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Robust and Non-conservative Control Barrier Functions for Stochastic Systems with Arbitrary Relative Degree

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Abstract— **This paper addresses the problem of safety-critical control for stochastic control systems. Constrained optimal control problems can be sub-optimally reduced to a sequence of quadratic programs by using Control Barrier Functions (CBFs). The recently proposed High Order CBFs (HOCBFs) can accommodate constraints of arbitrary relative degree. The main challenge of this HOCBF method for stochastic systems lies in the fact that intractable high-order derivatives of random variables will be involved. Meanwhile, the system tends to be very conservative such that the system state tends to stay far away from safe set boundary, which significantly limits the system performance. To avoid high-order derivatives of random variables, we propose a recursively robust HOCBF (rrHOCBF) that iteratively replace random variables by their bounds in the derivation of the HOCBF constraint. We further propose a non-conservative and robust HOCBF (nrHOCBF) to address the conservativeness issue in this robust control method by introducing adaptive terms to the bounds of random variables. We provably show the safety guarantees of the proposed rrHOCBFs and nrHOCBFs. A case study of 2D obstacle avoidance is presented to demonstrate the effectiveness and advantages of the proposed method when compared to existing approaches.**

I. INTRODUCTION

Constrained optimal control problems subject to safety requirements are central to rising safety-critical autonomous and cyber physical systems that can be mostly modeled by stochastic dynamics. In the state of the art, control barrier functions have received increasing attention in enforcing safety in recent years [1] [2] [3] [4] due to their high computational efficiency in dealing with nonlinear systems under nonlinear constraints.

Barrier functions (BFs) are widely used in optimization problems to enforce the satisfaction of constraints [5]. In control systems, BFs are Lyapunov-like functions [6], [7]. They have been used to prove set invariance [8], [9], [10], as well as for multi-objective control [11]. It was proved in [6] that if a BF for a given safe set satisfies Lyapunov-like conditions, then the set is forward invariant. A BF that is less restrictive in the sense that it is allowed to decrease when far away from the boundary of the set was proposed in [1]. Control BFs (CBFs) are the use of BFs for control systems, and they are employed to map a state constraint to a state-feedback control constraint. CBFs are originally proposed in [1] [2] to work for safety constraints that have relative degree one with respect to the system dynamics. Then, exponential CBFs [12] have been proposed for arbitrarily high relative degree constraints. The high order CBF (HOCBF) [3] is simpler (to define), less restrictive, and more general than the exponential CBF. All the CBFs mentioned above are for deterministic systems.

In the literature, there are different approaches to enforce safety for stochastic systems using CBFs. First, probability satisfaction guarantees, i.e., ensuring that the safety constraint will be satisfied with probability greater than a given value, have been proposed in the CBF framework [13] using Itô's lemma [14] or using chance constraints [15]. However, safety constraints can still be violated in such stochastic CBFs, and this method does not work well in scenarios where a single failure could lead to catastrophic results. To ensure the satisfaction of safety constraints for stochastic systems, robust control methods are widely adopted. There are typically two different policies to ensure robustness in the CBF method. The first approach is to use extended class $\mathcal{K}$ functions instead of class $\mathcal{K}$ functions when defining a CBF [16] [17] [18]. In this way, we can ensure the asymptotic stability of the safe set. In other words, the system state will always be stabilized to the safe set whenever the safety constraint is violated since the CBF can be viewed as a general form of Control Lyapunov Function (CLF) [1]. Although simple to implement, the safety constraint can still be violated in stochastic systems. Another way to use robust control methods in CBFs is by considering the bounds of random variables/uncertainties when deriving a CBF constraint [19] [20]. However, the system tends to be very conservative in such robust CBFs in the sense that the system state will stay far away from the safe set bound as we always consider the worst case values of random variables/uncertainties when enforcing safety. More importantly, we will have high order derivatives of random variables/uncertainties when enforcing safety using HOCBFs. These high order derivatives are usually intractable/difficult to evaluate in most scenarios.

In order to address the challenges of conservativeness and high order derivatives of random variables when enforcing safety using CBFs for stochastic systems, this paper contributes a general form of robust HOCBFs. Specifically, to avoid high order derivatives of random variables, we recur-

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sively define robust CBFs and substitute random variables by their bounds that are assumed to be known. However, each robust CBF may not be differentiable. We address this by over-approximating the non-differentiable components in robust CBF using differentiable functions. The conservativeness issue is even more challenge for such robust HOCBFs since we always consider bounds of random variables in the recursive derivation and due to the over-approximations. We propose an adaptive form of robust HOCBFs to address this issue. This is achieved by enforcing adaptive terms to the bounds of random variables that depend on the values of CBFs in the definition of a robust HOCBF. We provably show the safety guarantees of such non-conservative and robust HOCBFs. In summary, we make the following two contributions:

1) We propose a recursively robust HOCBF (rrHOCBF) to ensure safety for stochastic systems with arbitrary relative degree. The proposed rrHOCBF avoids high order derivatives of random variables that are intractable/difficult to evaluate.
2) We propose a non-conservative and robust HOCBF (nrHOCBF) to address the conservativeness of the robust control method. The conservativeness of the proposed nrHOCBF is tunable/trainable.

II. PRELIMINARIES

We introduce preliminaries on CBFs [1] and HOCBFs [3]. We start with some formal definitions. We omit the definitions of (extended) class $\mathcal{K}$ function, forward invariance, and relative degree. Please see [21] [3] for details if interested.

We consider an affine control system (assumed to be deterministic):

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u} \tag{1}$$

where $\boldsymbol{x} \in X \subset \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times q}$ are locally Lipschitz continuous, and $\boldsymbol{u} \in U \subset \mathbb{R}^q$ is the control constraint set that is defined as follows ($\boldsymbol{u}_{min} \in \mathbb{R}^q, \boldsymbol{u}_{max} \in \mathbb{R}^q$):

$$U := \{\boldsymbol{u} \in \mathbb{R}^q : \boldsymbol{u}_{min} \le \boldsymbol{u} \le \boldsymbol{u}_{max}\}, \tag{2}$$

where the inequalities are interpreted component-wise.

In this paper, we refer to the relative degree of b as the relative degree of the constraint as the function b is used to define a constraint $b(\boldsymbol{x}) \ge 0$. For a constraint $b(\boldsymbol{x}) \ge 0$ with relative degree $m \ge 1$, we define $\psi_0(\boldsymbol{x}) := b(\boldsymbol{x})$, and then we further define a sequence of CBFs $\psi_i : \mathbb{R}^n \to \mathbb{R}, i \in \{1, \dots, m\}$:

$$\psi_i(\boldsymbol{x}) := \dot{\psi}_{i-1}(\boldsymbol{x}) + \alpha_i(\psi_{i-1}(\boldsymbol{x})), i \in \{1, \dots, m\}, \tag{3}$$

where $\alpha_i, i \in \{1, \dots, m\}$ denote $(m-i)^{th}$ order differentiable class $\mathcal{K}$ functions.

Next, we define a sequence of safe sets $C_i, i \in \{1, \dots, m\}$ corresponding to CBFs (3):

$$C_i := \{\boldsymbol{x} \in \mathbb{R}^n : \psi_{i-1}(\boldsymbol{x}) \ge 0\}, i \in \{1, \dots, m\}. \tag{4}$$

***Definition** 1:* (*High Order Control Barrier Function (HOCBF)* [3]) Let $C_i, i \in \{1 \dots, m\}$ be defined by (4) and $\psi_i(\boldsymbol{x}), i \in \{1, \dots, m\}$ be defined by (3). A function $b : \mathbb{R}^n \to \mathbb{R}$ is a High Order Control Barrier Function (HOCBF) of relative degree m for system (1) if there exist $(m-i)^{th}$ order differentiable class $\mathcal{K}$ functions $\alpha_i, i \in \{1, \dots, m-1\}$ and a class $\mathcal{K}$ function α_m s.t.

$$\sup_{\boldsymbol{u}\in U}[L_f \psi_{m-1}(\boldsymbol{x}) + L_g \psi_{m-1}(\boldsymbol{x})\boldsymbol{u} + \alpha_m(\psi_{m-1}(\boldsymbol{x}))] \ge 0, \tag{5}$$

for all $\boldsymbol{x} \in \cap_{i=1}^m C_i$. In (5), the left part is actually $\psi_m(\boldsymbol{x})$, L_f (or L_g) denotes Lie derivatives along f (or g).

The HOCBF is a general form of the relative degree one CBF [1], [2]. If $m = 1$, a HOCBF reduces to the CBF form: $L_f b(\boldsymbol{x}) + L_g b(\boldsymbol{x})\boldsymbol{u} + \alpha_1(b(\boldsymbol{x})) \ge 0$,. We have the following theorem to show the safety guarantees of HOCBFs.

Theorem 1: ([3]) Given a HOCBF $b(\boldsymbol{x})$ from Def. 1 with the safe sets $C_i, i \in \{1, \dots, m\}$ defined by (4), if $\boldsymbol{x}(0) \in \cap_{i=1}^m C_i$, then any Lipschitz continuous controller $\boldsymbol{u}(t) \in U$ that satisfies the HOCBF constraint in (5), $\forall t \ge 0$ renders $\cap_{i=1}^m C_i$ forward invariant for system (1).

In the literature, one usually [1], [3] combines CBFs or HOCBFs with quadratic costs to reformulate constrained optimal control problems. We usually discretize the time, and an optimization problem with constraints given by the CBFs/HOCBFs is solved at each time step. Note that these constraints are linear in control since the state value is given and fixed at the beginning of the time interval, therefore, each optimization problem becomes a quadratic program (QP). This method works for deterministic systems. Otherwise, the HOCBF constraint will be involved with high order derivatives of random variables that are intractable/difficult to evaluate. In this paper, we show how to address this issue.

III. PROBLEM FORMULATION AND APPROACH

Consider a stochastic control system:

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u} + \boldsymbol{\epsilon}, \tag{6}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times q}, \boldsymbol{u} \in U \subset \mathbb{R}^q$ are defined similarly as in (1). $\boldsymbol{\epsilon} \in \mathbb{R}^n$ is a vector of random processes with finite support. In other words, we assume that $\boldsymbol{\epsilon}$ is bounded in the form:

$$|\boldsymbol{\epsilon}| \le E, \tag{7}$$

where $E \in \mathbb{R}^n$ is the stochastic bound, and the above inequality is interpreted component-wise. In (6), we are considering additive noise in the stochastic systems. However, the proposed methods in this paper can also work similarly for stochastic systems with multiplicative noise. The model (6) considers both matched and unmatched uncertainties.

Objective: (Minimizing cost) We consider an optimal control problem for system (6) with the cost defined as:

$$\min_{\boldsymbol{u}(t)} \int_0^T \mathcal{C}(||\boldsymbol{u}(t)||)dt + p_0||\boldsymbol{x}(T) - \boldsymbol{K}||^2 \tag{8}$$

where $T > 0, p_0 > 0$, $||\cdot||$ denotes the 2-norm of a vector, $\mathcal{C}(\cdot)$ is a strictly increasing function of its argument. $\boldsymbol{K} \in \mathbb{R}^n$ is a desired terminal state.

Safety requirements: System (6) should always satisfy a safety constraint:

$$b(\boldsymbol{x}(t)) \geq 0, \forall t \in [0, T], \tag{9}$$

where $b : \mathbb{R}^n \to \mathbb{R}$ is differentiable and has relative degree $m \in \mathbb{N}$ with respect to system (6). We may consider multiple constraints at the same time to make it general.

Control constraints: The control of stochastic system (6) should always satisfy the bound as defined in (2).

Problem 1: Find an optimal control policy for stochastic system (6) by solving the optimization (8) s.t. (9) and (2).

Approach: Our method to solve Problem 1 is based on the CBF-based QP introduced at the end of Sec. II. We use a HOCBF to enforce the safety constraint (9), and use a CLF to enforce the terminal state constraint in the cost (8). In order to avoid high order derivatives of the random process $\boldsymbol{\epsilon}$ for high relative degree safety constraints, we recursively define robust CBFs as in (3). Further, to ensure the differentiablity of each robust CBF in the definition of a HOCBF, we over-approximate the Absolute Value Function (AVF) that is widely used in robust control by a continuously differentiable function. Finally, to address the conservativeness issue in the robust CBFs and due to the over-approximation, we introduce an adaptive term in the definition of each robust CBF.

IV. Robust High-Order CBFs

In this section, we show how we can define a robust HOCBF that avoids the high-order derivatives of the random process in the stochastic system (6). We also propose an adaptive approach to address the conservativeness of robust HOCBFs. We start with a motivating example showing why existing HOCBF methods may fail to work for stochastic system (6).

A. Motivating Example

Consider a unicycle stochastic model defined in the form:

$$\begin{aligned}&\dot{x} = v\cos\theta + \epsilon_1, \dot{y} = v\sin\theta + \epsilon_2,\\&\dot{\theta} = u_1 + \epsilon_3, \dot{v} = u_2 + \epsilon_4,\end{aligned} \tag{10}$$

where $\boldsymbol{x} = (x, y, \theta, v)$, $(x, y) \in \mathbb{R}^2$ denotes the 2-D location of the vehicle, $v \in \mathbb{R}$ denotes its linear speed, $\theta \in \mathbb{R}$ denotes its heading, $u_1 \in \mathbb{R}, u_2 \in \mathbb{R}$ are the two controls corresponding to steering wheel angle and acceleration, respectively. $\epsilon_i, i \in \{1, 2, 3, 4\}$ denote random processes with finite support, and they are bounded by $E_i, i \in \{1, 2, 3, 4\}$.

Suppose system (10) has to satisfy a safety constraint:

$$(x - x_0)^2 + (y - y_0)^2 \geq r^2, \tag{11}$$

where $(x_0, y_0) \in \mathbb{R}^2$ denotes the 2-D location of a circular obstacle, and $r > 0$ denotes its radius.

The relative degree of the safety constraint (11) is two with respect to the dynamics (10). Thus, we may use a HOCBF with $m = 2$ as in Def. 1 to enforce this safety constraint. Suppose we choose the class $\mathcal{K}$ functions α_1, α_2 as linear functions. The corresponding HOCBF constraint (5) in this case becomes:

$$\begin{aligned}&2(v\cos\theta + \epsilon_1)^2 + 2(v\sin\theta + \epsilon_2)^2 + 2(x - x_0)\dot{\epsilon}_1\\&+2(y - y_0)\dot{\epsilon}_2 + (-2(x - x_0)\sin\theta + 2(y - y_0)\cos\theta)vu_1\\&+(2(x - x_0)\cos\theta + 2(y - y_0)\sin\theta)u_2 + 2\dot{b}(\boldsymbol{x}) + b(\boldsymbol{x}) \geq 0,\end{aligned} \tag{12}$$

where $b(\boldsymbol{x}) = (x - x_0)^2 + (y - y_0)^2 - r^2$ and $\dot{b}(\boldsymbol{x}) = 2(x - x_0)(v\cos\theta + \epsilon_1) + 2(y - y_0)(v\sin\theta + \epsilon_2)$.

Using existing robust control approaches [19] [20], the robust form of the HOCBF constraint (12) is obtained by replacing stochastic components by their bounds:

$$\begin{aligned}&2(v^2 - 2|v\cos\theta|E_1 - 2|v\sin\theta|E_2) + 2|x - x_0|\dot{E}_1\\&+2|y - y_0|\dot{E}_2 + (-2(x - x_0)\sin\theta + 2(y - y_0)\cos\theta)vu_1\\&+(2(x - x_0)\cos\theta + 2(y - y_0)\sin\theta)u_2 + 2\dot{\hat{b}}(\boldsymbol{x}) + b(\boldsymbol{x}) \geq 0,\end{aligned} \tag{13}$$

where $b(\boldsymbol{x}) = (x - x_0)^2 + (y - y_0)^2 - r^2$ and $\dot{\hat{b}}(\boldsymbol{x}) = 2(x-x_0)v\cos\theta + 2(y-y_0)v\sin\theta - 2|x-x_0|E_1 - 2|y-y_0|E_2$. $\dot{E}_i, i \in \{1, 2\}$ denote the bounds of $\dot{\epsilon}_i$. We can check that the satisfaction of (13) always implies the satisfaction of (12). Thus, we can find a robust controller using (13).

Note that the HOCBF constraint (12) contains $\dot{\epsilon}_1$ and $\dot{\epsilon}_2$ whose bounds $\dot{E}_i, i \in \{1, 2\}$ are intractable to evaluate. Therefore, we cannot use (13) to find a robustly safe controller. Moreover, since we always consider the bounds in the robust form, the system tends to be overly-conservative. We show how we may address these issues in this work.

B. Recursively Robust HOCBFs

In order to avoid high order derivatives of random variables $\epsilon_i, i \in \{1, 2, 3, 4\}$ in HOCBFs, we recursively construct robust CBFs instead of finding robust HOCBFs after constructing HOCBFs.

Given a safety constraint $b(\boldsymbol{x}) \geq 0$ whose relative degree is m for stochastic system (6), we first define a robust CBF $\phi_1 : \mathbb{R}^n \to \mathbb{R}$ in the form:

$$\begin{aligned}\phi_1(\boldsymbol{x}) := L_f\phi_0(\boldsymbol{x}) + L_g\phi_0(\boldsymbol{x})\boldsymbol{u} - \left|\frac{d\phi_0(\boldsymbol{x})}{d\boldsymbol{x}}\right| E\\+\alpha_1(\phi_0(\boldsymbol{x})) \geq 0,\end{aligned} \tag{14}$$

where $\phi_0(\boldsymbol{x}) = b(\boldsymbol{x})$ and $L_g\phi_0(\boldsymbol{x}) = 0, \forall \boldsymbol{x}$ if $m > 1$. $\alpha_1(\cdot)$ is a class $\mathcal{K}$ function. However, the $\phi_1(\boldsymbol{x})$ in the above is not differentiable since there is an AVF, which prevents us from further constructing a higher order CBF. We address this by introducing a logarithm function that is a differentiable over-approximation of the AVF, and have the following modified robust CBF:

$$\begin{aligned}\phi_1(\boldsymbol{x}) := L_f\phi_0(\boldsymbol{x}) + L_g\phi_0(\boldsymbol{x})\boldsymbol{u} + \alpha_1(\phi_0(\boldsymbol{x}))\\-\left(log\left(e^{2\frac{d\phi_0(\boldsymbol{x})}{d\boldsymbol{x}}} + 1\right) - \frac{d\phi_0(\boldsymbol{x})}{d\boldsymbol{x}}\right)E \geq 0,\end{aligned} \tag{15}$$

where the logarithm and exponential functions are operated component-wise. There are many other differentiable over-approximation functions that we can use to replace the AVF, such as $\sqrt{\left(\frac{d\phi_0(\boldsymbol{x})}{d\boldsymbol{x}}\right)^2 + \xi}$, where $\xi > 0$ is a small scalar. We use the logarithm function here for better clarification without loss of generality.

For a safety constraint with relative degree $m > 1$, we have that $L_g\phi_0(\boldsymbol{x}) = 0, \forall \boldsymbol{x}$ in (15). Following the definition of a HOCBF as in Def. 1, we are now ready to recursively define a sequence of differentiable and robust CBFs in the form:

$$\begin{aligned}\phi_i(\boldsymbol{x}) &:= L_f\phi_{i-1}(\boldsymbol{x}) + \alpha_i(\phi_{i-1}(\boldsymbol{x})) \\ &- \left(log\left(e^{2\frac{d\phi_{i-1}(\boldsymbol{x})}{d\boldsymbol{x}}} + 1\right) - \frac{d\phi_{i-1}(\boldsymbol{x})}{d\boldsymbol{x}}\right) E \geq 0.\end{aligned} \tag{16}$$

where $i \in \{1, \dots, m-1\}$ and $\alpha_i(\cdot), i \in \{1, \dots, m-1\}$ are class $\mathcal{K}$ functions. Since the relative degree of $b(\boldsymbol{x})$ is m, we would have the control show up in the derivative of $\phi_{m-1}(\boldsymbol{x})$. We further define a sequence of sets $\mathcal{C}_i, i \in \{1, \dots, m\}$ similarly as in (4):

$$\mathcal{C}_i := \{\boldsymbol{x} \in X : \phi_{i-1}(\boldsymbol{x}) \geq 0\}, i \in \{1, \dots, m\}. \tag{17}$$

Then, we define a recursively robust HOCBF based on $\phi_{m-1}(\boldsymbol{x})$ as follows.

***Definition** 2:* Let $\phi_i(\boldsymbol{x}), i \in \{1, \dots, m-1\}$ be defined as in (16) and the corresponding sets $\mathcal{C}_i, i \in \{1, \dots, m\}$ defined as in (17). A function $b : \mathbb{R}^n \to \mathbb{R}$ is a recursively robust HOCBF (rrHOCBF) if there exists differentiable class $\mathcal{K}$ functions $\alpha_i, i \in \{1 \dots, m\}$ such that

$$\begin{aligned}\sup_{\boldsymbol{u}\in U}[L_f\phi_{m-1}(\boldsymbol{x}) + L_g\phi_{m-1}(\boldsymbol{x})\boldsymbol{u} + \alpha_m(\phi_{m-1}(\boldsymbol{x})) \\ - \left|\frac{d\phi_{m-1}(\boldsymbol{x})}{d\boldsymbol{x}}\right| E] \geq 0,\end{aligned} \tag{18}$$

for all $\boldsymbol{x} \in \cap_{i=1}^m \mathcal{C}_i$.

Note that we do not use a logarithm function to over-approximate the AVF in (18) since the control has already show up in (18) and we do not need to define another robust CBF. Moreover, this over-approximation would introduce additional conservativeness. Given a rrHOCBF $b(\boldsymbol{x})$, we consider the set of control that satisfies:

$$\begin{aligned}U_{rcbf}(\boldsymbol{x}) := \{\boldsymbol{u} \in U : L_f\phi_{m-1}(\boldsymbol{x}) + L_g\phi_{m-1}(\boldsymbol{x})\boldsymbol{u} \\ + \alpha_m(\phi_{m-1}(\boldsymbol{x})) - \left|\frac{d\phi_{m-1}(\boldsymbol{x})}{d\boldsymbol{x}}\right| E \geq 0\},\end{aligned} \tag{19}$$

We have the following theorem to prove the safety guarantees of the proposed rrHOCBFs:

Theorem 2: Given a rrHOCBF $b(\boldsymbol{x})$ as in Def. 2 with the associated sets $\mathcal{C}_i, i \in \{1, \dots, m\}$ defined as in (17), if $\boldsymbol{x}(0) \in \cap_{i=1}^m \mathcal{C}_i$, then any Lipschitz continuous controller $\boldsymbol{u}(t) \in U_{rcbf}, \forall t \geq 0$ renders the set $\cap_{i=1}^m \mathcal{C}_i$ forward invariant for stochastic system (6).

Proof: Since $|\boldsymbol{\epsilon}| \leq E$, we can infer from the rrHOCBF constraint in (18) that:

$$\begin{aligned}L_f\phi_{m-1}(\boldsymbol{x}) + L_g\phi_{m-1}(\boldsymbol{x})\boldsymbol{u} + \alpha_m(\phi_{m-1}(\boldsymbol{x})) \\ \geq \left|\frac{d\phi_{m-1}(\boldsymbol{x})}{d\boldsymbol{x}}\right| E \geq -\frac{d\phi_{m-1}(\boldsymbol{x})}{d\boldsymbol{x}}\boldsymbol{\epsilon},\end{aligned} \tag{20}$$

Referring to the dynamics (6), the equation (20) is equivalent to

$$\dot{\phi}_{m-1}(\boldsymbol{x}, \boldsymbol{u}) + \alpha_m(\phi_{m-1}(\boldsymbol{x})) \geq 0, \tag{21}$$

Further, by Thm. 1, we have that $\phi_{m-1}(\boldsymbol{x}(t)) \geq 0, \forall t \geq 0$ since $\phi_{m-1}(\boldsymbol{x}(0)) \geq 0$, where $\phi_{m-1}(\boldsymbol{x})$ is defined as in (16).

If $\frac{d\phi_{m-2}(\boldsymbol{x})}{d\boldsymbol{x}} \geq 0$, we have that

$$log\left(e^{2\frac{d\phi_{m-2}(\boldsymbol{x})}{d\boldsymbol{x}}} + 1\right) \geq 2\frac{d\phi_{m-2}(\boldsymbol{x})}{d\boldsymbol{x}}, \tag{22}$$

otherwise, since $log\left(e^{2\frac{d\phi_{m-2}(\boldsymbol{x})}{d\boldsymbol{x}}} + 1\right) \geq 0$, we have that

$$log\left(e^{2\frac{d\phi_{m-2}(\boldsymbol{x})}{d\boldsymbol{x}}} + 1\right) - \frac{d\phi_{m-2}(\boldsymbol{x})}{d\boldsymbol{x}} \geq -\frac{d\phi_{m-2}(\boldsymbol{x})}{d\boldsymbol{x}}, \tag{23}$$

Therefore, we have that

$$log\left(e^{2\frac{d\phi_{m-2}(\boldsymbol{x})}{d\boldsymbol{x}}} + 1\right) - \frac{d\phi_{m-2}(\boldsymbol{x})}{d\boldsymbol{x}} \geq \left|\frac{d\phi_{m-2}(\boldsymbol{x})}{d\boldsymbol{x}}\right|. \tag{24}$$

Given $\phi_{m-1}(\boldsymbol{x})$ as in (16), we have that

$$\begin{aligned}&L_f\phi_{m-2}(\boldsymbol{x}) + L_g\phi_{m-2}(\boldsymbol{x})\boldsymbol{u} + \alpha_{m-1}(\phi_{m-2}(\boldsymbol{x})) \\ &\geq \left(log\left(e^{2\frac{d\phi_{m-2}(\boldsymbol{x})}{d\boldsymbol{x}}} + 1\right) - \frac{d\phi_{m-2}(\boldsymbol{x})}{d\boldsymbol{x}}\right) E \\ &\geq \left|\frac{d\phi_{m-2}(\boldsymbol{x})}{d\boldsymbol{x}}\right| E \geq -\frac{d\phi_{m-2}(\boldsymbol{x})}{d\boldsymbol{x}}\boldsymbol{\epsilon},\end{aligned} \tag{25}$$

which is equivalent to $\dot{\phi}_{m-2}(\boldsymbol{x}) + \alpha_{m-1}(\phi_{m-2}(\boldsymbol{x})) \geq 0$. Since we have that $\phi_{m-1}(\boldsymbol{x}(t)) \geq 0, \forall t \geq 0$ and $\phi_{m-2}(\boldsymbol{x}(0)) \geq 0$, by Thm. 1, we have that $\phi_{m-2}(\boldsymbol{x}(t)) \geq 0, \forall t \geq 0$. If we use other over-approximation functions (such as the square root function), the proof is similar.

Recursively, we can prove that $\phi_i(\boldsymbol{x}(t)) \geq 0, \forall t \geq 0, \forall i \in \{0, \dots, m-1\}$. Therefore, the set $\cap_{i=1}^m \mathcal{C}_i$ is forward invariant for stochastic system (6). ■

Remark 1: (**Conservativeness of rrHOCBFs**) As shown in the proof of Thm. 2, we always consider the bound E of random process $\boldsymbol{\epsilon}$ in defining a robust CBF $\phi_i(\boldsymbol{x})$. Although with safety guarantees, this could make the system very conservativeness, and thus significantly reduces the system performance. In addition, the over-approximation of the AVF using the logarithm function could also introduce additional conservativeness. We shown in the next subsection how to address this conservativeness issue.

Example revisited. Consider the example in Sec. IV-A, we can first define a differentiable robust CBF $\phi_1(\boldsymbol{x})$ to enforce the safety constraint $b(\boldsymbol{x}) \geq 0$ in the form:

$$\begin{aligned}\phi_1(\boldsymbol{x}) &= \dot{b}(\boldsymbol{x}) + \alpha_1(b(\boldsymbol{x})) \\ &= 2(x - x_0)v\cos\theta + 2(y - y_0)v\sin\theta + \alpha_1(b(\boldsymbol{x})) \\ &- (log(e^{4(x-x_0)} + 1) - 2(x - x_0))E_1 \\ &- (log(e^{4(y-y_0)} + 1) - 2(y - y_0))E_2,\end{aligned} \tag{26}$$

The relative degree of $b(\boldsymbol{x})$ is two in this case, then we would define another robust CBF $\phi_2(\boldsymbol{x}, \boldsymbol{u})$ based on $\phi_1(\boldsymbol{x})$ that is now differentiable. Then $\phi_2(\boldsymbol{x}, \boldsymbol{u}) \geq 0$ would be the recursively robust HOCBF that enforces safety.

C. Non-conservative rrHOCBFs

In this section, we show how we may address the conservativeness issue of rrHOCBFs. The conservativeness of rrHOCBFs mainly comes from the robust term in (16) since we always consider the bound E of the random process $\boldsymbol{\epsilon}$. Although the class $\mathcal{K}$ functions $\alpha_i, i \in \{1, \ldots, m-1\}$ could also introduce some conservativeness, this can be addressed using adaptive CBFs [4].

In order to address the conservativeness, we make the bound E of the random process $\boldsymbol{\epsilon}$ adaptive. In other words, given a safety constraint $b(\boldsymbol{x}) \geq 0$ for stochastic system (6), we make the bound dependent on the value of the CBF at each iteration, and define the following robust CBFs:

$$\begin{aligned}\phi_i(\boldsymbol{x}) &:= L_f\phi_{i-1}(\boldsymbol{x}) + \alpha_i(\phi_{i-1}(\boldsymbol{x})) \\ &- \left(log\left(e^{2\frac{d\phi_{i-1}(\boldsymbol{x})}{d\boldsymbol{x}}} + 1\right) - \frac{d\phi_{i-1}(\boldsymbol{x})}{d\boldsymbol{x}}\right) \\ &\times (E - \beta_i(\phi_{i-1}(\boldsymbol{x}) - \gamma)) \geq 0, i \in \{1, \ldots, m-1\},\end{aligned} \tag{27}$$

where $\beta_i, i \in \{1, \ldots, m-1\}$ are extended class $\mathcal{K}$ functions, and $\gamma > 0$ is a scalar.

In (27), we have that the bound E of the random process $\boldsymbol{\epsilon}$ is significantly relaxed by $\beta_i(\phi_{i-1}(\boldsymbol{x}) - \gamma)$ when $\phi_{i-1}(\boldsymbol{x}) > \gamma$, which allows the system state to get close to the safe set boundary. On the other hand, when $\phi_{i-1}(\boldsymbol{x}) < \gamma$, we have that the bound becomes larger in defining the robust CBFs. This could, of course, push the system state away faster from the safe set boundary. This shows the flexibility and adaptivity of the proposed robust CBFs.

We define a sequence of sets $\mathcal{C}_i, i \in \{1, \ldots, m\}$ similarly as in (17), and define a non-conservative and robust HOCBF:

***Definition** 3:* Let $\phi_i(\boldsymbol{x}), i \in \{1, \ldots, m-1\}$ be defined as in (27) and the corresponding sets $\mathcal{C}_i, i \in \{1, \ldots, m\}$ defined as in (17). A function $b : \mathbb{R}^n \to \mathbb{R}$ is a non-conservative and robust HOCBF (nrHOCBF) if there exists differentiable class $\mathcal{K}$ functions $\alpha_i, i \in \{1 \ldots, m\}$ such that

$$\begin{aligned}\sup_{\boldsymbol{u} \in U} [L_f\phi_{m-1}(\boldsymbol{x}) + L_g\phi_{m-1}(\boldsymbol{x})\boldsymbol{u} + \alpha_m(\phi_{m-1}(\boldsymbol{x})) \\ - \left|\frac{d\phi_{m-1}(\boldsymbol{x})}{d\boldsymbol{x}}\right| (E - \beta_m(\phi_{m-1}(\boldsymbol{x}) - \gamma))] \geq 0,\end{aligned} \tag{28}$$

for all $\boldsymbol{x} \in \cap_{i=1}^m \mathcal{C}_i$, where β_m is an extended class $\mathcal{K}$ function.

We also have the following theorem to prove the safety guarantees of the proposed nrHOCBFs:

Theorem 3: Given a nrHOCBF $b(\boldsymbol{x})$ as in Def. 3 with the associated sets $\mathcal{C}_i, i \in \{1, \ldots, m\}$ defined as in (17), if $\boldsymbol{x}(0) \in \cap_{i=1}^m \mathcal{C}_i$, then any Lipschitz continuous controller $\boldsymbol{u}(t)$ that satisfies (28) $\forall t \geq 0$ renders the set $\cap_{i=1}^m \mathcal{C}_i$ forward invariant for stochastic system (6).
The proof is similar to that of Thm. 2.

Remark 2: (**Inter-sampling of nrHOCBFs**) The inter-sampling effect, i.e., constraint satisfaction between discretized time instants, is prevalent in classical CBFs. This problem becomes more obvious in the proposed nrHOCBFs since we allow the system state to get close to the safe set boundary. In other words, we may need to employ a small enough discretized time interval to capture the evolution of state when $\phi_{m-1}(\boldsymbol{x}) \to \gamma$. One possible solution is to use the event-triggered approach proposed in [21]. This will be further studied in future work.

Remark 3: (**Auto-tuning of rrHOCBFs/nrHOCBFs**) There are many hyper parameters to tune in rrHOCBFs/nrHOCBFs, such as the parameters in class $\mathcal{K}$ functions α_i and extended class $\mathcal{K}$ functions β_i, as well as the γ in nrHOCBFs. These hyper parameters are crucial to system performance, and they are usually empirically determined. However, parameter-tuning is non-trivial, and they are application dependent. In order to address this, we may use the proposed BarrierNet [22] to automatically tune the parameters with data from desired system behavior.

Solution to Problem 1. To address Problem 1, we use a rrHOCBF or nrHOCBF to enforce the safety constraint (9), and use a CLF to enforce the terminal state constraint in the cost (8). Then, we can formulate a rrHOCBF/nrHOCBF-CLF based QP, and use the time discretization method introduced at the end of Sec. II to solve it.

Computation complexity. The computation complexity of the rrHOCBF and nrHOCBF based methods are similar to that of HOCBFs. The computation complexity is $\mathcal{O}(q^3)$.

V. CASE STUDIES

In this section, we consider a robot 2D obstacle avoidance example. We use the *Quadprog* to solve the QP, and use the *ODE45* to integrate the dynamics in *MATLAB*. All the computation runs on a *Intel(R) Core(TM) i7-10750H CPU @ 2.60GHz 2.59 GHz* computer. The computation time for each QP is less than $0.01s$.

We consider the stochastic dynamics as defined in (10) for the robot, and the safety constraint is defined as in (11). The objective function (8) is explicitly defined as $\min_{\boldsymbol{u}(t)} \int_0^T ||\boldsymbol{u}(t)||^2 dt + p_0||(x, y)) - (x_d, y_d))||^2$, where $(x_d, y_d) \in \mathbb{R}^2$ is a desired destination. We use a rrHOCBF or nrHOCBF to enforce the safety constraint (11), and use a CLF to drive the robot to the desired location (x_d, y_d).

Simulation parameters. We define all the class $\mathcal{K}$ and extended class $\mathcal{K}$ functions as linear functions. Other parameters are $\boldsymbol{x}(0) = (-20m, 0m, 0rad, 1m/s), (x_0, y_0) = (0, 0)m, (x_d, y_d) = (20, 0)m, r = 7m$, $u_{1,max} = -u_{1,min} = -1.8rad/s$, $u_{2,max} = -u_{2,min} = 2m/s^2$. All the class $\mathcal{K}$ functions are with slope 1, and the extended class $\mathcal{K}$ functions β_1, β_2 are with slope $0.1, 0.01$, respectively. $\gamma = 5, p_0 = 1, T = 50s$. The random processes ϵ_i take samples from $[-0.5, 0.5]$ with equal probability, and thus $E_i = 0.5$.

We consider four approaches to enforce the safety (11).

1) HOCBF: a HOCBF that employs extended class $\mathcal{K}$ functions to ensure robustness to noise, i.e., ignores all the random variables $\epsilon_i, i \in \{\{1, \ldots, 4\}$.
2) rrHOCBF-log: a recursively robust HOCBF that use a logarithm function to over-approximate the AVF.
3) rrHOCBF-sqrt: a recursively robust HOCBF that use a square-root function to over-approximate the AVF.
4) nrHOCBF: a non-conservative and robust HOCBF.

Results. The simulation results are shown in Figs. 1 - 2. The HOCBF method that employs extended class $\mathcal{K}$ functions to

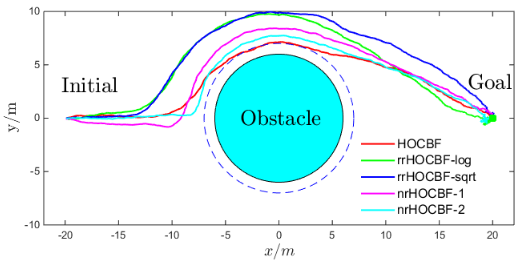

Fig. 1. Comparison of robot trajectories using different types of HOCBF in enforcing safety. Safety is violated in the HOCBF method, and the system tends to be conservative in rrHOCBFs. We can address the conservativeness in the nrHOCBF method, i.e., the system trajectory can stay close to the safe set boundary while guaranteeing safety.

ensure robustness to random process can still violate the safety constraint, as shown by the red trajectory in Figs. 1 and 2, although the system state will always be driven to the safe set whenever the safety constraint is violated. Both the rrHOCBFs employing logarithm and square-root approximation functions can make the system very conservative, as shown by the green and blue trajectories in Figs. 1 and 2. In nrHOCBFs, we can address the conservativeness by tuning the extended class $\mathcal{K}$ functions, as shown by the magenta (β_1 with slope 0.02) and cyan (β_1 with slope 0.1) trajectories in Figs. 1, and thus make the system state stay close to the safe set boundary while guaranteeing safety. This demonstrates the advantages of the proposed nrHOCBFs.

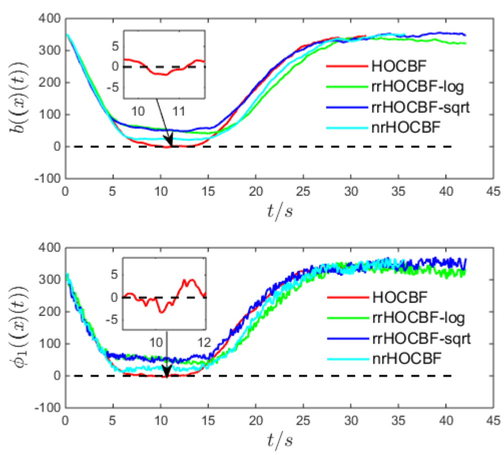

Fig. 2. Comparison of safety functions (CBFs) using different types of HOCBF in enforcing safety. $b(\boldsymbol{x}) \geq 0$ and $\phi(\boldsymbol{x}) \geq 0$ imply the forward invariance of the safe set $\mathcal{C}_1 \cap \mathcal{C}_2$.

VI. CONCLUSION & FUTURE WORK

This paper proposes a recursively robust high order control barrier function and a non-conservative and robust high order control barrier function to address challenges in enforcing safety for stochastic systems with arbitrary relative degree. The proposed methods can avoid high order derivatives of random variables that are intractable or difficult to evaluate, as well as address the conservativeness of the robust control method that is widespread in the literature. We validate the proposed methods on a 2D obstacle avoidance example with results showing tunable conservativeness in guaranteeing safety. Future work will focus on addressing the inter-sampling effect of the proposed methods by using event triggered approaches [21], as well as how to learn all the hyper parameters of the controller using machine learning.

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