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Constructible Canonical Form and High-gain Observer in Discrete Time

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Abstract— **This work presents a triangular form that is shown to be canonical for constructible discrete-time systems. For this form, we propose an observer that resembles the well-known high-gain observer in continuous time. This discrete-time observer exhibits exponential stability if its dynamics are picked sufficiently fast, as well as robustness against disturbances and measurement noise. We also study how to transform general discrete-time systems into this constructible form, under constructibility and backward distinguishability, and recover convergence in the given coordinates. Application to an electrical machine with comparison to the discretized version of the continuous-time high-gain observer illustrates our methods.**

I. INTRODUCTION

Observers are algorithms designed to estimate online a system's state from their known outputs and inputs [1]. From an application point of view, the implementation of observers in discrete time may be computation-wise lighter than in continuous time, especially when the system has known inputs that we need to store. The existing literature on discrete-time observers includes:

- LMI-based approaches [2], assuming detectability of the linear part and not guaranteeing the LMIs' solvability;
- Kalman(-like) designs: Global results for linear systems include the Kalman [3], [4] and Kalman-like [5] observers, exploiting uniform complete observability (UCO). However, when extended to nonlinear systems, the outcomes are constrained to be *local* [6]–[8]. Other local/linearization techniques include [9]–[13];
- Dead-beat estimators: These rely on the left inversion of the (forward) observability map [14], [15], providing instantaneous estimation but lacking noise filtering effects;
- Moving horizon state estimators [16], [17]: These minimize the estimation error with the observability map made of the m current and future outputs, requiring observability instead of backward distinguishability/constructibility;
- Kravaris-Kazantzis/Luenberger (KKL) designs [18], [19]: These propose transformations into some stable filter of the output where an observer is trivial, and inverting these to recover the estimate in the given coordinates, giving us a global result together with robustness. These rely on backward distinguishability, a fairly light condition. The difficulty lies in the implementation where it is hard to analytically compute these transformations, which is addressed by neural network-based approximations [20].

In discrete time, the notions of observability [21] and constructibility [22], [23] (or reconstructibility in some literature [24]) differ mainly in the invertibility of the dynamics. To be more precise, given a finite sequence of outputs (and possibly known inputs), the former corresponds to uniquely determining from these the initial state, and the latter typically means determining the final state. If we manage to retrieve the initial state, we can always proceed with the system's dynamics to reach the final state, indicating that observability classically implies constructibility. However, when the dynamics lack invertibility, the initial state may not be recoverable. It is crucial to emphasize that the observer does not necessarily require this capability, as its primary task is to estimate the current state. For linear systems, [25] distinguishes these notions and proposes a constructible canonical form with a linear output. For nonlinear systems, [22] introduced constructibility as a local property. The recent work [23] proposes in the nonlinear setting a notion of backward observability in the same spirit as constructibility but written mainly in terms of the observation space of the linearized systems (from a differential geometry point of view). Surprisingly, so far the only observer design under backward distinguishability or constructibility that we know of is the KKL design [18], [19]. The dead-beat or moving-horizon estimators could work in this framework, but they typically assume observability, which is stronger.

This work has three main contributions. First, it proposes a nonlinear *constructible canonical form* for nonlinear time-varying discrete-time systems, unlike [23] and [12, Section 4] where the forms are linear (modulo output injection). We redefine constructibility as the ability to express the current state as a function of a finite number of past outputs (and known inputs) and show that this is necessary and sufficient for a transformation into this canonical form. Second, we present a form of a *high-gain* observer in discrete time, where exponential stability and robustness can be achieved by pushing faster the convergence rate. This differs from those in [23] and [12, Section 4] where the observers are linear modulo output injection. We highlight that very few works in the literature propose global nonlinear discrete-time observer designs that exhibit robustness with respect to disturbances and measurement noise, e.g., [16], [17], [19]. Implementation-wise, this new design is more handy than the KKL one, asking for the same or even lighter observability conditions, and does not necessarily require invertibility of the dynamics. Third, we link this new notion of constructibil-

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ity with backward distinguishability to propose a transformation from a backward distinguishable system into a canonical form, providing a constructive design framework for general nonlinear systems. To illustrate our methods, we apply them to a strategically discretized permanent magnet synchronous motor (PMSM), comparing the proposed observer with a naively discretized version of the continuous-time high-gain observer, to show that it is better to directly design and then implement the observer in discrete time, something we have already seen in [19] for the KKL observer.

Notations: Let $\mathbb{R}$ (resp., $\mathbb{N}$) denote the set of real numbers (resp., natural numbers, i.e., $\{0,1,2,\ldots\}$), and $\mathbb{N}_{\geq m} = \{m, m+1, \ldots\}$ for some $m \in \mathbb{N}$. Let $\mathbb{R}^{m\times n}$ be the set of real $(m \times n)$-dimensional matrices. Denote $S + c$ as the set of points that lie within a distance less than or equal to $c > 0$ from a point in the set S. For a sequence $(x_k)_{k\in\mathbb{N}}$ of vectors in $\mathbb{R}^m$ indexed by the discrete time $k \in \mathbb{N}$, x_k is the vector at time k, while $x_{i,k}$ denotes its i^{th} component at time k. The vector norm is denoted as $|\cdot|$. Let Id be the identity map. For two functions f and g, $f \circ g$ is their composition, namely for all x in the domain of g, $g(x)$ is in the domain of f and $(f \circ g)(x) = f(g(x))$.

II. ON NOTION OF CONSTRUCTIBILITY

In this section, we introduce and analyze the notion of constructibility of nonlinear (time-varying) discrete-time systems. Consider systems of the form

$$x_{k+1} = f_k(x_k, y_k), \qquad y_k = h_k(x_k), \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ and $y_k \in \mathbb{R}^{n_y}$ are the state and the measured output at discrete time $k \in \mathbb{N}$; $f_k : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_x}$ and $h_k : \mathbb{R}^{n_x} \to \mathbb{R}^{n_y}$ are the dynamics and output maps. The time dependence of f_k and h_k may capture their dependence on inputs u_k, seen as known functions of time.

A. Constructibility and Constructible Canonical Form

This work is based on the following definition.

Definition 1. *System* (1) *is* constructible *(or in some literature,* reconstructible*) of order m if there exist $m \in \mathbb{N}$ and a map sequence $(\Psi_k)_{k\in\mathbb{N}_{\geq m}}$ such that for all solutions $k \mapsto x_k$ to system* (1)*, we have*

$$x_k = \Psi_k(y_{k-1}, \ldots, y_{k-m}), \qquad k \in \mathbb{N}_{\geq m}. \tag{2}$$

In the linear context, this property is known to be weaker than *observability* when the dynamics are not invertible [25]. Observability would instead require x_k to be a function of the *future* outputs (or x_{k-m} as a function of $(y_{k-1}, \ldots, y_{k-m})$ in (2)), which is easily checked with the Kalman criterion but is not necessary for observer design. In the nonlinear context, constructibility notions are studied in [23], but from the local point of view of differential geometry, and in [22, Proposition 1] as a local property.

Remark 1. *Note that, in Definition 1, if we write instead $x_k = \Psi'_k(y_k, \ldots, y_{k-(m-1)})$ for all $k \in \mathbb{N}_{\geq m-1}$ (as in [22]), it is implied that* (2) *holds with $\Psi_k = f_{k-1} \circ \Psi'_{k-1}$. The converse is true if each f_k is invertible. Also, observability, i.e., $x_k = \phi_k(y_k, y_{k+1}, \ldots, y_{k+(m-1)})$ for all $k \in \mathbb{N}$, implies constructibility. Indeed, we have for all $k \in \mathbb{N}_{\geq m}$,*

$$x_k = F_k(x_{k-m}) = (F_k \circ \phi_{k-m})(y_{k-m}, y_{k-(m-1)}, \ldots, y_{k-1}),$$

where $F_k = (f_{k-1} \circ f_{k-2} \circ \ldots \circ f_{k-m})$.

Lemma 1 below shows that the constructibility of system (1) is necessary and sufficient for it to be transformed into what we call a *constructible canonical form* (4).

Lemma 1. *The following statements are equivalent:*
1) System (1) *is constructible of order m;*
2) There exist map sequences $(\mathcal{T}_k)_{k\in\mathbb{N}_{\geq m}}$, $(\varphi_{i,k})_{k\in\mathbb{N}}$ with $i \in \{1, 2, \ldots, m\}$, and $(\gamma_k)_{k\in\mathbb{N}_{\geq m}}$ such that for all solutions $k \mapsto x_k$ to system (1)*, we have*

$$x_k = \mathcal{T}_k(z_k), \qquad k \in \mathbb{N}_{\geq m}, \tag{3}$$

with $k \mapsto z_k$ solution for all $k \in \mathbb{N}_{\geq m}$ to the dynamics

$$\begin{cases} z_{1,k+1} = \varphi_{1,k}(y_k) \\ z_{2,k+1} = \varphi_{2,k}(z_{1,k}, y_k) \\ \quad \cdots \\ z_{i,k+1} = \varphi_{i,k}(z_{1,k}, \ldots, z_{i-1,k}, y_k) \\ \quad \cdots \\ z_{m,k+1} = \varphi_{m,k}(z_{1,k}, \ldots, z_{m-1,k}, y_k), \end{cases} \tag{4a}$$

with the measured output

$$y_k = \gamma_k(z_k). \tag{4b}$$

Proof. First, if 1) holds, then by Definition 1, there exists $(\Psi_k)_{k\in\mathbb{N}_{\geq m}}$ and we define for all $k \in \mathbb{N}_{\geq m}$ the maps $\mathcal{T}_k = \Psi_k$ and $\gamma_k = h_k \circ \mathcal{T}_k$, and for all $k \in \mathbb{N}$ the maps $\varphi_{i,k}$, $i \in \{1, 2, \ldots, m\}$ as $\varphi_{1,k} = \mathrm{Id}, \varphi_{2,k}(z_1, y) = z_1, \varphi_{3,k}(z_1, z_2, y) = z_2$, etc., which gives us 2). On the other hand, if 2) holds, then because we have for all $k \in \mathbb{N}_{\geq m}$, $z_{1,k} = \varphi_{1,k-1}(y_{k-1})$, $z_{2,k} = \varphi_{2,k-1}(z_{1,k-1}, y_{k-1}) = \varphi_{2,k-1}(\varphi_{1,k-2}(y_{k-2}), y_{k-1})$, etc., we have $x_k = \mathcal{T}_k(z_k)$ as a function of only $(y_{k-1}, \ldots, y_{k-m})$ which gives us 1). ☐

The following example (see [26] for other ones inspired from [23]) illustrates that we can rely on constructibility to transform the system into the constructible canonical form (4).

Example 1. *Consider the system in [23, Section I]:*

$$\begin{cases} x_{1,k+1} = u_k \\ x_{2,k+1} = x_{3,k} \\ x_{3,k+1} = x_{1,k} + x_{2,k} u_k \end{cases} \qquad y_k = x_{3,k}, \tag{5}$$

where u_k is some known input. This system is not observable (see in [23, Example 1]), but it is constructible because x_k for all $k \in \mathbb{N}_{\geq 3}$ can be expressed as function of the past y_k and u_k. We see that $x_k = (z_{2,k}, z_{1,k}, z_{3,k})$, where z_k follows dynamics of the form (4)*:*

$$\begin{cases} z_{1,k+1} = y_k \\ z_{2,k+1} = u_k \\ z_{3,k+1} = z_{1,k} u_k + z_{2,k} \end{cases} \qquad y_k = z_{3,k}. \tag{6}$$

However, such transformations are situational since it is not clear how they can be obtained from constructibility. We then propose in the next part a constructive way, when possible, to find this transformation from the system's maps.

B. Transformation into a Constructible Canonical Form

We introduce the following definition.

Definition 2. *System* (1) *is* backward distinguishable *of order m if there exists $m \in \mathbb{N}$ and a sequence of maps $(\mathcal{O}_k^{bw})_{k\in\mathbb{N}_{\geq m}}$ with*

$$\mathcal{O}_k^{bw}(x) = (h_{-1,k}(x), h_{-2,k}(x), \dots, h_{-m,k}(x)), \tag{7a}$$

such that for all $k \in \mathbb{N}_{\geq m}$,

$$h_{-i,k}(x_k) = y_{k-i}, \qquad \forall i \in \{1, 2, \dots, m\}, \tag{7b}$$

along any solution $k \mapsto x_k$ to system (1)*, and $(\mathcal{O}_k^{bw})_{k\in\mathbb{N}_{\geq m}}$ is injective.*

Remark 2. *Definition 2 is the constructibility counterpart of the observability condition assumed in [14], [15], [21]. It does not require the maps f_k to be invertible. If f_k is independent of y_k and each map f_k is invertible with the corresponding inverse function f_k^{-1} defined on $\mathbb{R}^{n_x}$, we can define $(\mathcal{O}_k^{bw})_{k\in\mathbb{N}_{\geq m}}$ from these inverses as in [18], [19].*

Lemma 2. *If system* (1) *is* backward distinguishable *of order m, then the variable $z_k = \mathcal{O}_k^{bw}(x_k) \in \mathbb{R}^{n_z}$ where $n_z = mn_y$, along the solutions $k \mapsto x_k$ to system* (1)*, for all $k \in \mathbb{N}_{\geq m}$, is solution to the dynamics*

$$\begin{cases} z_{1,k+1} = y_k \\ z_{2,k+1} = z_{1,k} \\ \quad\dots \\ z_{i,k+1} = z_{i-1,k} \\ \quad\dots \\ z_{m,k+1} = z_{m-1,k}, \end{cases} \tag{8a}$$

with the measured output

$$y_k = h_k(\mathcal{O}_k^{bw,-1}(z_k)), \tag{8b}$$

where $\mathcal{O}_k^{bw,-1}$ is the left inverse of $\mathcal{O}_k^{bw}$ on $\mathbb{R}^{n_x}$. Moreover,

$$x_k = \mathcal{O}_k^{bw,-1}(z_k), \qquad \forall k \in \mathbb{N}_{\geq m}. \tag{9}$$

Proof. See in the full version [26]. □

This property is used in Section IV to design an observer for a discretized PMSM. The form (8), obtained through backward distinguishability, is a particular case of the form (4) and was also obtained under constructibility in the proof of Lemma 1. The difference is that in (7), the past outputs are expressed as an injective function of x_k (and thus z_k is a function of x_k in system (8) and vice-versa), while in (2), x_k is directly written as a function of the past outputs (and thus we only have x_k as a function of z_k in system (4)). In other words, backward distinguishability is sufficient, but not necessary, for constructibility. This is illustrated in Example 2.

Example 2. *Consider the system with dynamics and output*

$$x_{k+1} = x_k^2, \qquad y_k = x_k. \tag{10}$$

We have $x_k = y_{k-1}^2$ for all $k \in \mathbb{N}_{\geq 1}$ so that this system is constructible. However, given a current state x_k at a time $k \in \mathbb{N}_{\geq 1}$, there exist two corresponding past outputs y_{k-1} and $-y_{k-1}$, so that we cannot write the map (7) *and this system is therefore not backward distinguishable.*

III. OBSERVER FOR A CONSTRUCTIBLE FORM

A. Observer Design

Consider a system in the constructible canonical form (4) where $z_k \in \mathbb{R}^{n_z}$ is the state, and y_k is the measured output in $\mathbb{R}^{n_y}$. For system (4), we propose the following observer:

$$\begin{cases} \hat{z}_{1,k+1} = \bar{\varphi}_{1,k}(y_k) + \theta^m c_1(y_k - \bar{\gamma}_k(\hat{z}_k)) \\ \hat{z}_{2,k+1} = \bar{\varphi}_{2,k}(\hat{z}_{1,k}, y_k) + \theta^{m-1} c_2(y_k - \bar{\gamma}_k(\hat{z}_k)) \\ \quad\dots \\ \hat{z}_{i,k+1} = \bar{\varphi}_{i,k}(\hat{z}_{1,k}, \dots, \hat{z}_{i-1,k}, y_k) \\ \qquad\qquad + \theta^{m-i+1} c_i(y_k - \bar{\gamma}_k(\hat{z}_k)) \\ \quad\dots \\ \hat{z}_{m,k+1} = \bar{\varphi}_{m,k}(\hat{z}_{1,k}, \dots, \hat{z}_{m-1,k}, y_k) \\ \qquad\qquad + \theta c_m(y_k - \bar{\gamma}_k(\hat{z}_k)), \end{cases} \tag{11}$$

where $c_i \in \mathbb{R}$, $i \in \{1, 2, \dots, m\}$ and $\theta \in [0, 1]$ are design parameters to be selected, which may be 0, and the maps $\bar{\varphi}_{i,k}$, $i \in \{2, 3, \dots, m\}$ and $\bar{\gamma}_k$ are such that Item (A1.2) of Assumption 1 below holds.

Assumption 1. *Assume that:*

(A1.1) There exist sets $\mathcal{Z}_0 \subset \mathbb{R}^{n_z}$, $\mathcal{Y} \subset \mathbb{R}^{n_y}$, $\mathcal{W} \subset \mathbb{R}^{n_z}$, and $\mathcal{Z} \subset \mathbb{R}^{n_z}$ such that the solutions $k \mapsto z_k$ to system (4) *of interest are initialized in $\mathcal{Z}_0$, have outputs $y_k \in \mathcal{Y}$ and possibly disturbance $w_k \in \mathcal{W}$ for all $k \in \mathbb{N}$, remain in $\mathcal{Z}$ for all $k \in \mathbb{N}$;*

(A1.2) The maps $\varphi_{i,k}$, $\bar{\varphi}_{i,k}$, $i \in \{1, 2, \dots, m\}$ and γ_k, $\bar{\gamma}_k$ are such that there exist $L_{z,i}, L_{y,i} > 0$ (for each $i \in \{1, 2, \dots, m\}$) and $L_\gamma > 0$ such that for all $(z_k, \hat{z}_k, k, y_k, v_k) \in \mathcal{Z} \times \mathbb{R}^{n_z} \times \mathbb{N} \times \mathcal{Y} \times \mathbb{R}^{n_y}$,

$$|\varphi_{i,k}(z_k, y_k) - \bar{\varphi}_{i,k}(\hat{z}_k, y_k + v_k)| \leq L_{z,i}|z_k - \hat{z}_k| + L_{y,i}|v_k|,$$
$$|\gamma_k(z_k) - \bar{\gamma}_k(\hat{z}_k)| \leq L_\gamma |z_k - \hat{z}_k|.$$

Remark 3. *In the case where $\varphi_{i,k}$ (resp., γ_k) is globally Lipschitz with respect to z_k (uniformly in $k \in \mathbb{N}$), we can take $\bar{\varphi}_{i,k} = \varphi_{i,k}$ (resp., $\bar{\gamma}_k = \gamma_k$). In another case where the set $\mathcal{Z}$ is compact and the map $\varphi_{i,k}$ is locally Lipschitz with respect to z_k, uniformly in $k \in \mathbb{N}$ and $y_k \in \mathcal{Y}$ (resp., the map γ_k is locally Lipschitz with respect to z_k, uniformly in $k \in \mathbb{N}$), we take $\bar{\varphi}_{i,k}$ (resp., $\bar{\gamma}_k$) as a bounded map that coincides with $\varphi_{i,k}$ (resp., γ_k) for all $z_k \in \mathcal{Z} + c$, for some $c > 0$. This way, Item (A1.2) of Assumption 1 is satisfied.*

Because system (4) is constructible for all $k \in \mathbb{N}_{\geq m}$, we can neglect the correction terms in observer (11), i.e., pick $\theta = 0$ for an instantaneous convergence after m time steps. But the price to pay is the absence of filtering effects against disturbances and noise. On the other hand, the observer structure (11) somehow resembles the famous high-gain design in continuous time, the difference being that (i) the triangularity constraint is lower diagonal, (ii) all the maps may be nonlinear, and (iii) there are no constraints on the choice of the scalars c_i.

Theorem 1 shows exponential stability of the estimation error with an arbitrarily fast rate (by pushing θ smaller).

Theorem 1. *Under Assumption 1, for any choice of c_i, $i \in \{1,2,\dots,m\}$, there exists $\theta^\star > 0$ such that any solution $k \mapsto z_k$ to system* (4) *initialized in $\mathcal{Z}_0$ with $y_k \in \mathcal{Y}$ for all $k \in \mathbb{N}$ and any solution $k \mapsto \hat{z}_k$ to observer* (11) *with $0 < \theta < 1$, initialized in $\mathbb{R}^{n_z}$ and fed with y_k in* (4b)*, verify:*

$$|z_k - \hat{z}_k| \le \frac{1}{\theta^{m-1}} \left(\frac{\theta}{\theta^\star}\right)^k |z_0 - \hat{z}_0|, \qquad \forall k \in \mathbb{N}. \quad (12)$$

Proof. See in the full version [26]. Sketch of proof: i) Compute the dynamics of the estimation error $\tilde{z}_k = z_k - \hat{z}_k$ and of the re-scaled error ε_k where $\varepsilon_{i,k} = \theta^{i-1}\tilde{z}_{i,k}$ for $i \in \{1,2,\dots,m\}$, and ii) Establish the bounds of the nonlinear terms thanks to Item (A1.2) of Assumption 1 and come back to the z-coordinates. □

Remark 4. *Notice that, unlike the continuous-time high-gain observer, this observer is arbitrarily fast only after m steps. A special case of our observer has already been proposed in [23, Section V.A] and [12, Section 4]. Here, we try to be as general as possible by allowing $z_{i,k+1}$ to depend on not only $z_{i-1,k}$ but also the whole $(z_{1,k},\dots,z_{i-1,k})$, and the output y_k to be nonlinear in z_k.*

B. Robustness of the Observer

Consider system (4) with disturbance $w_k \in \mathbb{R}^{n_z}$

$$\begin{cases} z_{1,k+1} = \varphi_{1,k}(y_k) & + w_{1,k} \\ z_{2,k+1} = \varphi_{2,k}(z_{1,k}, y_k) & + w_{2,k} \\ \quad\cdots & \\ z_{i,k+1} = \varphi_{i,k}(z_{1,k},\dots,z_{i-1,k}, y_k) & + w_{i,k} \\ \quad\cdots & \\ z_{m,k+1} = \varphi_{m,k}(z_{1,k},\dots,z_{m-1,k}, y_k) & + w_{m,k}, \end{cases} \quad (13a)$$

and measurement noise $v_k \in \mathbb{R}^{n_y}$ added to the output

$$y_k + v_k. \quad (13b)$$

The disturbance $w_{i,k}$ could also model the non-Lipschitzness of $\varphi_{i,k}$. Theorem 2 shows the robustness of observer (11).

Theorem 2. *Under Assumption 1, for any choice of c_i, $i \in \{1,2,\dots,m\}$, there exists $\theta^\star > 0$ such that any solution $k \mapsto z_k$ to system* (13) *initialized in $\mathcal{Z}_0$ with $y_k \in \mathcal{Y}$ and $w_k \in \mathcal{W}$ for all $k \in \mathbb{N}$ and any solution $k \mapsto \hat{z}_k$ to observer* (11) *with $0 < \theta < 1$, initialized in $\mathbb{R}^{n_z}$ and fed with $y_k + v_k$ in* (13b)*, verify for each $i \in \{1,2,\dots,m\}$, for all $k \in \mathbb{N}$, and for all $j \in \{0,1,\dots,k-1\}$:*

$$\begin{aligned} |z_{i,k} - \hat{z}_{i,k}| &\le \frac{1}{\theta^{i-1}} \left(\frac{\theta}{\theta^\star}\right)^k |z_0 - \tilde{z}_0| \\ &\quad + \sum_{j=0}^{k-1} \left(\frac{\theta}{\theta^\star}\right)^{k-1-j} \sum_{q=1}^{m} \theta^{q-i} |w_{q,j}| \\ &+ \sum_{j=0}^{k-1} \left(\frac{\theta}{\theta^\star}\right)^{k-1-j} \sum_{q=1}^{m} (\theta^{q-i} L_{y,q} + \theta^{m-i+1}|c_q|)|v_j|. \end{aligned} \quad (14)$$

From Theorem 2, we see that with $0 < \theta < \min\{1, \theta^\star\}$:

- The estimation error is *robustly stable* with respect to disturbance and noise, in the sense of [27, Definition 2.3];
- Similarly to the high-gain design in continuous time (see e.g., [28]), the effects of $w_{q,j}$ (past disturbance on line q) on $\tilde{z}_{i,k}$ (current estimation error on line i) can either be magnified or reduced depending on q against i, because of the coefficient θ^{q-i}. Note that, however, the impact of the disturbance on the last line $(w_{m,j})$ can only be reduced (for θ sufficiently small) on the lines $i < m$, which does not give practical convergence, contrary to continuous time;
- In the proof of Theorem 2, it is not evident that we can attenuate the disturbance and noise by choosing the c_i in observer (11) because this proof is done conservatively for the general case of nonlinear maps $\varphi_{i,k}$. However, for the specific form (8) which is widely used in moving horizon schemes, by picking $c_1 < 0$ and $c_i = 0$ for $i \neq 0$, we get a penalization factor in front of the noise.

Proof. See in the full version [26]. □

C. Asymptotic Convergence in the Original x-Coordinates

In Section II, we have seen that constructible systems (1) can be linked to the constructible form (4) via some map (3), at least after m discrete steps. Then, an observer (11) can be designed, and we now study conditions to recover the asymptotic convergence in the x-coordinates. For this, the following assumption is made.

Assumption 2. *There exist a closed set $\mathcal{Z}$ and $m \in \mathbb{N}$ such that system* (1) *is constructible of order m, with $k \mapsto z_k$ in Lemma 2 such that $z_k \in \mathcal{Z}$ for all $k \in \mathbb{N}$, and with $(\mathcal{T}_k)_{k \in \mathbb{N}_{\ge m}}$ uniformly continuous on $\mathcal{Z}$ for all $k \in \mathbb{N}_{\ge m}$. More precisely, there exists a class-$\mathcal{K}$ function ρ such that for all $(z_a, z_b) \in \mathcal{Z} \times \mathcal{Z}$ and for all $k \in \mathbb{N}_{\ge m}$,*

$$|\mathcal{T}_k(z_a) - \mathcal{T}_k(z_b)| \le \rho(|z_a - z_b|). \quad (15)$$

Remark 5. *Assumption 2 is satisfied if either:*

- *System* (1) *is uniformly constructible of order m, i.e., the map sequence $(\Psi_k)_{k \in \mathbb{N}_{\ge m}}$ in* (2) *is uniformly continuous;*
- *Or system* (1) *has solutions remaining in $\mathcal{X}$ and is uniformly backward distinguishable of order m on $\mathcal{X}$, i.e., the map sequence $(\mathcal{O}_k^{bw})_{k \in \mathbb{N}_{\ge m}}$ in* (7) *is uniformly injective on $\mathcal{X}$ for all $k \in \mathbb{N}_{\ge m}$. More precisely, there exists a class-$\mathcal{K}$ function ρ such that $|\mathcal{O}_k^{bw}(x_a) - \mathcal{O}_k^{bw}(x_b)| \ge \rho(|x_a - x_b|)$ for all $(x_a, x_b) \in \mathcal{X} \times \mathcal{X}$ and for all $k \in \mathbb{N}_{\ge m}$.*

In these cases, we have $n_z = m n_y$.

The asymptotic convergence is then recovered in the x-coordinates as follows.

Lemma 3. *Under Assumption 2, any solution $k \mapsto x_k$ to system* (1) *and any solution $k \mapsto \hat{z}_k$ to observer* (11) *with $0 < \theta < \min\{1, \theta^\star\}$ fed with y_k in* (1) *verify $\lim_{k \to +\infty} |x_k - \hat{x}_k| = 0$, where $\hat{x}_k = \bar{\mathcal{T}}_k(\hat{z}_k)$, with $k \in \mathbb{N}_{\ge m}$, and $(\bar{\mathcal{T}}_k)_{k \in \mathbb{N}_{\ge m}}$ is a sequence of extensions of $(\mathcal{T}_k)_{k \in \mathbb{N}_{\ge m}}$ that is uniformly continuous on $\mathbb{R}^{n_z}$.*

Note that a uniformly continuous extension of $\mathcal{T}_k$ from $\mathcal{Z}$ to $\mathbb{R}^{n_z}$, with the same modulus of continuity for all $k \in \mathbb{N}_{\ge m}$, always exists thanks to (15) and [29].

IV. APPLICATION TO AN ELECTRICAL MACHINE

Consider a permanent magnet synchronous motor (PMSM) with model [30]

$$\dot{x} = u - Ri, \qquad y = |x - Li|^2 - \Phi^2 = 0, \tag{16}$$

where $x \in \mathbb{R}^2$ is the electromagnetic flux (in Vs); the voltages u (in V) and currents i (in A) are inputs in $\mathbb{R}^2$; the resistance $R = 1.45$ (Ω), the inductance $L = 0.0121$ (H), and the flux $\Phi = 0.1994$ (Vs) are constant parameters. Here, the value of y is always zero. Let us next build and compare for system (16) two observers: one designed in continuous time and then discretized; the other one designed in discrete time, on a discretized model of (16).

A. Euler Discretization of a Continuous-time Observer

It is known from [30] that the function consisting of $(y, \dot{y}, \ddot{y})$ is uniformly Lipschitz injective if the motor speed is uniformly bounded away from zero. Exploiting this, we perform the uniformly injective transformation

$$z_1 = y = |x - Li|^2 - \Phi^2, \tag{17a}$$
$$z_2 = \dot{y} = 2\eta^\top (x - Li), \tag{17b}$$
$$z_3 = \ddot{y} = 2\dot{\eta}^\top (x - Li) + 2\eta^\top \eta, \tag{17c}$$

where $\eta = u - Ri + L\frac{di}{dt}$. A high-gain observer for system (16) would be of the form

$$\begin{cases} \dot{\hat{z}}_1 = \hat{z}_2 + \ell k_1(\Phi^2 - |\hat{x} - Li|^2) \\ \dot{\hat{z}}_2 = \hat{z}_3 + \ell^2 k_2(\Phi^2 - |\hat{x} - Li|^2) \\ \dot{\hat{z}}_3 = 2\ddot{\eta}^\top(\phi(\hat{z}) - Li) + 2\dot{\eta}^\top\left(u - Ri - L\frac{di}{dt}\right) \\ \qquad + 4\dot{\eta}^\top \eta + \ell^3 k_3(\Phi^2 - |\hat{x} - Li|^2) \\ := \phi_3(\hat{z}) + \ell^3 k_3(\Phi^2 - |\hat{x} - Li|^2), \end{cases} \tag{18a}$$

with the output

$$\hat{x} = \phi(\hat{z}), \tag{18b}$$

where ϕ is a globally Lipschitz left inverse of (17), which depends on u, i, and their derivatives (which may introduce noise); (k_1, k_2, k_3) and ℓ are observer parameters, with $\ell > 0$ having to be pushed large enough. However, only a sampled version of the voltage and current is available and limited computations are possible, both at the fixed PWM rate. We thus have to implement a discrete-time observer. In general, we do not have clear ideas about how observer (18) should be discretized. So, we use a naive Euler discretization scheme with a given sampling period $\tau > 0$, giving us

$$\begin{cases} \hat{z}_{1,k+1} = \hat{z}_{1,k} + \tau(\hat{z}_{2,k} + \ell k_1(\Phi^2 - |\hat{x}_k - Li_k|^2)) \\ \hat{z}_{2,k+1} = \hat{z}_{2,k} + \tau(\hat{z}_{3,k} + \ell^2 k_2(\Phi^2 - |\hat{x}_k - Li_k|^2)) \\ \hat{z}_{3,k+1} = \hat{z}_{3,k} + \tau(\phi_{3,k}(\hat{z}_k) + \ell^3 k_3(\Phi^2 - |\hat{x}_k - Li_k|^2)), \end{cases} \tag{19a}$$

with the output

$$\hat{x}_k = \phi_k(\hat{z}_k). \tag{19b}$$

The results in Figure 1 (with only x_1, x_2 being similar) show an ineffective estimation performance, especially at high speeds, due to the lack of precision of the observer discretization, a phenomenon similar to [19, Section VI.B].

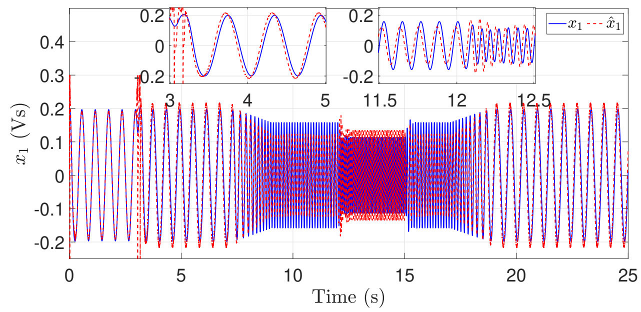

Fig. 1. Estimation results of the high-gain observer designed in continuous time then discretized (19), compared with the continuous-time trajectory.

B. Direct Design and Implementation in Discrete Time

We now propose to strategically discretize the PMSM model first, and then design and implement the discrete-time observer of this paper. For this, an appropriate method to discretize system (16) taking into account its rotating dynamics is [30]

$$x_{k+1} = x_k + \tau\Omega_k(u_k - Ri_k)\,\mathrm{sinc}(\varphi_k) =: x_k + g_k, \tag{20a}$$
$$y_k = |x_k - Li_k|^2 - \Phi^2 = 0, \tag{20b}$$

where $\Omega_k = \begin{pmatrix} \cos(\varphi_k) & -\sin(\varphi_k) \\ \sin(\varphi_k) & \cos(\varphi_k) \end{pmatrix}$, and $\varphi_k = \frac{\omega_k \tau}{2}$ where

$$\omega_k = \mathrm{sign}((u_k - u_{k-1})^\top u_{k-1}) \frac{|u_k - u_{k-1}|}{\tau |u_k|} \tag{20c}$$

is an approximation of the rotation speed of the motor, assuming that this speed does not vary too fast, and

$$\mathrm{sinc}(x) = \begin{cases} \frac{\sin(x)}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases} \tag{20d}$$

This scheme takes into account the physics of the system and so is much more precise compared to the general Euler one, which would be just $x_{k+1} = x_k + \tau(u_k - Ri_k)$. Note that system (20) does not follow a triangular form (4) because $x_{1,k+1}$ depends on $x_{1,k}$, so we deploy the transformation in Lemma 2. Based on the knowledge from continuous time in (17), we conjecture in the same way as in [19] that the maps $(\mathcal{O}_k^{bw})_{k\in\mathbb{N}}$ of order 3 should be uniformly Lipschitz injective if the sampling period τ is sufficiently small. We then perform the change of variables

$$z_{1,k} = |x_k - g_{k-1} - Li_{k-1}|^2 - \Phi^2, \tag{21a}$$
$$z_{2,k} = |x_k - g_{k-2} - g_{k-1} - Li_{k-2}|^2 - \Phi^2, \tag{21b}$$
$$z_{3,k} = |x_k - g_{k-3} - g_{k-2} - g_{k-1} - Li_{k-3}|^2 - \Phi^2, \tag{21c}$$

with g_k defined in (20), depending only on the inputs. Then, we implement observer (11) and obtain the results in Figure 2 with visibly better accuracy compared to Figure 1. This recalls the lesson we have drawn in [19]—instead of designing and then discretizing a continuous-time observer, we should properly discretize the system based on its physics and then build a discrete-time observer. Note that storing the past samples of the inputs as done in this discrete-time design is finite-dimensional and no derivatives need to be computed.

Unfortunately, with $c_1 = c_2 = c_3 = 1$, we need to select an exceedingly small value for θ (in particular, $7 \cdot 10^{-5}$) to make the observer work. This necessity arises from the large Lipschitz constant of the inverse map of (21). The reason behind this lies in the fact that with a low sampling rate of $\tau = 10^{-3}$ (s), past outputs closely resemble each other, resulting in a poorly conditioned transformation. This does not allow us to explore the filtering properties of the observer in this particular example. A way to improve this is to store more past outputs, namely take more dimensions in z_k like in moving horizon estimators [16], [17]. Another way could be to increase the sampling period to make the output different enough, risking the deviation of the discretized model from the real one and hence requiring an even more accurate discretization scheme. An alternative design is the KKL observer, which relies on a transformation that is capable of keeping memories of all the past outputs and which was shown to perform efficiently on this application in [19].

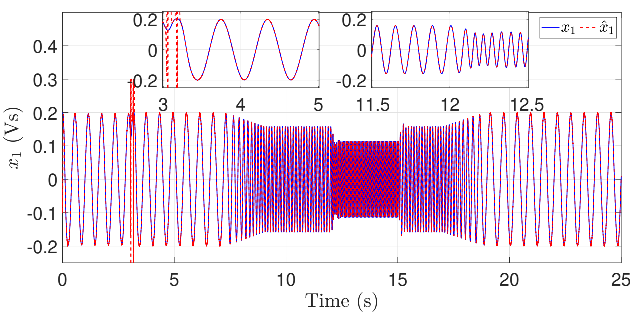

Fig. 2. Estimation results of the high-gain observer designed and implemented in discrete time (11), compared with the continuous-time trajectory.

V. Conclusion and Future Work

In this paper, we propose a constructible canonical form in discrete time and a corresponding high-gain observer. This design exhibits robust exponential stability if the rate is pushed fast enough. We also show how a system can be transformed into a constructible form under constructibility and backward distinguishability, and how convergence can be recovered in the original coordinates. Application to a PMSM and comparison with other designs illustrate our results.

Future work is to understand better the potential gain of performance in terms of noise/disturbance attenuation and figure out results relevant to the continuous-time high-gain observer.

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