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Joint weakly regularly persistent nonlinear observability for landmark-based SLAM with nonlinear relative measurements

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Abstract— **In this paper, we give sufficient conditions on the input for weak regular observability in the general case of landmark-based Simultaneous Localisation and Mapping (SLAM) both with a world-centric and a sensor-centric point of view. We show notably that in the sensor-centric point of view, the dynamics of the robot is not important for this notion of observability and only its state and input trajectories matter. Besides, we prove that tracking circular trajectories imply weak regular observability jointly for 2D systems with several types of commonly used measurements in SLAM.**

I. Introduction

The problem of Simultaneous Localization and Mapping (SLAM) consists of reconstructing the state of a robot and a map of its environment at the same time from nonlinear measurements. Note that SLAM is more and more treated using optimization-based methods, see [9] and in particular Moving Horizon Estimation (MHE) where one recovers a state trajectory by only using past measurements on a sliding time window of fixed size, see [11], [12], [14], [15], [16], [22]. Since SLAM is a nonlinear dynamical estimation problem, nonlinear observability is fundamental in guaranteeing of good estimation performance, see [1] for a review on classical nonlinear observability concepts and [13], [24], [17] for studies of observability properties in SLAM. However, it seems that observability conditions for MHE applied to SLAM has not been extensively studied. In this paper, we focus on the recently introduced notion of weak regular observability and weak regularly persistent inputs [6]. Its purpose is to ensure the well-posedness and stability under perturbation of MHE problems, which are necessary for good performance in practice. Therefore, *our first contribution is to give sufficient conditions on the input for weak regular observability in the general case of landmark-based SLAM. We show notably that in the case where the state of the robot is known (sensor-centric view), the state and input trajectories matter for weak regular observability*.

A common feature of the existing plethora of nonlinear observability concepts is their dependence on the input applied to the system. It can directly hinder estimation performance and prevent one from using the so-called separation principle for control design in a general nonlinear setting. In the case of SLAM, it has been noticed that having a robot following a circular path, which is referred to as circumnavigation, appears to improve observability in SLAM and related problems of localization of autonomous systems, see [4], [8], [7], [18], [20], [21]. Inspired by this, *we show that 2D SLAM systems with different types of sensors can be made simultaneously weakly-regularly observable if the robot tracks any circular trajectory in the position/velocity space.*

The rest of the paper is organized as follows: Section II recalls the notion of weak regular observability and weak regularly persistent inputs and introduces sufficient conditions for a general SLAM system. Section III shows that circular trajectories are joint weak regularly persistent inputs for a SLAM system in the sensor-centric view for several common types of measurements.

II. Weak regular observability in landmark-based SLAM problems

In classical SLAM, one's aim is to localise a mobile robot and reconstruct a map of its environment at the same time using measurements of the robot's pose and the environment. Landmark-based SLAM is a particular version of SLAM in which the environment is represented by a set of discrete landmarks associated with continuous measurements, see [23]. This representation has the advantage of being sufficiently general to match many realistic scenarios while being amenable to analysis from the system dynamics point of view. In this section, we focus on specifying the results from Proposition 2.1 in the case where each landmark is observed at all times.

A. Problem Formulation and Relevant Notions of Observability

1) Nonlinear Observability: In the following, we denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{R}^+$ the set of non-negative real numbers and by $\mathbb{R}$ the set of real numbers. We fix $(n_x, n_u, n_y) \in \mathbb{N}^3$. We consider the following general nonlinear system:

$$\begin{aligned} \dot{x} &= f(x,u), \\ y &= h(x,u), \end{aligned} \tag{1}$$

where $u : \mathbb{R}^+ \longrightarrow U \subset \mathbb{R}^{n_u}$ is a piecewise continuous input trajectory, x is the corresponding state trajectory valued in $\mathbb{R}^{n_x}$ and y the corresponding measurement (or output) trajectory valued in $\mathbb{R}^{n_y}$; and $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \longrightarrow \mathbb{R}^{n_x}$ is the controlled vector field of the system and $h : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \longrightarrow \mathbb{R}^{n_y}$ is the observation function, also called output

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function. Mappings f and h are both assumed to be three times continuously differentiable.

For $s_2 \geq s_1 \geq 0$, and $\xi \in \mathbb{R}^{n_x}$, we denote by $\phi_f(s_2; s_1, \xi, u)$ the solution flow of system (1) at time s_2 with initial condition ξ, initial time s_1 and input trajectory u. In the following, the reference trajectory is defined, for some input trajectory u, by $x(t) := \phi_f(t; 0, x_0, u)$.

Definition 2.1 (Cumulative output error): For $0 \leq t_1 \leq t_2$, an input trajectory u and a pair of states (ξ_1, ξ_2) we define the *cumulative output error* of system (1) on $[t_1, t_2]$ at (ξ_1, ξ_2) with input trajectory u, denoted by $l(t_1, t_2, \xi_1, \xi_2, u)$, as follows:

$$l(t_1, t_2, \xi_1, \xi_2, u) = \int_{t_1}^{t_2} \|h(\phi_f(s; t_1, \xi_1, u), u(s)) - h(\phi_f(s; t_1, \xi_2, u), u(s))\|^2 \mathrm{d}s,$$

where $\|\cdot\|$ denotes the Euclidian norm.

Definition 2.2 (Weakly regularly persistent input): Fix an initial condition $x_0 \in \mathbb{R}^{n_x}$. An input trajectory u is said to be *weakly regularly persistent* at x_0, if there exists $T > 0$, $R > 0$ and a continuous increasing function κ such that, $\kappa(0) = 0$ and for any $t \geq T$ and any $(\xi_1, \xi_2) \in (\bar{B}(x(t-T), R))^2$:

$$l(t-T, t, \xi_1, \xi_2, u) \geq \kappa(\|\xi_1 - \xi_2\|), \tag{2}$$

where $x(t-T) = \phi_f(t-T; 0, x_0, u)$ and $\bar{B}(x(t-T), R)$ denoted the closed ball centered at $x(t-T)$ of radius R. System (1) is said to be weakly regularly observable if for any $x_0 \in \mathbb{R}^{n_x}$ there exists a weakly regularly persistent input trajectory at x_0.

In the following, for any $(n, m, p) \in \mathbb{N}^3$ and for any differentiable function $\psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$, $\mathrm{d}_x\psi(x, y)$ denotes the differential of $\psi(\cdot, y)$ for any $y \in \mathbb{R}^m$. Note that $\mathrm{d}_y\psi(x, y)$ is defined similarly.

Definition 2.3 (Observability Gramian): Let $T > 0$ be a time horizon, $x_0 \in \mathbb{R}^{n_x}$ be an initial condition and u be an input trajectory. For $t \geq T$, the *Observability Gramian* of system (1) on $[t-T, t]$, denoted by $\mathcal{C}(t, T, x(t-T), u)$ is defined as half the Hessian of $l(t-T, t, x(t-T), \cdot, u)$ taken at $x(t-T)$ and reads:

$$\begin{aligned}\mathcal{C}(t, T, x(t-T), u) &= \frac{1}{2}\mathrm{d}^2_{\xi_2} l(t-T, t, x(t-T), x(t-T), u), \\ &= \int_{t-T}^{t} \Phi_f^T H^T(x(s), u(s)) H(x(s), u(s)) \Phi_f \mathrm{d}s, \end{aligned} \tag{3}$$

where $H(x(s), u(s)) = \mathrm{d}_x h(x(s), u(s))$ and $\Phi_f(s; t-T, x(t-T), u) = \mathrm{d}_x\phi_f(s; t-T, \cdot, u)$.

Proposition 2.1: Let $x_0 \in \mathbb{R}^{n_x}$ be an initial condition and u be an input trajectory. Assume that the set U is compact, that $\sup_{t\geq 0} \sup_{x_0 \in \mathcal{X}} \|\phi_f(t-T; 0, x_0, u)\| < +\infty$ for any compact set $\mathcal{X}$, and that there exist $T > 0$ and $\mu > 0$ such that for any $t \geq T$:

$$\mathcal{C}(t, T, x(t-T), u) \succeq \mu I_{n_x}. \tag{4}$$

Then, u is a weakly regularly persistent input trajectory at x_0.

2) SLAM model: In the rest of the paper, we consider the problem of landmark-based SLAM where one wants to reconstruct the state of a robot and the 2D position of J fixed landmarks denoted respectively by $z \in \mathbb{R}^{n_z}$ and $\ell = (\ell_j)_{1 \leq j \leq J} \in \mathbb{R}^{2J}$. In the following, we focus on the case the state of system (1) can be decomposed as $x = (z, \ell)$. We also assume that dynamics of (z, ℓ) is defined as follows for $1 \leq j \leq J$ and any initial condition (z_0, ℓ):

$$\dot{z} = g(z, u), \qquad \dot{\ell}_j = 0, \tag{5}$$

where g represents the robot dynamics and u is an input trajectory. It is assumed that z can be partitioned into the 2D position of the robot in the inertial frame denoted by χ and remaining variables denoted by ηsuch that

$$\chi = Pz, \qquad \eta = (I - P)z,$$

where P is a projection matrix from $\mathbb{R}^{n_z}$ to $\mathbb{R}^2$. We also assume that the robot state and the landmark positions are observed through observations, denoted by y, of the following type:

$$y = h(z, \ell, u) = \begin{bmatrix} h_0(z, u) \\ h_1(\chi - \ell_1, \eta, u) \\ \vdots \\ h_J(\chi - \ell_J, \eta, u) \end{bmatrix}, \tag{6}$$

h_0 denotes a direct measurement of the robot's state and $(h_j)_{1 \leq j \leq J}$ represent relative measurements between the robot position and the landmark positions that may also depend on the remaining variables η. The first contribution of this paper is to provide sufficient conditions for weak regular persistence of the landmark-based SLAM system (5)-(6) based on Proposition 2.1 both with a world-centric and sensor-centric view.

B. Observability conditions in the world-centric case

In the world-centric case, one considers the position of the robot and the landmarks in an fixed frame with no knowledge of the initial state of the robot.

Proposition 2.2: For any $t \geq T > 0$, the Observability Gramian of system (5)-(6) reads:

$$\mathcal{C}(t, T) = \begin{bmatrix} \mathcal{A}_0(t, T) + \sum_{j=1}^{J} \mathcal{D}_j(t, T) & -\mathcal{B}_1^T(t, T) & \cdots & -\mathcal{B}_J^T(t, T) \\ -\mathcal{B}_1(t, T) & \mathcal{A}_1(t, T) & & 0 \\ \vdots & & \ddots & \\ -\mathcal{B}_J(t, T) & 0 & & \mathcal{A}_J(t, T) \end{bmatrix}, \tag{7}$$

where for any $1 \leq j \leq J$

$$\mathcal{A}_0(t, T) = \int_{t-T}^{t} A_0(s, z(s), u(s)) ds,$$

$$\mathcal{A}_j(t, T) = \int_{t-T}^{t} A_j(\chi(s) - \ell_j, \eta(s), u(s)) ds,$$

$$\mathcal{B}_j(t,T) = \int_{t-T}^{t} B_j(\chi(s)-\ell_j,\eta(s),u(s))ds,$$
$$\mathcal{D}_j(t,T) = \int_{t-T}^{t} D_j(\chi(s)-\ell_j,\eta(s),u(s))ds,$$

$A_j(\chi-\ell_j,\eta,u) = H_j^T(\chi-\ell_j,\eta,u)H_j(\chi-\ell_j,\eta,u)$, $A_0(s,z,u)=\Phi_g(s,z;u)^T H_0^T(z,u)H_0(z,u)\Phi_g(s,z;u)$,

$$B_j(s,z,\ell_j,\eta,u) = -A_1(\chi-\ell_j,\eta,u)P\Phi_g(s,z;u)$$
$$-H_j^T(\chi-\ell_j,u)H_j'(\chi-\ell_j,\eta,u)(I-P)\Phi_g(s,z;u),$$

$$\begin{aligned} D_j(s,\chi-\ell_j,\eta,u) =& \Phi_g^T(s,z;u)[(I-P)H_j'(\chi-\ell_j,\eta,u) \\ &+PH_j(\chi-\ell_j,\eta,u)]^T \\ &\times[(I-P)H_j'(\chi-\ell_j,\eta,u) \\ &+PH_j(\chi-\ell_j,\eta,u)]\Phi_g(s,z;u). \end{aligned}$$

Proof Sketch: The proof involves calculating the the differentials of the observation function (5) and the joint dynamics of the robot and landmarks (6). □

The idea of the sequel is to separate the sufficient observability conditions from Proposition 2.1 into conditions that will be automatically satisfied in the sensor-centric view and a set of independent conditions on the observability of each landmarks for a fixed robot trajectory involving only the matrices $\mathcal{A}_j(t,T)$ for $1 \le j \le J$. This translates into Assumptions 2.1 and 2.2.

Assumption 2.1: For any $(z_0,\ell)\in\mathbb{R}^{n_z+2J}$, there exists an input trajectory u, $T>0$ and $\mu>0$ such that for any $t\ge T$ and any $1\le j\le J$:

$$\mathcal{A}_0(t,T)+\sum_{i=1}^{J}\mathcal{D}_i(t,T)-\mathcal{B}_i(t,T)^T\mathcal{A}_i(t,T)^{-1}\mathcal{B}_i(t,T)\succeq \mu I_{n_z}, \quad (8)$$

$$\mathcal{A}_j(t,T)\succeq \mu I_2. \quad (9)$$

Assumption 2.2: Tor any $(z_0,\ell)\in\mathbb{R}^{n_z+2J}$ and any $T>0$ there exists $\sigma>0$ such that and $t\ge T$:

$$\sum_{j=1}^{J}\mathcal{B}_j(t,T)^T\mathcal{B}_j(t,T)\preceq \sigma I_{n_z}.$$

Consequently, the following proposition states that, in order to obtain weak regular observability in the world-centric view, one can consider the persistence with respect to the state of the robot z and the landmarks ℓ_j in a split way through Schur complemenents.

Proposition 2.3: Let Assumptions 2.1 and 2.2 hold. For any $(z_0,\ell)\in\mathbb{R}^{n_z+2J}$, there exist (u,T,μ) satisfying (8)-(9), such that such that for any $0<\mu_0<\mu$ and $t\ge T$,

$$\mathcal{C}(t,T)\succeq \mu_0 I_{n_z+2J}. \quad (10)$$

Moreover, if $\sup_{t\ge0}\sup_{z_0\in\mathcal{X}}\|\phi_g(t-T;0,z_0,u)\|< +\infty$, then system (5) is weakly regularly observable.

Proof Sketch: The proof involves fixing $(z_0,\ell)\in\mathbb{R}^{n_z+2J}$, the associated T and μ from Assumption 2.1, $t\ge0$ and $0<\mu_0<\mu$ and establishing $\mathcal{C}(t,T)-\mu_0 I_{n_z+2J}\succeq 0$. □

C. Observability conditions in the sensor-centric case

In the sensor-centric case, one considers that z_0 is known so that the frame of study is centered at χ_0. In our deterministic framework, since the input u is known, this is equivalent to having $h_0(z,u)=z$. Therefore, Assumption 2.1 can be weakened to recover the result of Proposition 2.3.

Assumption 2.3: For any $(z_0,\ell)\in\mathbb{R}^{n_z+2J}$, there exists an input trajectory u, $T>0$ and $\mu>0$ such that for any $t\ge T$ and any $1\le j\le J$:

$$\mathcal{A}_j(t,T)\succeq \mu I_2. \quad (11)$$

Finally, Proposition 2.4 states that in the sensor-centric view, only the weak regular persistence with respect to the position of the landmarks is required.

Proposition 2.4: Under Assumptions 2.2 and 2.3, if $h_0(z,u)=z$ then for any $(z_0,\ell)\in\mathbb{R}^{n_z+2J}$, with (u,T,μ) satisfying (9), there exists μ_0 such that for any $t\ge0$,

$$\mathcal{C}(t,T)\succeq \mu_0 I_{n_z+2J}. \quad (12)$$

Furthermore, under Assumption 2.1, system (5) is weakly regularly observable.

Proof: Under the assumptions of the proposition, for any $(z_0,\ell)\in\mathbb{R}^{n_z+2J}$, there exists an input trajectory u, $T>0$ and $\mu>0$ such that for any $t\ge T$ and any $1\le j\le J$:

$$\begin{aligned} &\mathcal{A}_0(t,T)=TI_{n_z}, \\ &\mathcal{A}_j(t,T)\succeq \mu I_2. \end{aligned} \quad (13)$$

Besides, for any $1\le j\le J$,

$$\mathcal{D}_j-\mathcal{B}_j^T\mathcal{A}_j^{-1}\mathcal{B}_j\succeq 0,$$

as $\begin{bmatrix}\mathcal{A}_j & \mathcal{B}_j\\ \mathcal{B}_j^T & \mathcal{D}_j\end{bmatrix}\succeq 0$ and $\mathcal{A}_j\succ 0$. This leads to $\mathcal{A}_0+\sum_{j=1}^{J}\mathcal{D}_j-\mathcal{B}_j^T\mathcal{A}_j^{-1}\mathcal{B}_j\succeq TI_{n_z}$. Thus, (8) and (9) from Assumption 2.1 are satisfied with $\mu'=\min(T,\mu)$. The rest follows from Proposition 2.3. ■

Remark 2.1: For simplicity of the presentation, it is assumed in this section that the landmarks are observed at all times by the robot. Yet, it is not completely realistic as one needs in practice to match the measurements with the landmarks. This is represented by a familiy of data association functions $(a_j(\cdot))_{1\le j\le J}$ such that $a_j:\mathbb{R}^+\longrightarrow\{0,1\}$ takes the value 1 when the landmark j is seen by robot and 0 if not. However, if we consider fixed and known data association functions $(a_j(\cdot))_{1\le j\le J}$, the result from this section can be recovered by defining $\tilde{h}_j(t,\chi-\ell_1,\eta,u)=a_j(t)h_1(\chi-\ell_1,\eta,u)$. In particular, $\widetilde{\mathcal{A}}_j$ can be written as

$$\widetilde{\mathcal{A}}_j(t,T)=\int_{t-T}^{t}a_j(s)H_j^T(s)H_j(s),$$

where $H_j(s)=H_j^T(\chi(s)-\ell_j,u(s))$. One can then recover Assumption 2.3 and Proposition 2.4. In this case, Assumption 2.3 requires, broadly speaking, that the input u only be persistent for the landmark j only when it is seen by the robot.

Remark 2.2: Assumption 2.2 is not restrictive as in most SLAM problems the area to explore is bounded a priori as well as the state and input trajectories of the system. This will be illustrated later in Section III

Remark 2.3: One can notice that Assumption 2.3 does not depend on the dynamics of system (5). This means that, in the sensor-centric case, the study of weak regular observability can be decomposed in two steps. First, for fixed state and input spaces, one can look for state and input trajectories that satisfy Assumption 2.3. Secondly, for some robot dynamics g one can check if the previous trajectories can be tracked by state and input trajectories (or state trajectories only if the measurement does not depend explicitly on u) that are compatible with the corresponding dynamical constraint (5).

Following from Remark 2.1, the last contribution of this paper is to show that several relevant simple landmark-based SLAM systems with different observation types are made weakly regularly observable by tracking circular paths.

III. Joint weak regular observability by circumnavigation for second order SLAM problems with different sensor modalities

Circumnavigation to ensure observability in localisation and SLAM problems with bearing measurements or range measurements has notably been studied in [4], [20]. Optical flow measurements are also well-known in SLAM, see [3]. An observability analysis of optical flow measurement for inertial navigation can be found in [5], [25]. Observability of Doppler-shift measurements in SLAM and localisation problems have been studied in [10], [19]. The goal of this section is to shed light on the similarities of these four types of measurements by carrying out a joint observability analysis in the framework developed in the previous sections. This joint study does not seem to be present in the existing literature. In particular, we show that, in the case of a second order SLAM system with one landmark, Assumption 2.2 and 2.3 are jointly satisfied for the four types of measurements when the robot tracks circular paths around any point. As discussed below, it is without loss of generality that we can consider only one landmark.

A. Model description

In this section we are interested in a 2D SLAM system where the state of the robot is represented by position and velocity variables, $z = (\chi, v) \in \mathbb{R}^4$. Since Assumption 2.3 is distributed over the landmarks, we can assumed without loss of generality that $J = 1$ and our SLAM system has only one unknown landmark position $\ell \in \mathbb{R}^2$. Our system can be written in the following form:

$$\dot{\chi} = v, \quad \dot{v} = g(\chi, v, u), \quad \dot{\ell} = 0, \tag{14}$$

where u is some input trajectory and g is twice continuously differentiable. In this application, the remaining variables η are exactly the velocity variables so that $\eta = v$. The state of the robot is supposed to be observed through a relative measurement in position that can also depend on the velocity variables. In the sequel, we consider the following types of measurements:

1) Bearing measurements where one measures the direction from the robot to the landmark such that:
$$h^{(1)}(\chi - \ell, v) = p, \tag{15}$$
where $p = \frac{\ell - \chi}{\|\ell - \chi\|}$.
2) Range measurements where one measures the distance between the robot and the landmark such that:
$$h^{(2)}(\chi - \ell, v) = r, \tag{16}$$
where $r = \|\ell - \chi\|$.
3) Optical flow measurements where one measures the angular velocity of the landmark in the referential of the robot such that:
$$h^{(3)}(\chi - \ell, v) = \frac{\langle v, Qp \rangle}{r}, \tag{17}$$
where $\langle \cdot \rangle$ denotes the canonical scalar product on $\mathbb{R}^2$ and Q is the rotation matrix of angle $\frac{\pi}{2}$.
4) Doppler shift measurements where one measures a frequency shift between the landmark and the robot such that:
$$h^{(4)}(\chi - \ell, v) = \alpha \langle v, p \rangle, \tag{18}$$
where $\alpha > 0$ is a constant.

We consider this SLAM system in the sensor-centric view so the trajectory of the robot can be seen as fully observed:

$$h_0(z) = z. \tag{19}$$

In particular, the specific choices of g and u do not matter in the verification of Assumption 2.3 as stated in Remark 2.1. Only the solution flow ϕ_g representing the trajectory of the robot in the position/velocity space and the fixed position of the landmark ℓ are important. As a consequence, satisfying Assumption 2.2 and 2.3 in this case boils down to finding a bounded position and velocity trajectory for the robot, denoted by $z(s) = (\chi(s), v(s))$, and positive real numbers $\mu > 0$ and $T > 0$ such that for any $t \geq T$ and $i \in \{1, 2, 3, 4\}$:

$$\mathcal{A}^{(i)}(t, T) = \int_{t-T}^{t} H^{(i)T}(s) H^{(i)}(s) ds \succeq \mu I_2, \tag{20}$$

where $H^{(i)}(s) = H^{(i)}(\chi(s) - \ell, v(s))$.

The topic of the next section is precisely to show that if the robot travels in circle around any point then (20) is satisfied simultaneously for the aforementioned four types of relative measurements.

B. Main results

To define the circular paths considered in this section, we fix $\chi_c = (\chi_{c,1}, \chi_{c,2}) \in \mathbb{R}^2$, $\bar{\chi}_0 = (\bar{\chi}_{0,1}, \bar{\chi}_{0,2}) \in \mathbb{R}^2$, $\omega > 0$ and $r_c > 0$. Then, the circular position and velocity trajectories around χ_c of radius r_c, denoted by $(\bar{\chi}, \bar{v})$, read for any $s \geq 0$:

$$\bar{\chi}(s) = \chi_c + r_c \begin{bmatrix} \cos(\omega s + \bar{\psi}(0)) \\ \sin(\omega s + \bar{\psi}(0)) \end{bmatrix}, \tag{21}$$

$$\bar{v}(s) = \omega r_c \begin{bmatrix} -\sin(\omega s + \bar{\psi}(0)) \\ \cos(\omega s + \bar{\psi}(0)) \end{bmatrix}, \tag{22}$$

where $\bar{\psi}(0) = \text{atan2}(\bar{\chi}_{0,1} - \chi_{c,1}, \bar{\chi}_{0,2} - \chi_{c,2})$. From this, one can define, $\bar{\mathcal{A}}^{(i)}(t,T)$ as the Observability Grammian with respect to the landmark position associated with the circular trajectory for measurement of type i. It reads, for $\ell \in \mathbb{R}^2$, $T > 0$, $t \geq T$, and $i \in \{1,2,3,4\}$:

$$\bar{\mathcal{A}}^{(i)}(t,T) = \int_{t-T}^{t} \bar{H}^{(i)T}(s)\bar{H}^{(i)}(s)ds, \tag{23}$$

where $\bar{H}^{(i)}(s) = H^{(i)}(\bar{\chi}(s) - \ell, \bar{v}(s))$.

One can now state the first result of the section which is contained in Proposition 3.1.

Proposition 3.1: For any $r_c > 0$, $\omega > 0$, $\chi_c \in \mathbb{R}^2$, $\chi_0 \in \mathbb{R}^2$, $\ell \in \mathbb{R}^2$ and any $i \in \{1,2,4\}$ there exist $T^{(i)} > 0$ and $\mu^{(i)} > 0$ such that, for any $t \geq T^{(i)}$

$$\bar{\mathcal{A}}^{(i)}(t, T^{(i)}) \succeq \mu^{(i)} I_2. \tag{24}$$

Additionally, if $r_c \neq \|\chi_c - \ell\|$, then (24) also holds for $i = 3$.
Proof Sketch: The proof requires establishing a time-invariant lower bound for the integrals of the form (20) to prove the weak regular observability of (5). □

Set $\|z\| = (\|\chi\|^2 + \|v\|)^{\frac{1}{2}}$. We can now state the main result of the section.

Theorem 3.2: For any $(\chi_0, v_0, \ell) \in \mathbb{R}^6$, $r_c > 0$, $\omega > 0$, $\chi_c \in \mathbb{R}^2$, $\bar{\chi}_0 \in \mathbb{R}^2$, and bounded input trajectory, u, for the dynamics (14) such that $r_c \neq \|\chi_c - \ell\|$, there exists $\bar{\epsilon} > 0$ such that for any $0 < \epsilon < \bar{\epsilon}$, if the associated circular path, $(\bar{\chi}, \bar{v})$, defined by (22) and the corresponding position and velocity solution flow, (χ, v), defined, for any $s \geq 0$ by $(\chi(s), v(s)) = \phi_g(s; 0, (\chi_0, v_0), u)$ satisfy:

$$\sup_{s \geq 0} \|z(s) - \bar{z}(s)\| < +\infty, \tag{25}$$

$$\int_0^{+\infty} \|z(s) - \bar{z}(s)\| ds \leq \epsilon, \tag{26}$$

then u is a weakly regularly persistent input trajectory at (χ_0, v_0, ℓ) for the systems (14) and (19) with anyone of the measurements (15)-(18). In particular, the previous systems are weakly regularly observable with a joint input trajectory.
Proof: See Appendix A. ■

Remark 3.1: Informally, Theorem 3.2 states that for a fixed initial condition, if a controlled trajectory of system (14) tracks any circular trajectory of the form (22) then the corresponding input trajectory is weakly regularly persistent at this initial state. As an immediate corollary, the result also holds if $\int_0^{+\infty} \|z(s) - \bar{z}(s)\| ds = 0$, which corresponds to the case where the state trajectory z is exactly a circular path.

Remark 3.2: Theorem 3.2 has a direct application in control design. In fact, as any circular trajectory makes the system weakly regularly observable then one does not need to know the position of ℓ to ensure observability through an adequate choice of u. In other words, only the level of observability, represented by λ in (30), depends on ℓ but not the qualitative property of weak regular observability. This is to be nuanced in the case of optical flow measurement as the result does not hold if $\|\ell - \chi_c\| = r_c$ even if this condition seems not to be satisfied generically.

CONCLUSION

In this paper, we first prove sufficient conditions for weak regular observability landmark-based SLAM systems both in the world-centric and sensor-centric case and provided several sufficient conditions for weak regular observability. Secondly, we show these conditions are simultaneously satisfied in a SLAM problem with a second order dynamics and various measurements when the robot trajectory tracks a circular path.

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APPENDIX

A. Proof of Theorem 3.2

Proof: Since systems (14), (19), (15)-(18) evolve in the sensor-centric view, the result can be obtained by applying Proposition 2.4. To do so, one need to check if Assumption 2.2 and 2.3 are satisfied with the settings of the theorem. Therefore, we first fix $(\chi_0, v_0, \ell) \in \mathbb{R}^6$, $r_c > 0$, $\omega > 0$, $\chi_c \in \mathbb{R}^2$, $\chi_0 \in \mathbb{R}^2$, $\ell \in \mathbb{R}^2$ and an input trajectory, u, for the dynamics (14) such that the corresponding circular path, $\bar{z} = (\bar{\chi}, \bar{v})$, defined by (22) and the corresponding position and velocity solution flow, $z = (\chi, v)$, defined, for any $s \geq 0$ by $(\chi(s), v(s)) = \phi_g(s; 0, (\chi_0, v_0), u)$ satisfies (25) and (26).

Concerning Assumption 2.2, as $(\bar{\chi}, \bar{v})$ is bounded by nature and (χ, v) verifies (25), (χ, v) is also bounded. We recall that u is also bounded. Note that the observation functions defined by (15)-(18) are continuously differentiable. Then, keeping the notations from (7) and by boundedness of z and u, we have that for any $T > 0$ there exists $L > 0$ such that, for any $t \geq T$: $\|\mathcal{B}(t,T)\| \leq L_1 \int_{t-T}^{t} \|\Phi_f(s; t - T, z(t-T), u)\| \mathrm{d}s$. Let's define $f(\chi, v, u) = \begin{bmatrix} v \\ g(\chi, v, u) \end{bmatrix}$. According to Theorem 2.3.2 of [2], for any $T > 0$, $t \geq T$ and $s \in [t-T, t]$, $\Phi_f(s; t-T, z(t-T), u) = M(s, t-T)$ is the solution of the following matrix-valued linear Cauchy problem:

$$\mathrm{d}_s M(s, t-T) = \mathrm{d}_z f(\chi(s), v(s), u(s)) M(s, t-T),$$
$$M(t-T, t-T) = I_6.$$

By integrating on $[t-T, t]$ and taking the norm, one gets for any $T > 0$, $t \geq T$ and $s \in [t-T, t]$:

$$\|M(s, t-T)\| \leq \|M(t-T, t-T)\| + \int_{t-T}^{t} \|\mathrm{d}_z f(\chi(s), v(s), u(s))\| \|M(s, t-T)\| ds \tag{27}$$

By assumption, $\mathrm{d}_z f$ is continuous and the trajectories (χ, v, u) are bounded so there exists $\sigma_1 > 0$ such that for any $T > 0$, $t \geq T$ and $s \in [t-T, t]$, i.e., $\|\mathrm{d}_z f(\chi(s), v(s), u(s))\| \leq \sigma_1$. Thus, $\|M(s, t-T)\| \leq 1 + \sigma_1 \int_{t-T}^{t} \|M(s, t-T)\| ds$. Thus, applying Gronwall Lemma yields $\|M(s, t-T)\| \leq \exp(\sigma_1 T)$. Integrating again on $[t-T, t]$ results in $\int_{t-T}^{t} \|M(s, t-T)\| ds \leq T \exp(\sigma_1 T)$ and $\|\mathcal{B}(t,T)\| \leq L_1 T \exp(\sigma_1 T)$.

Thus, Assumption 2.2 holds. Concerning Assumption 2.3, from Proposition 3.1, for $i \in \{1,2,3,4\}$ there exist $T^{(i)} > 0$ and $\mu^{(i)} > 0$ such that, for any $t \geq T^{(i)}$:

$$\bar{\mathcal{A}}^{(i)}(t, T^{(i)}) \succeq \mu^{(i)} I_2. \tag{28}$$

Recall the definition of $\mathcal{A}^{(i)}(t, T^{(i)})$ for any $i \in \{1,2,3,4\}$ and $t \geq T^{(i)}$:

$$\mathcal{A}^{(i)}(t, T^{(i)}) = \int_{t-T^{(i)}}^{t} H^{(i)T}(s) H^{(i)}(s) ds,$$

where $H^{(i)}(s) = H^{(i)}(\chi(s) - \ell, v(s))$ Moreover, since the observation functions defined by (15)-(18) are twice continuously differentiable then for any $i \in \{1,2,3,4\}$ $H^{(i)T} H^{(i)}$ is locally Lipschitz. We recall that the trajectories z and $\bar{z}$ are bounded. Thus, there exists $L_1 > 0$ such that for any $i \in \{1,2,3,4\}$, $t \geq T^{(i)}$ and $s \in [t-T, t]$:

$$\|H^{(i)T}(s) H^{(i)}(s) - \bar{H}^{(i)T}(s) \bar{H}^{(i)}(s)\| \leq L_1 \|z(s) - \bar{z}(s)\|.$$

Therefore, from (26), one has for any $i \in \{1,2,3,4\}$ and $t \geq T^{(i)}$,

$$\|\mathcal{A}^{(i)}(t, T^{(i)}) - \bar{\mathcal{A}}^{(i)}(t, T^{(i)})\| \leq L \int_0^{+\infty} \|z(s) - \bar{z}(s)\| ds,$$
$$\|\mathcal{A}^{(i)}(t, T^{(i)}) - \bar{\mathcal{A}}^{(i)}(t, T^{(i)})\| \leq L\epsilon, \tag{29}$$

where $L = T L_1$. Finally by substituting (29) in (28), one gets the following matrix inequality, for any $i \in \{1,2,3,4\}$ and $t \geq T^{(i)}$:

$$\mathcal{A}^{(i)}(t, T^{(i)}) \succeq \lambda I_2, \tag{30}$$

where $\lambda = \mu - L\epsilon$ with $\mu = \min_i(\mu^{(i)})$. Set $\bar{\epsilon} = \frac{\mu}{L}$. Then, for any $0 < \epsilon < \bar{\epsilon}$, $\lambda > 0$ and the input trajectory u satisfies Assumption 2.3. As mentioned at the beginning of the proof, the result follows from Proposition 2.4. ∎