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Structural Conditions for Leak Localization in Potential Flow Networks

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***Abstract*— In this paper, we analyze the leak localization problem in potential flow networks. We present a general model encompassing various physical systems, for example, water distribution networks. In contrast to conventional water distribution network models, our model is neither restricted to any particular type of potential loss function nor to vertex leaks. We consider leak localization via *vertex potential analysis*, a general description of methods that work by comparing vertex potentials calculated under a leak location hypothesis to measured vertex potentials. We derive conditions on the graph structure of the network and the potential sensor placement, under which it is guaranteed that the leak can be limited, via vertex potential analysis, to a small set of locations. We suggest a bisection method to utilize our conditions. Our conditions are based on *one-way edges* in graphs, a concept that we introduce. In extension, our results can be used for sensor placement.**

I. Introduction

Potential flow networks model various physical systems, such as water distribution networks, DC electrical circuits, steady-state AC electrical networks, heat networks, and gas networks. In these systems, particularly in water distribution networks, our primary focus, leaks present a significant problem.

In urban areas, access to clean water is inhibited as a substantial proportion, estimated around 30% globally [1], of the treated drinking water is lost through leaks. However, in addition to the reduced access and wasted monetary and energy resources, water pipe leaks may undermine and damage infrastructure [2] and allow pathogens to enter the drinking water [3]. For these reasons, leak localization (to close the broken pipe and repair the leak) is a critical operational task in managing water distribution systems.

The development of modern, communicating integrated sensor technology opens up the possibility of automating leak localization in water distribution systems. The survey [4] outlines a wide range of approaches for this task. Although many of these localization schemes yield impressive numerical results—such as those seen in the competition [5]—we recognize two matters of interest within the research field that deserve further attention.

1) In most of the present literature, for example [6], [7], [8], [9], [10], which are covered in [4], leaks are modeled to appear in a-priori fixed network vertices only. In practice, leaks often appear along water pipes (see the report [11]), and it is interesting to see how the leak localization problem is affected by including edge leaks. For special types of networks, single pipelines and branched networks, there is an extensive theory for water leaks placed in pipes [12], [13]. We generalize by considering arbitrary network topologies.
2) Theoretical aspects such as the well-posedness of the leak localization problem and localization guarantees of specific algorithms are seldom treated. An interesting exception is the work [14], which analyzes the relation between the leak location and the network pressures. However, again, only for vertex leaks.

In this paper, we approach both of these points. We define a potential flow network with a leak along an edge. We consider a specific algorithm class for leak localization within our model, which we call *vertex potential analysis*. Vertex potential analysis compares measured vertex potentials to calculated vertex potentials to deduce the leak location. Although the criteria for comparison are varied among the different works, the potential residual framework includes the leak localization solutions presented in [6], [7], [8], [9] and [10]. Our contribution consists of conditions on network structure and potential sensor placement sufficient for vertex potential analysis to restrict the leak to a small set of locations. We introduce a *one-way* property of network edges to do this. Our conditions are general in that they are independent of particular potential loss functions and network state.

First, in Section II, we introduce our potential flow network model and define the leak localization problem. In Section III, we present the *vertex potential analysis* method for leak localization. We account for two theoretical concerns with this method in Problem 1 and Problem 2. In Section IV, we describe how the potential difference over the leaking edge depends on the position of the leak in this edge. Here, we also introduce some graph-theoretic notation. In Section V, we introduce the concept of one-way edges between two vertices v_i and v_l. Our main result, Therorem 1, says that if e_j is a one-way edge between v_i and v_l, the potential difference between v_i and v_l depends predictably on the leak position in e_j. In Section VI, we summarize and mention how we plan to extend and apply the presented research. Proofs of theoretical results can be found in [15].

II. Potential Flow Network With a Leak

In this section, we present our potential flow network model. We have a leaky water distribution network in mind as the main application. However, we use general language so that our model describes other physical systems, such as

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$\text{in}(1) = v_4$
$\text{out}(1) = v_1$
$\text{in}(2) = v_1$
$\text{out}(2) = v_2$
$\text{in}(3) = v_1$
$\text{out}(3) = v_3$

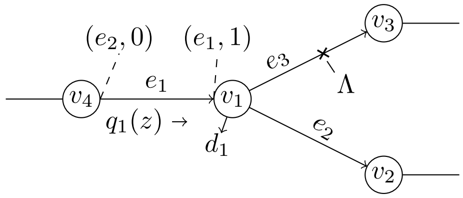

Fig. 1: A part of water distribution network. This part contains the leak.

DC electrical circuits, steady-state AC electrical networks, heat networks, and gas networks.

We consider a flow network model consisting of vertices $V = \{v_i\}_{i=1}^n$ and edges $E = \{e_j\}_{j=1}^m$ (n and m are finite). We refer to the tuple of vertices and edges, (V, E), as a *graph*. Each edge e_j in the graph has a flow reference direction, from the inlet vertex $\text{in}(j) \in V$ to the outlet vertex $\text{out}(j) \in V$. Disregarding the reference directions, the undirected graph is connected. A part of a network is shown in Fig. 1.

Now, every vertex v_i has an associated *demand* $d_i \in \mathbb{R}$. A positive demand represents consumption, and a negative demand represents production. Every vertex also has a *potential* $h_i \in \mathbb{R}$. Similarly, in every *position* $z \in [0, 1]$, along every edge e_j, there is a *flow* $q_j(z) \in \mathbb{R}$ and a potential $p_j(z) \in \mathbb{R}$. Demand and flow share the same unit, and the same holds for vertex and edge potential. Using different notation, d, q, h, and p is a choice we have made for presentation purposes.

In every vertex v, there is a conservation of mass equation, analogous to Kirchhoff's current law from electric circuit theory [20],

$$\sum_{j:\text{out}(j)=i} q_j(1) - \sum_{j:\text{in}(j)=i} q_j(0) = d_i. \tag{1}$$

Further, corresponding to Ohm's law from circuit theory and head-loss equations from water pipe modeling, every edge e_j has a *potential loss* function $u_j(z, q)$, such that

$$p_j(z_1) - p_j(z_2) = \int_{z_1}^{z_2} u_j(z, q_j(z))\, dz, \qquad z_1, z_2 \in [0, 1].$$

All potential loss functions are differentiable with respect to flow, with positive derivatives $\frac{\partial u_j(z,q)}{\partial q} > 0$ for all $z \in [0, 1]$. We define the *edge potential loss functional*

$$U_j(q_j) = \int_0^1 u(z, q_j(z))dz = p_j(0) - p_j(1). \tag{2}$$

Further, potential is continuous around the vertices, $p_j(0) = p_l(1) = h_i$ for all $j : \text{in}(j) = v_i$ and $l : \text{out}(l) = v_i$.

We assume there is a (single) leak in the network, in position x along edge e_k:

$$q_k(z) = \begin{cases} q_k(0), & 0 \le z \le x, \\ q_k(1) = q_k(0) - d_{\text{leak}}, & z < x \le 1. \end{cases}$$

Here, the leak demand $d_{\text{leak}} = -\sum_{v_j \in \mathcal{V}} d_j$ corresponds to the discrepancy between production and consumption. All other q_j ($j \neq k$) are independent of z. We use $\Lambda^* = (e_k, x)$ to denote the leak location.

Remark 1. *The model we describe includes leaky steady-state water distribution systems, as is the focus of the survey [4] and as used in the popular simulation software EPANET [16]. In the steady-state water network setting, flow is interpreted as pipe water flow and potential as pressure or hydraulic head. The potential loss functions are head-loss functions per normalized pipe length. Our model also describes DC electrical networks. In this case, flow is interpreted as electrical current and potential as voltage. The potential loss functions describe voltage drop per normalized conductor length.*

Remark 2. *An important contribution is that, in contrast to many of the works on water distribution systems covered in [4], which assume the leak to be located in a vertex, our model allows the leak to be located anywhere along any edge. However, this does not mean we neglect the possibility of a leak in a vertex. It can be shown that our convention of placing the leak in an edge includes the case of a leak in a vertex v_i, by letting $\Lambda^* = (e_j, 0)$ for any $j : in(j) = v_i$ or $\Lambda^* = (e_j, 1)$ for any $e_j : out(j) = v_i$.*

We assume the system operator has perfect model knowledge, i.e., knows all u_j. This means the potential loss functions can be used to calculate solutions to the model equations. However, our theoretical results do not rely on any specific form of function but only on differentiability and a positive derivative, as stated above. Full model knowledge is a strong assumption in many scenarios, which may limit the immediate application of our results in physical systems. However, when a perfect model is not available, there may be an uncertain one. Our results can be extended to uncertain models. If, for example, model parameters are given within uncertainty intervals (as in the water leak localization paper [17]), then the estimated leak location will fall within uncertainty intervals as well.

Furthermore, we assume the system operator has access to sensor measurements. In particular, all demands d_i are measured. Finally, we assume the system operator has access to potential sensor measurements of h_i in a subset of vertices, $v_i \in V_{\text{sens}} \subseteq V$.

It's important to note that in some practical flow networks, other types of sensors may be installed besides the ones we are considering. For example, there may be water pipe flow or electrical current sensors. In water distribution systems, microphones are sometimes installed to detect the sound of leaks [18]. In general, having more sensors provides more information, which makes leak localization easier. When installing sensors during system development, it's essential to carefully balance the economic costs against the effectiveness of each sensor in aiding leak localization. Our work provides theoretical guarantees for identifying which leaks can be localized with the available sensors. However, a broader assessment of the effectiveness of different sensor types presents an interesting direction for future research.

Algorithm 1 Vertex potential analysis.

for each leak location hypothesis H_Λ : *the leak is in* $\Lambda = (e_j, x_j)$ **do**
  Solve all instances of (1) and (2) by utilizing knowledge of $\{u_j\}_{e_j \in E}$ and $\{d_i\}_{v_i \in V}$ to calculate $h_V^\Lambda = \{h_i^\Lambda\}_{v_i \in V}$.
end for
Choose the leak location Λ which gives zero error between the measured $h_{V_{\text{sens}}}$ and calculated $h_{V_{\text{sens}}}^\Lambda$.

By leak localization, in our current work, we refer to the system operator's task of determining Λ^*, given $\{u_j\}_{e_j \in E}$, $\{d_i\}_{v_i \in V}$, and $\{h_i\}_{v_i \in V_{\text{sens}}}$.

III. The Vertex Potential Analysis approach to leak localization

The unifying components of the water network leak localization methods in [6], [7], [8], [9], [10] are the steps in Algorithm 1. In this section, we discuss some theoretical considerations regarding this approach.

The rationale behind Algorithm 1 follows from [19]. According to Theorem 3' from [19], when all demands are known, and there are no leakages, it is possible to solve all instances of (1) and (2), thus calculating all flows and potentials (potentials are determined up to an additive constant). When we make a leak location hypothesis and assign the leak demand d_{leak} to this position, we attain a network where all demands are known, for which we can calculate a solution.

In particular, for a leak location hypothesis, we construct an augmented graph (V_Λ, E_Λ) with an added vertex in $\Lambda = (e_j, x_j)$, so that $V_\Lambda = V \cup \{v_\Lambda\}$, $E_\Lambda = (E \setminus \{e_j\}) \cup \{e_{j,\text{in}}, e_{j,\text{out}}\}$ where $\text{in}(e_{j,\text{in}}) = \text{in}(j)$, $\text{out}(e_{j,\text{in}}) = \text{in}(e_{j,\text{out}}) = v_\Lambda$, $\text{out}(e_{j,\text{out}}) = \text{out}(j)$ and $u_{j,\text{in}}(z,q) = x_j u_j(z x_j, q)$, $u_{j,\text{out}}(z,q) = (1-x_j) u_j(x_j + z(1-x_j), q)$. The scaling here, with x and $(1-x)$ is such that $U_{j,\text{in}}(q) + U_{j,\text{out}}(q) = U_j(q)$. This means that the two new edges connected in series would behave like the original edge if there was no extraction in v_Λ.

The difference between measured production and consumption in the whole network is assigned as the output of the leak vertex, $d_\Lambda = -\sum_{v_i \in V} d_i$.

For this augmented graph network, all demands are known, and the solution to all instances of (1) and (2) can be calculated.

With the calculated solution, the hypothesis H_Λ can be evaluated by comparing $h_{V_{\text{sens}}}^\Lambda$ to the measured $h_{V_{\text{sens}}}$. If we formulate the correct hypothesis, $j = k$, $x_j = x$, the calculated solution will match the true solution, and we will have zero error between the calculated and measured potentials in V_{sens}.

Remark 3. *In all practical scenarios, measurement errors and model parameter uncertainties are involved in the leakage localization problem, and the theoretical analysis becomes more involved. When errors are present, it is, for instance, generally not guaranteed that the correct hypothesis will produce a zero potential residual. However, this paper does not delve into the disturbed case; we assume error-free measurements and perfect parameter knowledge. It turns out that leak localization can be challenging even in undisturbed scenarios, see, for example, [21]. Importantly, if the leak localization problem is hard under these conditions, it will also be hard when disturbances are present. Accordingly, we focus on the undisturbed case.*

Remark 4. *Our formulation of Algorithm 1 is elementary. We make simple assumptions and consider measurements from a single steady-state instant only. This makes the theoretical analysis easier. In practice, often, one can assume a correlation of data in time. For example, there may be long recorded data trajectories before a leak occurs. The system response can then potentially be used to localize the leakage. Analysis of multiple steady states for leak localization can be found in [21].*

In this paper, we investigate two problems related to the leakage localization scheme in Algorithm 1.

We notice that it is not guaranteed that $j = k$, $x_j = x$ is the only hypothesis that will give zero error. From [21], we know that for a special network parallel edge network topology, multiple leakage locations on different edges may lead to zero error between measured and calculated solutions to model equations. In this paper, we determine if, when, and how similar phenomena occur in arbitrary network topologies. We formulate problem 1

Problem 1. *Which leakage location hypotheses* H_Λ *lead to a zero potential difference between* $h_{V_{sens}}$ *and* $h_{V_{sens}}^\Lambda$*, i.e., a zero potential* residual*?*

As mentioned, in our error-free scenario, H_{Λ^*} leads to a zero potential residual. However, more leakage location hypotheses may lead to zero potential residual. We investigate where these may appear.

The second question related to Algorithm 1 we investigate concerns infinitely many possible hypotheses. Since every position $z \in [0,1]$ on every edge e_j is considered a potential leak location, it is impossible to calculate the network solution for each. However, there may be some structure that we can use to rule out continua of hypotheses at a time. We formulate Problem 2.

Problem 2. *Which leakage location hypotheses* H_Λ *can be rejected without calculating* h_V^Λ *for each of them?*

The rest of the paper deals with Problem 1 and Problem 2. We do not present complete solutions to the problems but provide conditions for when the number of leakage locations can be efficiently reduced.

IV. Potential over hypothesized edge

We aim to derive predictable relations between the calculated solution to (1) and (2), and the hypothesized leak location $\Lambda = (e_j, x_j)$, that we can use to evaluate and reject sets of hypotheses simultaneously, thereby approaching Problem 2.

We begin by looking at the calculated flows and potentials in the edge e_j when $\Lambda = (e_j, x_j)$, and how these depend on x_j. The derivative with respect to x_j describes the sensitivity of the calculated solution relative to an infinitesimal change of the leak position x_j. Our first result, Lemma 1, concerns the potential loss over the hypothesized leaking edge e.

Lemma 1. *If $\Lambda = (e_j, x_j)$ then*

$$\frac{d}{dx_j}\left(h^{\Lambda}_{in(j)} - h^{\Lambda}_{out(j)}\right) = J_1(x_j, q^{\Lambda}_{j,in}, q^{\Lambda}_{j,out}) + J_2(x_j, q^{\Lambda}_{j,in}, q^{\Lambda}_{j,out})\frac{dq^{\Lambda}_{j,in}}{dx_j},$$

where

$$J_1(x_j, q^{\Lambda}_{j,in}, q^{\Lambda}_{j,out}) = u_j(x_j, q^{\Lambda}_{j,in}) - u_j(x_j, q^{\Lambda}_{j,out}),$$
$$J_2(x_j, q^{\Lambda}_{e_{j,in}}, q^{\Lambda}_{e_{j,out}}) = \int_0^{x_j} \frac{\partial u_j}{\partial q}(z, q^{\Lambda}_{j,in})\ dz + \int_{x_j}^1 \frac{\partial u_j}{\partial q}(z, q^{\Lambda}_{j,out})\ dz.$$

Lemma 1 describes how the calculated potential difference over edge e_j, under H_Λ, $\Lambda = (e_j, x_j)$, depends on the hypothesised relative leak position x_j.

To obtain the exact numerical values of the J_1 and J_2 function expressions, we must know the calculated edge flows $\{q^{\Lambda}_l\}_{e_l \in E}$, which we obtain from (1) and (2). However, we may make useful quantitative observations before solving (1) and (2). This is indeed what we are after, namely, a way to predict how the calculated solution depends on Λ without having to compute $\{q^{\Lambda}_l\}_{e_l \in E}$ and h^{Λ}_V.

We now present a simple example involving Lemma 1. The considered network example was treated in [12] and [21].

Example 1. *We consider a single-edge graph $(V, E) = (\{v_1, v_2\}, \{e_1\})$ with a leak, $d_1 + d_2 \neq 0$. We hypothesize a leak location $\Lambda = (e_1, x_1)$. According to Lemma 1,*

$$\frac{d}{dx_1}\left(h^{\Lambda}_1 - h^{\Lambda}_2\right) = u_1(x_1, q^{\Lambda}_{1,in}) - u_1(x_1, q^{\Lambda}_{1,out}) + \left(\int_0^{x_1} \frac{\partial u_1}{\partial q}(z, q^{\Lambda}_{1,in})\ dz + \int_{x_1}^1 \frac{\partial u_1}{\partial q}(z, q^{\Lambda}_{1,out})\ dz\right)\frac{dq^{\Lambda}_{1,in}}{dx_1}.$$

Applying (1) *to v_1 and v_2 gives $q_{1,in} = -d_1$, which is constant, so $\frac{dq^{\Lambda}_{1,in}}{dx_1} = 0$. Also* (1) *applied to v_2 gives $q_{1,out} = d_2$. Consequently, we get $\frac{d}{dx_1}(h^{\Lambda}_1 - h^{\Lambda}_2) = u_1(x_1, -d_1) - u_1(x_1, d_2)$. Now if $-d_1 > d_2$, we get $\frac{d}{dx_1}(h^{\Lambda}_1 - h^{\Lambda}_2) > 0$, due to the positive derivative of u_1 in the second argument, q. If instead $-d_1 < d_2$, we get $\frac{d}{dx_1}(h^{\Lambda}_1 - h^{\Lambda}_2) < 0$. Either way, $h^{\Lambda}_1 - h^{\Lambda}_2$ depends monotonically on x_1, and so there can only be one x_1 such that $h^{\Lambda}_1 - h^{\Lambda}_2 = h_1 - h_2$ (measured values). If $-d_1 = d_2$, there is no leakage to find. In the following, without loss of generality, we will consider only positive leakage, $\sum_{v_i \in d_i} < 0$.*

We see that if V_{sens}, the true leak position x is recognized as the only x_1 that gives $h^{\Lambda}_1 - h^{\Lambda}_2 = h_1 - h_2$. This is a positive result in relation to Problem 1.

The monotonicity is also positive in relation to Problem 2. Once we have calculated $h^{\Lambda}_1 - h^{\Lambda}_2 \neq$, we know immediately on which side of x_1 the leakage truly lies, and we can reject all alternatives on the other side.

This is to be expected for the relatively simple considered single-edge example. Indeed, there is a formula for x in terms of d_1, d_2, h_1, h_2 in [12]. The example serves as a special case of, and an easy introduction to, a more general computational framework for and perspective of this type of problem.

The computational aspect of finding x depends on the potential loss function u_1. The formula in [12] holds under the assumption of uniform edges. In general, we can always use a bisection method. We suggest such a scheme in Algorithm 2.

Our first consequence of Lemma 1 is that the potential over e_j (in the solution under $\Lambda = (e_j, x_j)$) always increases with x_j, i.e., as the leak is moved towards out(j). We formulate this in Corollary 1. However, we first introduce some graph theoretic notation in Definition 1.

Definition 1. *For a natural number N, we let $[N] = \{1, 2, \ldots, N\}$ and $[N]^- = \{-N, \ldots, -1, 1, \ldots, N\}$. We consider the function $\pi : [\mu] \to [m]^-$.*

We say that:

- *π is a* walk *on (V, E) from in$(\pi(1))$ to out$(\pi(\mu))$ if in$(\pi(\iota + 1))$ = out$(\pi(\iota))$ for each $\iota = 1, \ldots, \mu - 1$.*
- *A walk π on (V, E) is a* path *from in$(\pi(1))$ to out$(\pi(\mu))$ if all vertices in$(\pi(\iota))$ $\iota = 1, \ldots, \mu$, as well as out$(\pi(\mu))$, are distinct.*
- *A walk π on (V, E) is a* cycle *if all in$(\pi(\iota)))$ $\iota = 1, \ldots, \mu$ are distinct, and out$(\pi(\mu))$ = in$(\pi(1))$.*
- *$\pi' : [\mu'] \to [m]^-$ is a* sub-walk *of π (denoted $\pi' \subseteq \pi$) if there is a $\zeta \in \{0, \ldots, \mu - \mu'\}$ such that $\pi'(\iota) = \pi(\iota + \zeta)$, $\iota = 1, \ldots \mu'$. A sub-walk of a path is called a sub-path.*
- *π contains e_j ($e_j \in \pi$) if $\pi(\iota) = j$ for some ι.*

We use π and the generated sequence of edges $(e_{\pi(\iota)})_{\iota=1}^{\mu}$ interchangeably.

Finally, walks (including paths and cycles) can traverse the edges in E in both the positive and negative reference directions. We use e_{-j} to denote the reversed perspective on e_j, with in$(-j)$ = out(j), out$(-j)$ = in(j).

Corollary 1. *Suppose there is a positive leak in the network, $\sum_{v_i \in V} d_i < 0$. Then for $\Lambda = (e_j, x_j)$ it holds that*

$$\frac{dq^{\Lambda}_{j,in}}{dx_j} = \frac{dq^{\Lambda}_{j,out}}{dx_j}\begin{cases} < 0, & \text{if there is a cycle containing } e_j, \\ = 0, & \text{if there is no cycle containing } e_j. \end{cases}$$

Either way (cycle or no cycle), $\frac{d}{dx_j}\left(h^{\Lambda}_{in(j)} - h^{\Lambda}_{out(j)}\right) > 0$.

Remark 5. *We want to emphasize that the derivatives of $q^{\Lambda}_{j,in}$ and $q^{\Lambda}_{j,out}$ with respect to x_j in Corollary 1 do not represent*

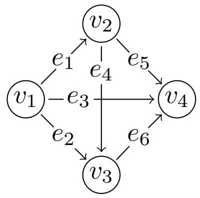

Fig. 2: A potential flow network with four vertices and six edges.

the flow variation along the length of the edge. All derivatives with respect to x_j represent the sensitivity of the calculated solution to all instances (1) *and* (2) *under the hypothesis H_Λ, where $\Lambda = (e_j, x_j)$, to a variation in the leak position x_j.*

Corollary 1 is interesting because it tells us something about what can be expected to happen with the calculated solution in a certain edge as we perturb the leakage location hypothesis, without having to calculate the solution. It is also remarkable that we need not assume anything about the topology of the rest of the network to know that the potential difference over the considered edge will increase.

In Section V, we extend the analysis to see how $h_i^\Lambda - h_l^\Lambda$, $\Lambda = (e_j, x_j)$, depends on a variation in x_j when v_i and v_l are not necessarily in(j) and out(j). We show that when e_j is a *one-way* edge between v_i and v_l, we can make certain conclusions regarding $h_i^\Lambda - h_l^\Lambda$.

V. One-way Edges

In this section, we investigate what happens if the hypothesized leak location is not in an edge that directly connects v_i and v_l, but rather in a *one-way* edge between these nodes.

Definition 2. *The edge e_j is* one-way *between v_i and v_l if both of the following hold.*

- *There is a path π from v_i to v_l which passes through e_j ($e_j \in \pi$).*
- *No path from v_i to v_l passes backwards through e_j (no path from v_i to v_l contains e_{-j}).*

Remark 6. *One-way does not mean that the flowing medium can not flow in both directions through an edge. It is a concept related to the pair of nodes v_i and v_l, which says that on the way from v_i to v_l, the pipe e_j is always passed through in the same direction.*

Example 2. *Consider the graph $(V, E) = (\{v_1, v_2, v_3, v_4\}, \{e_1, e_2, e_3, e_4, e_5, e_6\})$, displayed in Figure 2, with in$(1) = v_1$, out$(2) = v_2$, in$(2) = v_1$, out$(2) = v_3$, in$(3) = v_1$, out$(2) = v_4$, in$(4) = v_2$, out$(4) = v_3$, in$(5) = v_2$, out$(5) = v_4$, in$(6) = v_3$, out$(6) = v_4$.*

The set of all paths from v_1 to v_2, is $\Pi(v_1, v_2) = \{(e_1), (e_2, e_{-4}), (e_3, e_{-5}), (e_2, e_6, e_{-5}), (e_3, e_{-6}, e_4)\}$. The edges e_1, e_2, e_3, e_{-4} and e_{-5} are all one-way between v_1 and v_2. However, e_6, which is traversed in opposite directions in the paths (e_2, e_6, e_{-5}) and (e_3, e_{-6}, e_4), is not one-way between v_1 and v_2.

The useful property of a one-way edge e_j between a pair of nodes, v_i and v_l, is that the calculated potential difference $h_i^\Lambda - h_l^\Lambda$, where $\Lambda = (e_j, x_j)$, behaves similarly to the calculated potential difference over e_j, as described in Section IV. We formulate this in our main result, Theorem 1.

Algorithm 2 Bisection method to search a one-way edge for the plausible leak location.

for every pair $v_i, v_l \in V_{\text{sens}}$ **do**
  for every one-way edge e_j between v_i and v_l **do**
    let $\Lambda_0 = (e_j, 0)$ and $\Lambda_1 = (e_j, 1)$.
    calculate $h_i^{\Lambda_0} - h_l^{\Lambda_0}$ and $h_i^{\Lambda_1} - h_l^{\Lambda_1}$
    if $h_i^{\Lambda_0} - h_l^{\Lambda_0} > h_i - h_l$ **then**
      reject H_Λ for all $\Lambda = (e_j, x_j)$
    else if $h_i^{\Lambda_1} - h_l^{\Lambda_1} < h_i - h_l$ **then**
      reject H_Λ for all $\Lambda = (e_j, x_j)$
    else
      use bisection method on e_j to find the one leak location Λ such that $h_i^\Lambda - h_l^\Lambda = h_i - h_l$.
      reject $H_{\tilde{\Lambda}} = (e_j, \tilde{x}_j)$ for all $\tilde{\Lambda} \neq \Lambda$
    end if
  end for
end for

Theorem 1. *Suppose there is a positive leak in the network, $\sum_{v_{\tilde{i}} \in V} d_{\tilde{i}} < 0$. Then for every one-way edge, e_j, between the pair of edges v_i and v_l, it holds that*

$$\frac{d}{dx_j}\left(h_i^\Lambda - h_l^\Lambda\right) > 0,$$

where $\Lambda = (e_j, x_j)$.

Theorem 1 is useful when testing leakage location hypotheses H_Λ because it says there can be at most one plausible leakage location in any one-way edge between a set of potential sensor nodes, and it thus allows us to reject continua of hypotheses at a time. Thus, it is a positive result for both Problem 1 and Problem 2.

If there are potential sensors installed in v_i and v_l and e_j is a one-way edge between v_i and v_l, we proceed as follows to test hypotheses in e_j. We let $\Lambda_0 = (e_j, 0)$ and $\Lambda_1 = (e_j, 1)$.

If either $h_i^{\Lambda_0} - h_l^{\Lambda_0} > h_i - h_l$ or $h_i^{\Lambda_1} - h_l^{\Lambda_1} < h_i - h_l$ we can immediately reject all H_Λ where $\Lambda = (e_j, x_j)$. If instead $h_i^{\Lambda_0} - h_l^{\Lambda_0} < h_i - h_l < h_i^{\Lambda_1} - h_l^{\Lambda_1}$, then, as we vary x_j there will be exactly one $\Lambda = (e_j, x_j)$ for which $h_i^\Lambda - h_l^\Lambda = h_i - h_l$. This position can be found, for example, with the help of an ordinary bisection method [22]. We summarize the steps in Algorithm 2.

Remark 7. *In Algorithm 2, the bisection method will not find the exact leak location for which $h_i^\Lambda - h_l^\Lambda = h_i - h_l$. However, the interval of hypotheses that have not yet been rejected is halved at every iteration, and this fast exponential convergence rate is good enough in practice.*

VI. Conclusion and Future Work

This paper regards leakage localization in potential flow networks, a general model encompassing water distribution networks and electrical networks.

We have considered leakage localization by the scheme described in Algorithm 1. Two central problems are associated with this leak localization approach: Problem 1 and Problem 2. We introduced the concept of *one-way* edges between pairs of potential sensor nodes. One-way edges admit some useful properties in relation to Problem 1 and Problem 2.

In [15], we extend the one-way concept to paths and sub-paths. Furthermore, we are currently working on using the results for one-way edges to guide potential sensor placement. For example, placing potential sensors to maximize the number of one-way edges in the network could be beneficial, as each one-way pipe can have at most one plausible leakage location. We plan to develop algorithms for this task and compare the resulting potential sensor placement to existing works from water potential sensor placement such as [23], [24]. In this endeavor, we will apply our analysis to commonly used benchmark networks, for example, the Modena network [25].

Further, we noticed that e_6 in Example 2 was not one-way between v_1 and v_2. In more extensive networks, often only the pipes close to v_i and v_l are one-way between these vertices. Our results for localizing leakages along one-way edges are only sufficient, i.e., we do not say anything of general sense for edges that are not one-way. Thus, there is plenty of analysis to be made for edges that are not one-way.

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