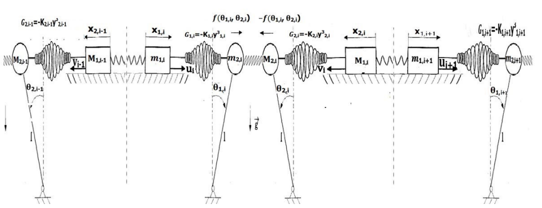
$\max\{\bar{K}_{1,i}r^{1-\theta_1}, \bar{K}_{2,i}r^{1+\theta_2}\}$ in Item (V) instead of $\alpha_i(r) \sim r^{1-\theta_1}$ as $r \to +0$ Item (IV) similarly to the Proof of Lemma 1 from [27]. This completes the proof of Theorem 1, whereas the proof of statements (VI)-(VIII) of Theorem 2 is the same as above with the identical $\varrho_i(r) := r$.

VI. EXAMPLE

Consider the following infinite network composed of a countable set of interconnected nodes, each of which has the same dynamics as the single control systems studied in [30] and in [29] (Section 4 "Application"),

$$\begin{cases} \ddot{\theta}_{1,i} = \frac{g}{l}\theta_{1,i} + \frac{K_{1,i}}{m_{2,i}l}(x_{1,i} - l\theta_{1,i})^3 + f(\theta_{1,i}, \theta_{2,i}) \\ \ddot{x}_{1,i} = -\frac{k_i(x_{1,i}-x_{2,i-1})}{m_{1,i}} - \mu_i^\star(x_{1,i})\dot{x}_{1,i} \\ -\frac{K_{1,i}(x_{1,i}-l\theta_{1,i})^3}{m_{1,i}} + \frac{\omega_i}{m_{1,i}} \\ \ddot{\theta}_{2,i} = \frac{g}{l}\theta_{2,i} + \frac{K_{2,i}}{M_{2,i}l}(x_{2,i} - l\theta_{2,i})^3 + f(\theta_{1,i}, \theta_{2,i}) \\ \ddot{x}_{2,i} = -\frac{k_{i+1}(x_{2,i}-x_{1,i+1})}{M_{1,i}} - \hat{\mu}_i(x_{2,i})\dot{x}_{2,i} \\ -\frac{K_{2,i}(x_{2,i}-l\theta_{2,i})^3}{M_{1,i}} + \frac{v_i}{M_{1,i}} \end{cases} \tag{40}$$

with states $(\theta_{1,i}, \dot{\theta}_{1,i}, x_{1,i}, \dot{x}_{1,i})$ and $(\theta_{2,i}, \dot{\theta}_{2,i}, x_{2,i}, \dot{x}_{2,i})$, and controls (ω_i, v_i), $i \in \mathbb{Z}$, whose model is depicted on the following Fig. 1.



Each mass $m_{1,i}$ or $M_{1,i}$, $i \in \mathbb{Z}$, models an "one-dimensional car" interconnected with the corresponding inverted pendulum of length l with mass $m_{2,i}$ or $M_{2,i}$ respectively by an elastic spring with its elasticity force $G_{1,i} = -K_{1,i}y_{1,i}^3$ or $G_{2,i} = -K_{2,i}y_{2,i}^3$ respectively as in [30], [29], $y_{1,i}$, $y_{2,i}$ being the deformations of these springs. To satisfy (4), and eventually the conditions of Theorems 1, 2, we assume that all the coefficients of (40) have uniform upper and lower (away from zero) boundaries.

Note that (40) is a *countably infinite* set of interconnected couples "car plus pendulum plus pendulum plus car" considered in Example 3 and Fig. 1 from [25] with new additional coupling terms $k_i(x_{1,i}-x_{2,i-1})$ denoting elasticity force between each "car" $m_{1,i}$ with the "previous car" $M_{1,i-1}$. In addition, motivated by [13], [32], we assume that, for each $i \in \mathbb{Z}$, the "one-dimensional cars" with masses $m_{1,i}$ and $M_{1,i}$ are also affected by viscous friction forces represented by the terms "$-\mu_i^\star(x_{1,i})\dot{x}_{1,i}$" and "$-\hat{\mu}_i(x_{2,i})\dot{x}_{2,i}$". Furthermore, we can assume that functions $\mu_i^\star(x_{1,i})$ and $\hat{\mu}_i(x_{2,i})$ are not continuous, but piecewise continuous, for instance, the "cars" can be affected by non-symmetric viscous friction as in Eq. (1),(2) of [32].

As in [25], we want to find a decentralized feedback $\omega_i = \omega_i(\theta_{1,i}, \dot{\theta}_{1,i}, x_{1,i}, \dot{x}_{1,i})$, $v_i = v_i(\theta_{2,i}, \dot{\theta}_{2,i}, x_{2,i}, \dot{x}_{2,i})$, which renders (40) ℓ_∞-finite-time US. As in [30], [29], [25], we use the same feedback transformations as in Eqs. (12)-(16) from [25], and finally bring (40) to the following form

$$\begin{cases} \dot{\xi}_{j,1}=\xi_{j,2}, & \dot{\xi}_{j,2}=\frac{g}{l}\xi_{j,1}+\xi_{j,3}^3+f(\xi_{j,1}, \xi_{m,1}), \\ \dot{\xi}_{j,3}=x_{j,4}, & \dot{\xi}_{j,4}=u_j+\alpha_j\xi_{\varkappa,1}+\beta_j\xi_{\varkappa,3}, \end{cases} \quad j\in\mathbb{Z}, \tag{41}$$

where $m=j\pm1$, $\varkappa=j\mp1$, (in contrast to Eq. (17)-(16) from [25], the $\xi_{j,4}$-equation in (41) contains additional coupling terms $\alpha_j\xi_{\varkappa,1}$ and $\beta_j\xi_{\varkappa,3}$ due to additional springs between $M_{1,i}$ and $m_{1,i+1}$). Then, for each node number $j \in \mathbb{Z}$ in (41), we follow the algorithm from Section 5, Proof of Theorem 3 in [25]. Note that each node number $j \in \mathbb{Z}$ of *infinite* network (41) has the same form as one of the two nodes of *finite* network (11) from [25] except the last $\xi_{j,4}$-equation, which has new additional coupling terms $\alpha_j\xi_{\varkappa,1}$, $\beta_j\xi_{\varkappa,3}$ in (41). Therefore, we repeat the first three steps corresponding to the $\xi_{j,1}$-, $\xi_{j,2}$-, $\xi_{j,3}$ - equations of the backstepping design from Example 3 in [25], Eqs. (139)-(167) (pp.275-281). Using the notation from [25], we have $m_0 = 1$, $m_1 = 1$, $m_2 = 3$, $m_3 = 1$, $m_4 = 1$; and $\nu = -\frac{2}{17}$, and $r_1 = 1$, $r_2 = \frac{15}{17}$, $r_3 = \frac{13}{51}$, $r_4 = \frac{7}{51}$, $\beta_0 = r_2 = \frac{15}{17}$, $\beta_1 = \frac{17}{15}$, $\beta_2 = \frac{83}{39}$, $\beta_3 = \frac{89}{7}$. Then, as in [25], we design (by induction on $l = 1, 2, 3, 4$) a state transformation

$$\begin{array}{ll} w_{j,1} := \xi_{j,1}^{\frac{15}{17}}, & v_{j,1}(\xi_{j,1}) = -w_{j,1}\lambda_{j,1}(\xi_{j,1}), \\ w_{j,2} := \xi_{j,2}^{\frac{17}{15}} - v_{j,1}^{\frac{17}{15}}(\xi_{j,1}), & v_{j,2}(X_{j,2}) = -w_{j,2}^{\frac{13}{17}}\lambda_{j,2}(X_{j,2}), \\ w_{j,3} := \xi_{j,3}^{\frac{83}{13}} - v_{j,2}^{\frac{83}{39}}(X_{j,2}), & v_{j,3}(X_{j,3}) = -w_{j,3}^{\frac{7}{83}}\lambda_{j,3}(X_{j,3}), \\ w_{j,4} := \xi_{j,4}^{\frac{89}{7}} - v_{j,3}^{\frac{89}{7}}(X_{j,3}), & v_{j,4}(X_{j,4}) = -w_{j,4}^{\frac{1}{89}}\lambda_{j,4}(X_{j,4}), \\ \text{where} & X_{j,l} := [\xi_{j,1}, \xi_{j,2}, \ldots, \xi_{j,l}]^\top \end{array} \tag{42}$$

with $\lambda_{j,l}(X_{j,l})>0$ in C^∞, $l=\overline{1,4}$, s.t. the Lyapunov functions

$$\begin{array}{l} V_{j,1}(\xi_{j,1}) = \frac{17}{32}\xi_{j,1}^{\frac{32}{17}}, \quad V_{j,k}(X_{j,l}) = V_{j,l-1}(X_{j,l-1}) \\ +W_{j,l}(X_{j,l}), \quad l = 2, 3, 4, \quad \text{where} \\ W_{j,2}(X_{j,2}) = \int_{v_{j,1}}^{\xi_{j,2}}(s^{\frac{17}{15}}-[v_{j,1}(\xi_{j,1})]^{\frac{17}{15}})ds, \\ W_{j,3}(X_{j,3}) = \int_{v_{j,2}^{\frac{1}{3}}}^{\xi_{j,3}}(s^{\frac{83}{13}}-[v_{j,2}(X_{j,2})]^{\frac{83}{39}})ds, \\ W_{j,4}(X_{j,4}) = \int_{v_{j,3}}^{\xi_{j,4}}(s^{\frac{89}{7}}-[v_{j,3}(X_{j,3})]^{\frac{89}{7}})ds. \end{array} \tag{43}$$

satisfy the Eq. (48) in Theorem 3 in [25] with

$$\begin{array}{l} Q_{j,1}^{1+r_2+\nu}=w_{j,1}^2, \quad Q_{j,2}^{1+r_2+\nu}=w_{j,1}^2+w_{j,2}^{\frac{30}{17}}, \\ Q_{j,3}^{1+r_2+\nu}=Q_{j,2}^{1+r_2+\nu}+w_{j,3}^{\frac{90}{83}}, \;\; Q_{j,4}^{1+r_2+\nu}=Q_{j,3}^{1+r_2+\nu}+w_{j,4}^{\frac{90}{89}}. \end{array} \tag{44}$$

Thus, we take any $\bar{\lambda}_{j,1} > \bar{\lambda}_{j,2} > \bar{\lambda}_{j,3} > \bar{\lambda}_{j,4} > 0$ and $\bar{\varepsilon}_3 \in]0, \frac{\bar{\lambda}_{j,4}}{8}[$, then we follow the design from pp.275-281 in [25] (See [25], Eqs. (139)-(167)), but instead of (167) from [25], we now obtain

$$\begin{array}{l} \dot{V}_{j,4}\big|_{(41)} \leq -\bar{\lambda}_{j,3}\Big(w_{j,1}^2 + w_{j,2}^{\frac{30}{17}} + w_{j,3}^{\frac{90}{83}}\Big) + w_{j,4}\Big[u_j + \alpha_j\xi_{\varkappa,1} \\ +\beta_j\xi_{\varkappa,3} + w_{j,4}^{\frac{1}{89}}\lambda_{j,4}^\star(X_{j,4})\Big] - \bar{\lambda}_{j,4}V_{j,4} + \frac{\bar{\varepsilon}_3}{2}V_{m,1}. \end{array} \tag{45}$$

Note that (45) is similar to (167) from [25], but, in contrast to (167) from [25], the new additional coupling terms $\alpha_j w_{j,4}\xi_{\varkappa,1}$ and $\beta_j w_{j,4}\xi_{\varkappa,3}$ appear in the last line of our estimate (45). We obtain their upper estimate by using the Young's inequality, more specifically, similarly to (64) from

[25], for each fixed $A \geq 0$, $c_1 > 0$ and $l \in \{1, \ldots, 4\}$, we have

$$A|x_{\varkappa,l}||w_{j,4}| \leq \frac{(c_1)^{-\alpha_{1,j}}}{\alpha_{1,j}}|Aw_{j,4}|^{\alpha_{1,l}} + \frac{(c_1)^{\alpha_{2,j}}}{\alpha_{2,j}}|x_{\varkappa,l}|^{\alpha_{2,l}}, \\ \text{where} \quad \alpha_{1,l} := \frac{1+r_2}{1+r_2-r_l}, \;\; \alpha_{2,l} := m_{l-1}\beta_{l-1} + 1. \tag{46}$$

In addition, by Lemma 7 from [25], there are functions $c_{j,l}(X_{j,l}) > 0$, $j \in \mathbb{Z}$, $l = 1, 2, 3, 4$, of class C^∞ such that $|x_{j,l}|^{m_{l-1}\beta_{l-1}+1} \leq c_{j,l}(X_{j,l})V_{j,l}(X_{j,l})$ for all $X_{j,l} \in \mathbb{R}^l$, $j \in \mathbb{Z}$, $l = 1, 2, 3, 4$. Then, arguing as in [25] (Proof of Theorem 3, Inductive Step, or Example 3, pp. 281-282), we find a positive function $\mathbb{R}^4 \ni X_{i,4} \mapsto \lambda_{i,4}(X_{i,4})$ of class $C^\infty(\mathbb{R}^4;]0, +\infty[)$ such that

$$\dot{V}_{j,4}\big|_{(41),u_j=v_{j,4}(X_{j,4})} \leq -\bar{\lambda}_{j,4}Q_{j,4}^{1+r_2+\nu} - \frac{3\bar{\lambda}_{j,4}}{4}V_{j,4}(X_{j,4}) \\ +\bar{\varepsilon}_3[V_{m,1}(X_{m,1}) + V_{\varkappa,1}(X_{\varkappa,1}) + V_{\varkappa,3}(X_{\varkappa,3})], \;\; j \in \mathbb{Z}, \tag{47}$$

where $v_{j,4}(X_{j,4})$ is defined by $\lambda_{i,4}(X_{i,4})$ in (42). Note that $\mu_i^\star(x_{1,i})$ and $-\hat{\mu}_i(x_{2,i})$ and the transformation $\{(\omega_i, v_i)\}_{i\in\mathbb{Z}} \mapsto \{u_j\}_{j\in\mathbb{Z}}$ are discontinuous. Therefore, the final closed-loop system will have discontinuous right-hand side and we cannot apply the previous small gain theorems for infinite networks. However, all the conditions of both Item (IV) of Theorem 1 and Item (VII) of Theorem 2 are satisfied and our closed-loop system will be ℓ_∞-finite-time US. Let us finally note that it is also possible to update the designed feedback in order to satisfy Items (V),(VIII) and to obtain the ℓ_∞-fixed-time US by using the method from [26].

VII. Conclusion

We proved two small gain theorems for infinite networks of nonlinear systems whose dynamics is discontinuous and demonstrated their applications to decentralized control of infinite networks on a benchmark example.

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