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A Game-Theoretic Approach for Optimal Dispatch of Building Thermal Loads Subject to Linear-Plus-Exponential Marginal Price

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Abstract— **This paper introduces a game-theoretical strategy for optimal dispatch of building thermal loads, based on a marginal price model derived from an actual dispatch curve. A non-cooperative game is formulated, and the existence and uniqueness of the Nash equilibrium solution are proved aided by the variational inequality theory. A game solution algorithm is presented in this paper to solve the control problem with guaranteed convergence. The proposed game-theoretical control technique was evaluated against a baseline energy minimization strategy and a socially optimal solution, through a simulation test of a virtual market comprised of six buildings. The results show that the proposed game-theoretical strategy could achieve performance very close to the social optimum with a Price of Anarchy of 1.0041 and a 24% cost reduction compared to the baseline energy-priority strategy.**

I. INTRODUCTION

The operational efficiency, cost, and carbon emissions of the electrical grid are significantly impacted by daily fluctuations of electricity demand where the marginal price is highly dependent on the total generation capacity [1]. In the U.S., buildings consume 75% of electricity and contribute to a similar peak electrical demand [2]. Flexible loads within buildings, such as air conditioning, space heating, and refrigeration, can be actively managed to shift energy usage, aiding the grid to maintain stable and efficient operations. While centralized strategies for aggregate load control have been extensively studied [3–5], they are often impractical to implement due to players' self-interestedness, as individuals tend to prioritize their own benefits over those of the group. To address these challenges, the game-theoretic approach has been investigated, particularly the non-cooperative game framework for demand-side management [6–10]. In the demand management game, energy users participate as players, aiming to minimize their electricity costs by optimizing their electricity use profiles. Within this framework, users can access information of other players as well as the energy price from the electricity market to refine their control actions. Past research on game-theoretic control for demand-side management typically focused on linear [10,11] or logarithmic [12] marginal price models that are mostly hypothetical.

This study, for the first time, addresses the load dispatch problem in a non-cooperative game framework subject to a more realistic price model. We have derived an approximation fit to a more realistic dispatch curve using a linear-plus-exponential function. Under this price model, the load dispatch game is proved to admit a unique Nash equilibrium (NE) solution through the Variational Inequality (VI) theory. A game solution algorithm is proposed to solve the non-cooperative game with guaranteed convergence to the unique NE. Simulation tests of a six-building case study were carried out for the proposed strategy along with two benchmarking control methods and key findings from the simulation results are reported.

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II. PROBLEM FORMULATION

This study focuses on the demand-side management problem involving N buildings, denoted as $\mathcal{N} := \{1, ..., N\}$. The control time horizon is partitioned into T time slots, denoted as $\mathcal{T} := \{1, ..., T\}$. For each building $n \in \mathcal{N}$, x_n^t represents its power consumption at time slot $t \in \mathcal{T}$ and we use vector $\mathbf{x}_n = (x_n^t)_{t\in\mathcal{T}} \in \mathbb{R}^T$ to denote the power profile of building n over all time slots. $\mathbf{x}_{-n} = (\mathbf{x}_m)_{m\in\mathcal{N}\setminus\{n\}} \in \mathbb{R}^{(N-1)\times T}$ represents the power consumption of all other buildings except building n. At each time slot, $x^t = \sum_{n\in\mathcal{N}} x_n^t$ represents the total power consumption of all buildings. The aggregate control actions of the entire market are represented by $\mathbf{x} = (\mathbf{x}_n, \mathbf{x}_{-n})$. This study focuses on managing the Heating, ventilation, and air conditioning (HVAC) loads of buildings, as they are the most accessible demand-side flexibility resource.

A. Energy cost and pricing model

The US Energy Information Administration released a report [1] containing a dispatch curve for a hypothetical collection of electric generators. The marginal electricity cost can be approximated as a sum of an exponential and a linear function of the generation power over a wide range of generation capacity, as illustrated by Fig. 1. The approximate

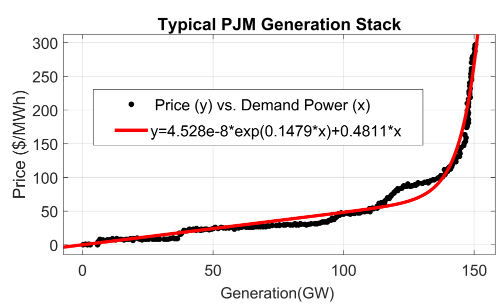

Fig. 1: Approximation fit to the EIA dispatch curve

model for the marginal electricity cost p^t (in $/kWh) at each

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time step t can be expressed as:

$$p^t = \bar{a}_h \cdot \exp\Big(a_e \cdot \sum_{n=1}^{N} x_n^t\Big) + a_h \cdot \Big(\sum_{n=1}^{N} x_n^t\Big) \tag{1}$$

where $\bar{a}_h$, a_e, a_h and a_p are positive scalar parameters.

B. Feasible set

Maintaining indoor temperature within a comfortable range is a major function of HVAC systems. In this context, the thermal dynamic behavior of a building can be characterized by a thermal network model, which is widely used for control and optimization of HVAC systems. The thermal network model represents a building as a network of interconnected thermal resistors and capacitors, with nodal temperatures as state variables. The model captures the transient heat transfer processes within the building driven by external weather conditions and internal heat gains. A linear state-space representation of the thermal network model is given in Eq. (2) and Eq. (3). The state-space matrices $\mathbf{A}_n$, $\mathbf{B}_w$, $\mathbf{B}_u$, and $\mathbf{C}_n$ depend on the thermal resistances and capacitances of the thermal network model, with details provided in [13]. The vector $\mathbf{r}$ is the state vector comprised of nodal temperatures of a building, $\mathbf{u}$ represents uncontrollable thermal inputs that are caused by external factors such as outdoor weather conditions and solar radiation, and T_n^t denotes the indoor temperature of the building n at time t.

$$\mathbf{r}_n^{t+1} = \mathbf{A}_n \mathbf{r}_n^t + \mathbf{B}_{w,n}\mathbf{u}_n^t + \mathbf{B}_{u,n} x_n^t \tag{2}$$

$$T_n^{t+1} = \mathbf{C}_n \mathbf{r}_n^{t+1} \tag{3}$$

This model assumes a single zone temperature for each building. For a building with multiple temperature zones, the global zone temperature (weighted average of all zone temperatures) can be used. To ensure indoor comfort, Eq. (4) presents the temperature constraints that must be satisfied within the control horizon, where $\underline{T}_n$ and $\overline{T}_n$ denote the lower and upper bounds of the comfort temperature zone.

$$\underline{T}_n^t \leq T_n^t \leq \overline{T}_n^t, \quad \forall t \in \mathcal{T}, \quad \forall n \in \mathcal{N} \tag{4}$$

Additionally, the HVAC power must be bounded by the system capacity $\overline{x}_n$, as expressed in Equation (5).

$$0 \leq x_n^t \leq \overline{x}_n, \quad \forall t \in \mathcal{T}, \quad \forall n \in \mathcal{N} \tag{5}$$

For ease of analysis, the building control constraints can be expressed as a set of linear inequalities, i.e., $\mathbf{g}_n(\mathbf{x}_n) := \mathbf{D}_n\mathbf{x}_n - \mathbf{b}_n \geq \mathbf{0} \in \mathbb{R}^{m_n}$, where $\mathbf{D}_n$ and $\mathbf{b}_n$ are dependent on the state-space matrices, as well as the power and temperature upper and lower bounds. The set of feasible actions for building n is denoted by $\mathcal{X}_n = \{\mathbf{x}_n | \quad \mathbf{g}_n(\mathbf{x}_n) \geq \mathbf{0}\}$. It is important to note that the feasible sets are completely decoupled across the different buildings. For case studies with cross-building thermal couplings, generalized Nash equilibrium problems will be obtained [14,15].

C. Game theoretical formulation

Non-cooperative game theory studies the behavior of independent decision-makers in situations where the outcome of each individual's choice depends on the choices made by others. In a non-cooperative game, each player is assumed to act in a self-interested manner to maximize their own payoff. One of the most important concepts in non-cooperative game theory is the Nash equilibrium (NE). A NE is a situation in which no player can benefit by changing their strategy, given the strategies chosen by the other players. The demand-side management problem can be framed as a non-cooperative game, where each building in the set $\mathcal{N}$ is a player and the HVAC power control actions represent the players' strategies. The objective of each player is to minimize their energy cost, which is influenced by the control actions of other players through the marginal price. Each building can determine its optimal control strategy given the control actions of the other buildings, i.e.,

$$\begin{aligned}\min_{\mathbf{x}_n \in \mathcal{X}_n} w_n(\mathbf{x}_n, \mathbf{x}_{-n}) &= \sum_{t=1}^{T} p^t x_n^t \\ &= \sum_{t=1}^{T} \bar{a}_h \cdot \exp(a_e \cdot \sum_{n=1}^{N} x_n^t) \cdot x_n^t \\ &\quad + \sum_{t=1}^{T} a_h \Big(\sum_{n=1}^{N} x_n^t\Big) \cdot x_n^t \end{aligned} \tag{6}$$

The strategy set of the Nash equilibrium problem (NEP) is the Cartesian product of the individual strategy sets of all players, denoted as $\mathcal{X} = \prod_{n\in\mathcal{N}} \mathcal{X}_n \subseteq \mathbb{R}^{NT}$. The game problem described above, denoted by $\mathcal{G} = \langle \mathcal{X}, \mathbf{w} \rangle$, is a NEP with the following specifications:

- Players: buildings in the set $\mathcal{N}$.
- Cost function: $w_n(\mathbf{x}_n, \mathbf{x}_{-n})$ represents the cost function for each building player n.
- Strategy set: The set of feasible strategies for all players is $\mathcal{X}$.

A vector $\mathbf{x}^* = (\mathbf{x}_n^*, \mathbf{x}_{-n}^*) \in \mathcal{X}$ is called a NE if $w_n(\mathbf{x}_n^*, \mathbf{x}_{-n}^*) \leq w_n(\mathbf{x}_n, \mathbf{x}_{-n}^*), \forall \mathbf{x}_n \in \mathcal{X}_n$ and $\forall n \in \mathcal{N}$.

D. Reformulate NEP as VI problem

The VI theory provides a powerful mathematical framework for characterizing and computing NE in non-cooperative games. The basic structure of a VI($\mathcal{X}, \mathbf{F}$) is expressed as the problem of finding a point $\mathbf{x}^*$ that satisfies the following condition:

$$(\mathbf{x} - \mathbf{x}^*)^\intercal \mathbf{F}(\mathbf{x}^*) \geq 0 \qquad \forall \mathbf{x} \in \mathcal{X} \tag{7}$$

where $\mathbf{F} : \mathcal{X} \to \mathbb{R}^n$ is a nonlinear operator, and $\mathcal{X}$ is a closed and convex subset of $\mathbb{R}^n$. For further details on VI problems, interested readers may refer to [16]

In the load dispatch game, the goal of player n is to minimize their cost while ensuring their decision variables satisfy the constraints given by $\mathbf{D}_n\mathbf{x}_n - \mathbf{b}_n \geq \mathbf{0}$.

$$\min_{\mathbf{x}_n} \{w_n(\mathbf{x}_n, \mathbf{x}_{-n}) | \mathbf{D}_n\mathbf{x}_n - \mathbf{b}_n \geq \mathbf{0}\} \tag{8}$$

Since the individual's objective function $w_n(\mathbf{x}_n, \mathbf{x}{-}n)$ is continuously differentiable for any $\mathbf{x} \in \mathcal{X}$ and convex in $\mathbf{x}_n$ for every fixed $\mathbf{x}_{-n}$, and the strategy set of player n is both compact and convex, the optimal strategy $\mathbf{x}_n^*$ satisfies the following optimality conditions [17]

$$\nabla_{\mathbf{x}_n} w_n(\mathbf{x}_n^*, \mathbf{x}_{-n}^*)^\top (\mathbf{x}_n - \mathbf{x}_n^*) \geq 0, \quad \forall \mathbf{x}_n \in \mathcal{X}_n \tag{9}$$

for all $n \in \mathcal{N}$. Let $\mathbf{F}(\mathbf{x}) = (\nabla_{\mathbf{x}_n} w_n(\mathbf{x}))_{n=1}^N$. It follows that $\mathbf{x}^*$ is a NE if and only if

$$\mathbf{F}(\mathbf{x}^*)^\top (\mathbf{x} - \mathbf{x}^*) \geq 0, \qquad \forall \mathbf{x} \in \mathcal{X}. \tag{10}$$

E. Existence and uniqueness of the NE

The existence and uniqueness of the NE are critical questions to address first for both game analysis and NE computation. Since the feasible set $\mathcal{X}_n$ is a bounded polytope and w_n is convex in $\mathbf{x}_n$, NE solutions of the game under study exist by Theorem II.1. The NE uniqueness is guaranteed by Lemma II.2.

Theorem II.1. *If $\forall n \in \mathcal{N}$, $\mathcal{X}_n$ is nonempty, convex and compact, $w_n : \mathcal{X} \to \mathbb{R}$ is continuous within $\mathcal{X}$ and $\forall \mathbf{x}_{-n} \in \mathcal{X}_{-n}$, w_n is convex in $\mathbf{x}_n$ on $\mathcal{X}_n$, then there exists a NE. [18]*

Lemma II.2. *If the marginal price is a weighted sum of a linear and an exponential function, i.e., $p(x^t) = a_h \cdot x^t + \bar{a}_h \cdot exp(a_e \cdot x^t)$ where a_h, $\bar{a}_h$ and a_e are positive parameters, then the load dispatch game under study $\mathcal{G} = \langle \mathcal{X}, \mathbf{w} \rangle$ has a unique Nash Equilibrium.*

Proof: See Appendix V-A.

F. Solution Algorithm for the NEP

In the following, we present a solution algorithm for the given NEP. First we need to reformulate the problem to address the explicit constraints through a primal-dual approach.

1) Reformulate NEP: The Lagrangian for player n's problem (8) is given by Eq. (11), which involves the introduction of Lagrange multipliers λ_n.

$$L_n = w_n(\mathbf{x}_n, \mathbf{x}_{-n}) - \lambda_n^\intercal (\mathbf{D}_n \mathbf{x}_n - \mathbf{b}_n), \lambda_n \in \mathbb{R}_+^{m_n} \tag{11}$$

Let $(\mathbf{x}_n^*, \lambda_n^*)$ denote the saddle point of the following min-max problem for given $\mathbf{x}_{-n}^*$:

$$\max_{\lambda_n} \min_{\mathbf{x}_n} \{L_n(\mathbf{x}_n, \mathbf{x}_{-n}^*, \lambda_n) | \lambda_n \geq \mathbf{0}\} \tag{12}$$

which is equivalent to the following pair of inequalities:

$$L_n(\mathbf{x}_n^*, \mathbf{x}_{-n}^*, \lambda_n) \leq L_n(\mathbf{x}_n^*, \mathbf{x}_{-n}^*, \lambda_n^*), \forall \lambda_n \in \mathbb{R}_+^{m_n} \tag{13}$$

$$L_n(\mathbf{x}_n^*, \mathbf{x}_{-n}^*, \lambda_n^*) \leq L_n(\mathbf{x}_n, \mathbf{x}_{-n}^*, \lambda_n^*), \forall \mathbf{x}_n \in \mathbb{R}^T. \tag{14}$$

Substituting the expression for L_n in Eq. (11) into (13) and (14), respectively, yields:

$$\begin{aligned} & L_n(\mathbf{x}_n^*, \mathbf{x}_{-n}^*, \lambda_n^*) - L_n(\mathbf{x}_n^*, \mathbf{x}_{-n}^*, \lambda_n) \\ = \; & w_n(\mathbf{x}_n^*, \mathbf{x}_{-n}^*) - (\lambda_n^*)^\intercal (\mathbf{D}_n \mathbf{x}_n^* - \mathbf{b}_n) \\ & - w_n(\mathbf{x}_n^*, \mathbf{x}_{-n}^*) + \lambda_n^\intercal (\mathbf{D}_n \mathbf{x}_n^* - \mathbf{b}_n) \\ = \; & (\lambda_n - \lambda_n^*)^\intercal (\mathbf{D}_n \mathbf{x}_n^* - \mathbf{b}_n) \\ \geq \; & 0, \forall \lambda_n \in \mathbb{R}_+^{m_n} \end{aligned} \tag{15}$$

and

$$\begin{aligned} & L_n(\mathbf{x}_n, \mathbf{x}_{-n}^*, \lambda_n^*) - L_n(\mathbf{x}_n^*, \mathbf{x}_{-n}^*, \lambda_n^*) \\ = \; & w_n(\mathbf{x}_n, \mathbf{x}_{-n}^*) - (\lambda_n^*)^\intercal (\mathbf{D}_n \mathbf{x}_n - \mathbf{b}_n) \\ & - w_n(\mathbf{x}_n^*, \mathbf{x}_{-n}^*) + (\lambda_n^*)^\intercal (\mathbf{D}_n \mathbf{x}_n^* - \mathbf{b}_n) \\ = \; & w_n(\mathbf{x}_n, \mathbf{x}_{-n}^*) - w_n(\mathbf{x}_n^*, \mathbf{x}_{-n}^*) - (\mathbf{x}_n - \mathbf{x}_n^*)^\intercal \mathbf{D}_n^\intercal \lambda_n^* \\ \geq \; & 0, \forall \mathbf{x}_n \in \mathbb{R}^T \end{aligned} \tag{16}$$

Let $\mathbf{y}_n = (\mathbf{x}_n^\intercal, \lambda_n^\intercal)^\intercal$ defines a vector that combines the decision variables and Lagrange multipliers of player n. Then (17) is obtained by adding the left and right sides of (15) and (16), respectively, which represents a necessary condition for the optimal strategy of each player in the game $\mathcal{G} = \langle \mathcal{X}, \mathbf{w} \rangle$.

$$\begin{aligned} & w_n(\mathbf{x}_n, \mathbf{x}_{-n}^*) - w_n(\mathbf{x}_n^*, \mathbf{x}_{-n}^*) + (\mathbf{y}_n - \mathbf{y}_n^*)^\intercal \mathbf{\Phi}_n(\mathbf{y}_n^*) \\ & \geq 0, \quad \forall \lambda_n \in \mathbb{R}_+^{m_n}, \forall \mathbf{x}_n \in \mathbb{R}^T \end{aligned} \tag{17}$$

where

$$\mathbf{\Phi}_n(\mathbf{y}_n^*) = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_n^\intercal \\ \mathbf{D}_n & \mathbf{0} \end{bmatrix} \mathbf{y}_n^* - \begin{bmatrix} \mathbf{0} \\ \mathbf{b}_n \end{bmatrix} \tag{18}$$

$$= \mathbf{\Upsilon}_n \mathbf{y}_n^* - \mathbf{\Sigma}_n. \tag{19}$$

Due to the convexity of the function w_n with respect to $\mathbf{x}_n$, w_n satisfies the first order condition for convexity [17]:

$$\begin{aligned} & w_n(\mathbf{x}_n, \mathbf{x}_{-n}^*) - w_n(\mathbf{x}_n^*, \mathbf{x}_{-n}^*) \\ & \geq (\mathbf{x}_n - \mathbf{x}_n^*)^\intercal \nabla_{\mathbf{x}_n} w_n(\mathbf{x}_n^*, \mathbf{x}_{-n}^*), \forall \mathbf{x}_n \in \mathbb{R}^T \end{aligned} \tag{20}$$

Therefore, the following inequality is a sufficient condition for (17).

$$\begin{aligned} & (\mathbf{x}_n - \mathbf{x}_n^*)^\intercal \nabla_{\mathbf{x}_n} w_n(\mathbf{x}_n^*, \mathbf{x}_{-n}^*) + (\mathbf{y}_n - \mathbf{y}_n^*)^\intercal \mathbf{\Phi}_n(\mathbf{y}_n^*) \\ & \geq 0, \quad \forall n \in \mathcal{N}, \forall \lambda_n \in \mathbb{R}_+^{m_n}, \forall \mathbf{x}_n \in \mathbb{R}^T \end{aligned} \tag{21}$$

Since all players must satisfy (21) in their optimal play, finding a vector $\mathbf{y}^* = (\mathbf{y}_n^*)_{n=1}^N$ that satisfies (22) is equivalent to finding the NE of the game problem. Eq. (22) is a compact form of (21) for all players.

$$\begin{aligned} & (\mathbf{x} - \mathbf{x}^*)^\intercal \mathbf{F}(\mathbf{x}^*) + (\mathbf{y} - \mathbf{y}^*)^\intercal \mathbf{\Psi}(\mathbf{y}^*) \geq 0, \\ & \qquad \forall \mathbf{x} \in \mathbb{R}^{NT}, \forall \mathbf{y} \in \mathbb{R}^{NT+M} \end{aligned} \tag{22}$$

where $\mathbf{F}(\mathbf{x}^*) = (\nabla_{\mathbf{x}_n} w_n(\mathbf{x}^*))_{n=1}^N$, and $\mathbf{\Psi}(\mathbf{y}^*)$ is a linear function that is constructed from $\mathbf{\Upsilon}_n$ and vectors $\mathbf{\Sigma}_n$ below.

$$\mathbf{\Psi}(\mathbf{y}^*) = \begin{bmatrix} \mathbf{\Upsilon}_1 & & & \\ & \mathbf{\Upsilon}_2 & & \\ & & \ddots & \\ & & & \mathbf{\Upsilon}_N \end{bmatrix} \mathbf{y}^* - \begin{bmatrix} \mathbf{\Sigma}_1 \\ \mathbf{\Sigma}_2 \\ \vdots \\ \mathbf{\Sigma}_N \end{bmatrix} \tag{23}$$

2) Game solution algorithm: This section presents a game solution algorithm based on the VI method, which offers guaranteed global convergence. The algorithm is shown in Algorithm 1. At each iteration, the new iterate $\mathbf{x}^{k+1}$ is generated by finding the root of Eq. (28), and the new iterate's Lagrange multiplier λ^{k+1} is generated using a max function that ensures $\lambda^{k+1} \in \mathbb{R}_+^M$. Note that $\mathbf{I}^n$ is an $n \times n$ identity matrix. The convergence of the algorithm is discussed in Appendix V-B

Algorithm 1 Game solution algorithm

Initialize $j = 0$, $\mathbf{y}^j \in \mathbb{R}^{TN+M}$, $\epsilon_1 \geq 0$, $\epsilon_2 \geq 0$, $r_n = (\|\mathbf{D}_n^\intercal \mathbf{D}_n\| + \epsilon_1)^{\frac{1}{2}}, \forall n \in \mathcal{N}$, $\omega_z > \epsilon_2$

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_1 & & & \\ & \mathbf{D}_2 & & \\ & & \ddots & \\ & & & \mathbf{D}_N \end{bmatrix} \tag{24}$$

$$\mathbf{E} = \begin{bmatrix} r_1\mathbf{I}^T & & & \\ & r_2\mathbf{I}^T & & \\ & & \ddots & \\ & & & r_N\mathbf{I}^T \end{bmatrix} \tag{25}$$

$$\lambda = \begin{bmatrix} \lambda_1^\intercal & \lambda_2^\intercal & \dots & \lambda_N^\intercal \end{bmatrix}^\intercal \tag{26}$$

$$\mathbf{P}^{-1} = \begin{bmatrix} \frac{1}{r_1}\mathbf{I}^{m_1} & & & \\ & \frac{1}{r_2}\mathbf{I}^{m_2} & & \\ & & \ddots & \\ & & & \frac{1}{r_N}\mathbf{I}^{m_N} \end{bmatrix} \tag{27}$$

while $\omega_z > \epsilon_2$ **do**

Newton's method to find $\mathbf{x}^{k+1}$ that satisfies:

$$\mathbf{F}(\mathbf{x}^{k+1}) - \mathbf{D}^\intercal \lambda^k + \mathbf{E}(\mathbf{x}^{k+1} - \mathbf{x}^k) = \mathbf{0} \tag{28}$$

$$\lambda^{k+1} = \max\left(\lambda^k - \mathbf{P}^{-1}(\mathbf{D}(2\mathbf{x}^{k+1} - \mathbf{x}^k) - \mathbf{b}), \mathbf{0}\right) \tag{29}$$

$$\omega_z = \|\mathbf{x}^{k+1} - \mathbf{x}^k\| + \|\lambda^{k+1} - \lambda^k\| \tag{30}$$

end

III. Case Study

To evaluate the effectiveness of the proposed game-theoretic control strategy for building HVAC loads, we conducted a simulation case study to compare its performance against two benchmarking control strategies. These benchmarking strategies are described in the following subsections. The case study involved six commercial buildings of different types and with different occupancy schedules. The power capacity of the six buildings was scaled to the generation capacity in the energy dispatch curve shown in Fig. 1. To ensure indoor comfort during occupied hours, the proposed control strategy maintained a tighter temperature range of 21.5°C to 22.5°C, while relaxed temperature bounds of 19°C to 25°C were used during unoccupied hours to reduce energy consumption and costs.

A. Baseline control

During the cooling season, higher zone air temperature (ZAT) settings can reduce HVAC energy consumption and lower power demand. Therefore, the baseline control strategy assumes the ZAT setpoint to be maintained at the upper bound of the comfort zone to minimize cooling energy consumption, which best represents the current practice. The cooling power can be estimated using the ZAT setpoint and load model in Eq. (31)

$$x_n^t = \max\left(0, \mathbf{B}_u^{-1}(\mathbf{C}^{-1}\overline{T}_n^{t+1} - \mathbf{A}\mathbf{r}_n^t - \mathbf{B}_w\mathbf{u}_n^t)\right) \tag{31}$$

B. Centralized control

The centralized control strategy assumes that all players are fully cooperative and aim to minimize the collective electricity cost. This strategy represents the social optimum, where all buildings' load flexibility can be used to achieve the best benefit for the group. In contrast, the proposed game-theoretic control approach assumes that all customers are selfish and aim to reduce their individual electricity bills. The deviations between the game-theoretic and centralized control represent performance loss due to user self-interestedness. The centralized control problem can be formulated as a convex problem that can be solved easily.

C. Case study results

Fig. 2 shows the simulation test results for three representative buildings with distinct occupancy schedules, demonstrating the diverse control behaviors of the various players. The top subplots exhibit the zone temperature trajectories along with the upper and lower bounds of the comfort zone, represented by blue shaded areas, while the lower subplots display the cooling power associated with the different control strategies. Fig. 3 plots the variations of the aggregate power of all six buildings under the three control strategies. Tab. I summarizes the costs of each control strategy.

The baseline control strategy maintained the zone temperature at the upper bound whenever mechanical cooling was required, resulting in the lowest energy consumption at each time step. During unoccupied hours, the air-conditioning system was off for a majority of the time with zone temperatures floating in the comfort zone. Although this approach resulted in the lowest energy consumption, it led to the highest peak demand and total operation cost among the three control strategies, due to the use of expensive peaker plants.

The centralized control strategy assumes all buildings to be fully cooperative, which enables the optimal use of all customers' flexibility to minimize the collective cost. Cost savings are achieved through early morning precooling of buildings to flatten the overall load profile and reduce the operation time of peaker plants. This strategy aims to strike a balance between total electricity usage and peak demand, as aggressive precooling can reduce peak demand but may lead to higher overall electricity usage. The centralized control strategy achieves significant total operation cost savings exceeding 24% and a peak demand reduction of 28.4 % compared to the baseline strategy.

The proposed game-theoretic control approach results in very different behaviors in individual building loads compared to those obtained by the centralized controller. More specifically, the game-theoretic control strategy results in smooth precooling power profiles for all buildings, as any increase in individual demand not only increases the marginal price but also increases his/her utility cost. Although the individual loads are distinct, the collective demand profile of the game-theoretic control case is very similar to that of the centralized results. The aggregate electricity cost is reduced by 24% compared to the baseline strategy, and a Price of Anarchy of 1.0041 is obtained for this specific case study. This result indicates that the selfish behaviors of participants have only caused a 0.41% performance degradation compared to the social optimum. Although the total operation cost and energy consumption are slightly higher than the social optimum, the game-theoretic approach achieves a lower peak-to-average ratio (PAR), as shown in Table I.

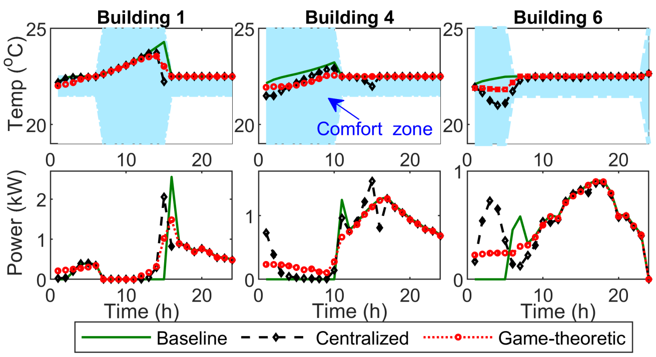

Fig. 2: Simulation results for buildings

TABLE I: The electricity costs and other performance metrics for the various strategies

	Total charge ($)		
	Baseline	Centralized	Game-theoretic
Building #1	0.453	0.285	0.280
Building #2	0.265	0.222	0.223
Building #3	0.147	0.146	0.152
Building #4	0.472	0.407	0.408
Building #5	0.515	0.310	0.316
Building #6	0.360	0.311	0.312
Aggregate	2.212	1.681	1.688
Total energy use (kWh)	58.31	61.344	61.85
PAR	3.465	2.397	2.35

IV. Conclusion

This paper presented a non-cooperative game-theoretic control strategy for scheduling building thermal loads under a more realistic marginal price model. Through reformulation of the NEP as a VI problem, the existence and uniqueness of the NE solution were proved. A VI algorithm was proposed to find the NE solution with guaranteed global convergence. Our simulation test, which used a case study of six buildings, demonstrated that the proposed game-theoretic control strategy was effective in reducing total electricity costs while ensuring lower peak to average ratio (PAR). The control performance of the game-theoretic strategy was very close to the social optimum, with a Price of Anarchy of 1.0041. Future research shall investigate the scalability of the proposed approach to a larger number of buildings and explore the effects of market uncertainty on the performance of the proposed strategy.

V. Appendix

A. Proof of lemma II.2

Here we utilize the monotonicity of the formulated VI to prove the uniqueness of the NE solution. The role of monotonicity in a VI problem is similar to that of convexity in convex optimization analysis. The monotonicity can be established by investigating the positive-definiteness of the Jacobian matrix $\mathbf{JF}$ of $\mathbf{F}$, per the following theorem.

Theorem V.1.

If $\mathcal{X}$ is convex, $\mathbf{F}$ is continuously differentiable on $\mathcal{X}$ and the Jacobian matrix $\mathbf{JF}$ is positive-semidefinite (positive-definite), then $\mathbf{F}$ is monotone (strictly monotone). (Proposition 2.3.2 of [16])

In the following, we will prove that for the exponential-plus-linear marginal price model, the corresponding Jacobian $\mathbf{JF}$ is positive definite. From Eq. (6), $\mathbf{JF}$ can be separated into a Jacobian of the exponential component ($\mathbf{JF_e}$) and a Jacobian of the linear counterpart ($\mathbf{JF_p}$). $\mathbf{JF_p}$ has been shown to be positive definite in [19] and also earlier work in [20]. Therefore, we only need to prove $\mathbf{JF_e}$ is also positive definite. Dropping the linear price term, we have

$$w_{e,n}(\mathbf{x}_n, \mathbf{x}_{-n}) = \sum_{t=1}^{T} \bar{a}_h \cdot \exp\Big(a_e \cdot \sum_{n=1}^{N} x_n^t\Big) \cdot x_n^t \tag{32}$$

where the superscript e indicates the cost with a purely exponential price. Then we have

$$\mathbf{F_e}(\mathbf{x}) = \begin{pmatrix} \nabla_{\mathbf{x_1}} w_{e,1}(\mathbf{x_1}, \mathbf{x_{-1}}) \\ \vdots \\ \nabla_{\mathbf{x}_N} w_{e,N}(\mathbf{x}_N, \mathbf{x}_{-N}) \end{pmatrix} = \begin{pmatrix} \nabla_{x_1^1} w_{e,1}(\mathbf{x_1}, \mathbf{x_{-1}}) \\ \nabla_{x_1^2} w_{e,1}(\mathbf{x_1}, \mathbf{x_{-1}}) \\ \vdots \\ \nabla_{x_1^T} w_{e,1}(\mathbf{x_1}, \mathbf{x_{-1}}) \\ \vdots \\ \nabla_{x_N^1} w_{e,N}(\mathbf{x}_N, \mathbf{x}_{-N}) \\ \nabla_{x_N^2} w_{e,N}(\mathbf{x}_N, \mathbf{x}_{-N}) \\ \vdots \\ \nabla_{x_N^T} w_{e,N}(\mathbf{x}_N, \mathbf{x}_{-N}) \end{pmatrix} \tag{33}$$

To prove the positive definiteness of the Jacobian matrix of $\mathbf{F_e}(\mathbf{x})$, it is necessary to reorder the elements of the vector $\mathbf{x}$ based on the time index instead of user ID. The reordered vector is named as $\hat{\mathbf{x}} = (x_1^1, x_1^2, ...x_1^T, \ldots, x_N^1, x_N^2, \ldots x_N^T)$.

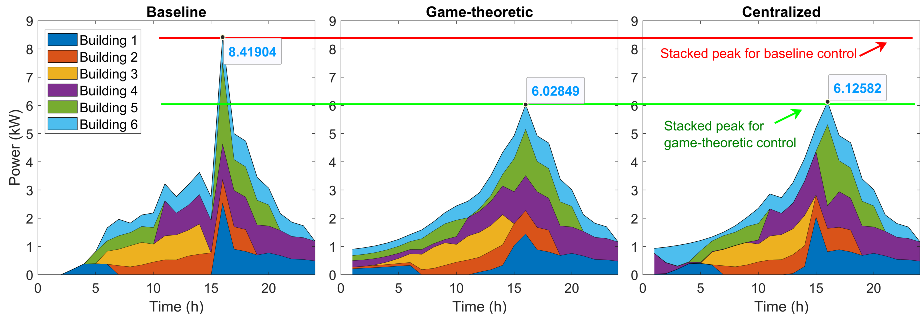

Fig. 3: Aggregate power consumption under various controls.

Then the vector operator $\mathbf{F_e}(\mathbf{x})$ becomes

$$\mathbf{F_e}(\hat{\mathbf{x}}) = \begin{pmatrix} \nabla_{x_1^1} w_{e,1}(\mathbf{x_1}, \mathbf{x_{-1}}) \\ \nabla_{x_2^1} w_{e,1}(\mathbf{x_1}, \mathbf{x_{-1}}) \\ \vdots \\ \nabla_{x_N^1} w_{e,1}(\mathbf{x_1}, \mathbf{x_{-1}}) \\ \vdots \\ \nabla_{x_1^T} w_{e,N}(\mathbf{x}_N, \mathbf{x}_{-N}) \\ \nabla_{x_2^T} w_{e,N}(\mathbf{x}_N, \mathbf{x}_{-N}) \\ \vdots \\ \nabla_{x_N^T} w_{e,N}(\mathbf{x}_N, \mathbf{x}_{-N}) \end{pmatrix} \tag{34}$$

With these re-arrangements, the Jacobian matrix $\mathbf{JF_e}(\hat{\mathbf{x}})$ becomes a block diagonal matrix:

$$\mathbf{JF_e}(\hat{\mathbf{x}}) = \begin{bmatrix} \underbrace{\begin{bmatrix} \bar{f}_{11}^1 & \dots & \bar{f}_{1N}^1 \\ \vdots & \ddots & \vdots \\ \bar{f}_{N1}^1 & \dots & \bar{f}_{NN}^1 \end{bmatrix}}_{\bar{f}^1} & 0 & 0 & \dots & 0 \\ 0 & \bar{f}^2 & 0 & \dots & 0 \\ 0 & 0 & \bar{f}^3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \bar{f}^T \end{bmatrix} \tag{35}$$

where

$$\begin{aligned} \bar{f}^t &= (\bar{f}_{ij}^t) \in \mathbb{R}^{N \times N}, \forall t \in \mathcal{T} \\ \bar{f}_{ij}^t &= \begin{cases} \bar{a}_h \cdot a_e^2 \cdot \exp\Big(a_e \cdot \sum_{n=1}^N x_n^t\Big) \cdot x_i^t & \\ +\bar{a}_h \cdot a_e \cdot \exp\Big(a_e \cdot \sum_{n=1}^N x_n^t\Big) & i \neq j \\ \bar{a}_h \cdot a_e^2 \cdot \exp\Big(a_e \cdot \sum_{n=1}^N x_n^t\Big) \cdot x_i^t & \\ +2\bar{a}_h \cdot a_e \cdot \exp\Big(a_e \cdot \sum_{n=1}^N x_n^t\Big) & i = j \end{cases}, \\ & \forall i, j \in \mathcal{N} \end{aligned} \tag{36}$$

Since $\exp(a_e \cdot \sum_{n=1}^N x_n^t) \geq 1$ in the feasible set, $\bar{f}^t$ is the sum of a positive semidefinite matrix of rank 1 and the identity matrix up to a positive scalar, and thereby is positive definite. This holds for all $t \in \mathcal{T}$ and therefore, $\mathbf{JF_e}$ is positive definite and so is $\mathbf{JF}$. This ensures the uniqueness of the NE solution by Theorem V.2 below. □

Theorem V.2. *If $\mathcal{X}$ is a closed bounded convex set and $\mathbf{F}$ is strictly monotone on $\mathcal{X}$, the solution of the VI problem is unique. [16]*

B. Proof of convergence of the game solution algorithm

The computed $(\mathbf{x}^{k+1}, \lambda^{k+1})$ using equations Eq. (28) and Eq. (29) satisfies (37) and (38).

$$(\mathbf{x} - \mathbf{x}^{k+1})^\intercal \left(\mathbf{F}(\mathbf{x}^{k+1}) - \mathbf{D}^\intercal \lambda^k + \mathbf{E}(\mathbf{x}^{k+1} - \mathbf{x}^k)\right) \geq 0 \\ , \forall \mathbf{x} \in \mathbb{R}^{TN} \tag{37}$$

$$(\lambda - \lambda^{k+1})^\intercal \left(\mathbf{D}\mathbf{x}^{k+1} - \mathbf{b} + \mathbf{D}(\mathbf{x}^{k+1} - \mathbf{x}^k) \\ + \mathbf{P}(\lambda^{k+1} - \lambda^k)\right) \geq 0, \forall \lambda \in \mathbb{R}_+^M \tag{38}$$

With simple manipulations, (37) will become

$$\begin{aligned} &(\mathbf{x} - \mathbf{x}^{k+1})^\intercal \Big(\mathbf{F}(\mathbf{x}^{k+1}) - \mathbf{D}^\intercal \lambda^{k+1} \\ &+ \mathbf{D}^\intercal (\lambda^{k+1} - \lambda^k) + \mathbf{E}(\mathbf{x}^{k+1} - \mathbf{x}^k)\Big) \geq 0, \\ &\forall \mathbf{x} \in \mathbb{R}^{TN} \end{aligned} \tag{39}$$

The following compact expression represents a necessary condition for (39) and (38)

$$\begin{aligned} & (\mathbf{x} - \mathbf{x}^{k+1})^\intercal \mathbf{F}(\mathbf{x}^{k+1}) + (\mathbf{y} - \mathbf{y}^{k+1})^\intercal \mathbf{\Psi}(\mathbf{y}^{k+1}) \\ \geq \ & (\mathbf{y} - \mathbf{y}^{k+1})^\intercal \mathbf{\Omega}(\mathbf{y}^k - \mathbf{y}^{k+1}), \\ & \forall \mathbf{x} \in \mathbb{R}^{TN}, \forall \mathbf{y} \in \mathbb{R}^{TN+M} \end{aligned} \tag{40}$$

where

$$\mathbf{\Omega} = \begin{bmatrix} \mathbf{\Theta}_1 & & & \\ & \mathbf{\Theta}_2 & & \\ & & \ddots & \\ & & & \mathbf{\Theta}_N \end{bmatrix}, \mathbf{\Theta}_n = \begin{bmatrix} r_n \mathbf{I} & \mathbf{D}_n^\intercal \\ \mathbf{D}_n & r_n \mathbf{I} \end{bmatrix} \tag{41}$$

If r_n satisfies the following criterion, $\mathbf{\Theta}_n$ is guaranteed to be a positive definite matrix

$$r_n^2 > \|\mathbf{D_n}^\intercal \mathbf{D}_n\|, \forall n \in \mathcal{N}, \tag{42}$$

where $\|\cdot\|$ is the spectral norm of the matrix, and thus $\mathbf{\Omega}$ is a positive definite matrix. Note that if $\mathbf{y}^k = \mathbf{y}^{k+1}$, the right-hand side of (40) becomes zero, (40) and (22) become identical, and $\mathbf{x}^{k+1}$ is the NE $(\mathbf{x}^*)$.

The following inequality can be derived by substituting $\mathbf{x} = \mathbf{x}^*$ into (40)

$$\begin{aligned} & (\mathbf{x}^* - \mathbf{x}^{k+1})^\intercal \mathbf{F}(\mathbf{x}^{k+1}) + (\mathbf{y}^* - \mathbf{y}^{k+1})^\intercal \mathbf{\Psi}(\mathbf{y}^{k+1}) \\ \geq \ & (\mathbf{y}^* - \mathbf{y}^{k+1})^\intercal \mathbf{\Omega}(\mathbf{y}^k - \mathbf{y}^{k+1}) \end{aligned} \tag{43}$$

Multiplying both sides of (43) by a negative sign, the following inequality can be obtained:

$$\begin{aligned} & (\mathbf{y}^{k+1}-\mathbf{y}^*)^\intercal \mathbf{\Omega}(\mathbf{y}^k-\mathbf{y}^{k+1}) \\ \geq \quad & (\mathbf{x}^{k+1}-\mathbf{x}^*)^\intercal \mathbf{F}(\mathbf{x}^{k+1}) + (\mathbf{y}^{k+1}-\mathbf{y}^*)^\intercal \mathbf{\Psi}(\mathbf{y}^{k+1}) \end{aligned} \tag{44}$$

Since $\mathbf{F}$ and $\mathbf{\Psi}$ are both monotone, the following inequality can be obtained:

$$\begin{aligned} & (\mathbf{x}^{k+1}-\mathbf{x}^*)^\intercal \mathbf{F}(\mathbf{x}^{k+1}) + (\mathbf{y}^{k+1}-\mathbf{y}^*)^\intercal \mathbf{\Psi}(\mathbf{y}^{k+1}) \\ \geq \quad & (\mathbf{x}^{k+1}-\mathbf{x}^*)^\intercal \mathbf{F}(\mathbf{x}^*) + (\mathbf{y}^{k+1}-\mathbf{y}^*)^\intercal \mathbf{\Psi}(\mathbf{y}^*) \\ \geq \quad & 0 \end{aligned} \tag{45}$$

From (45) and (44), the following inequality can be obtained:

$$(\mathbf{y}^{k+1}-\mathbf{y}^*)^\intercal \mathbf{\Omega}(\mathbf{y}^k-\mathbf{y}^{k+1}) \geq 0 \tag{46}$$

Note that if two vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ satisfy $\hat{\mathbf{b}}^\intercal \mathbf{M}(\hat{\mathbf{a}}-\hat{\mathbf{b}}) \geq 0$ and $\mathbf{M}$ is positive definite, the following property can be derived:

$$\begin{aligned} \|\hat{\mathbf{a}}\|_{\mathbf{M}}^2 &= \|\hat{\mathbf{b}}+(\hat{\mathbf{a}}-\hat{\mathbf{b}})\|_{\mathbf{M}}^2 \\ &= \|\hat{\mathbf{b}}\|_{\mathbf{M}}^2 + \|\hat{\mathbf{a}}-\hat{\mathbf{b}}\|_{\mathbf{M}}^2 + \hat{\mathbf{b}}^\intercal \mathbf{M}(\hat{\mathbf{a}}-\hat{\mathbf{b}}) \\ &\geq \|\hat{\mathbf{b}}\|_{\mathbf{M}}^2 + \|\hat{\mathbf{a}}-\hat{\mathbf{b}}\|_{\mathbf{M}}^2 \end{aligned} \tag{47}$$

where $\|\mathbf{x}\|_{\mathbf{M}}^2 = \mathbf{x}^\intercal \mathbf{M}\mathbf{x}$. The following inequality can be obtained by moving $\|\hat{\mathbf{a}}-\hat{\mathbf{b}}\|_{\mathbf{M}}^2$ from the right-hand side of (47) to the left side:

$$\|\hat{\mathbf{b}}\|_{\mathbf{M}}^2 \leq \|\hat{\mathbf{a}}\|_{\mathbf{M}}^2 - \|\hat{\mathbf{a}}-\hat{\mathbf{b}}\|_{\mathbf{M}}^2 \tag{48}$$

Letting $\hat{\mathbf{a}} = \mathbf{y}^k - \mathbf{y}^*$ and $\hat{\mathbf{b}} = \mathbf{y}^{k+1} - \mathbf{y}^*$ leads to:

$$\|\mathbf{y}^{k+1}-\mathbf{y}^*\|_{\mathbf{\Omega}}^2 \leq \|\mathbf{y}^k-\mathbf{y}^*\|_{\mathbf{\Omega}}^2 - \|\mathbf{y}^k-\mathbf{y}^{k+1}\|_{\mathbf{\Omega}}^2 \tag{49}$$

From this inequality, we can conclude that (1) $\|\mathbf{y}^k - \mathbf{y}^{k+1}\| \to 0$ as $k \to \infty$ and (2) $\{\mathbf{y}^k\}$ is a Cauchy sequence (proof omitted due to space limitation) and also a convergent sequence with limit point $\mathbf{y}^\infty$. We assume the iteration operator associated with Eq. (28) is $\mathcal{P}$, i.e., $\mathbf{y}^{k+1} = \mathcal{P}(\mathbf{y}^k)$. Then it is clear that $\mathbf{y}^k - \mathcal{P}(\mathbf{y}^k)$ is continuous in $\mathbf{y}^k$ and since $\mathbf{y}^k \to \mathbf{y}^\infty$ as $k \to \infty$, we can obtain $\mathbf{y}^\infty - \mathcal{P}(\mathbf{y}^\infty) = 0$. This indicates $\mathbf{y}^\infty$ is a solution of (22) since the right-hand side of (40) vanishes when plugging in $\mathbf{y}^k = \mathbf{y}^\infty$, which means $\mathbf{y}^\infty = \mathbf{y}^*$. $\square$

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