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On Stability Analysis of Predictive Flocking Using N-Paths

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Abstract— **Most publications in the field of model predictive flocking (MPF) present frameworks without providing stability analyses for the proposed schemes. In those providing stability results, one specific line of reasoning is based on the geometric properties of the optimal state sequences, so-called N-paths. This method is used in several publications to show stability for centralized and distributed MPF schemes. In this paper, we critically discuss this line of reasoning and point out several errors in the N-path-based analysis. As incorrect statements in assumptions and lemmas cause the lines of reasoning to break, this raises the question of whether the N-path-based line of reasoning is suitable for the MPF stability analysis in general.**

I. INTRODUCTION

Flocking is a multi-agent system (MAS) control technique where the control laws are designed based on the desired swarm behavior instead of the objectives for individual agents. This is achieved by formulating the rules for the agents' behavior inspired by biological swarms [1]. One direction of flocking research focuses on implementing flocking control in a model predictive control (MPC) framework. In MPC frameworks, the agents' decisions are calculated by solving optimization problems based on predictions of the agents' future states. In general, this approach results in an increased control performance compared to non-predictive flocking algorithms. Additionally, predictive frameworks can handle constraints on states and inputs explicitly.

The first model predictive flocking (MPF) algorithms were proposed by Zhan and Li in 2011 [2], [3]. In [2], a scheme for centralized MPF (CMPF) is presented, in which one large optimization problem is solved for the whole MAS. In [3], the authors propose a distributed MPF (DMPF) approach where each agent solves a local optimization problem based on the information of neighboring agents only. Since then, several MPF algorithms including input and state constraints, obstacle avoidance, and reference tracking have been proposed (see [4]–[6]). DMPF has also successfully been applied in outdoor drone experiments [7].

All the aforementioned publications demonstrate the performance of their schemes in simulation, showing superior performance compared to non-predictive flocking algorithms, as, for example, the one proposed in [1]. However, only a minority of the papers published in the field of MPF provide stability analyses. One reason for this is that the analysis of MPF algorithms, especially in the case of DMPF, is a challenging task due to the nonlinear nature of the flocking constraints and setpoints that are not known a priori [8], [9].

The first stability analysis for MPF was presented in [10]. In this paper, the authors present a CMPF and a sequential DMPF scheme for single-integrator agents. The line of reasoning in the stability analysis is based on the geometric properties of the optimal state sequence. These sequences are also referred to as N-paths and were introduced in [11] to show stability for a distributed MPC consensus problem with single-integrator agents. Using the properties of N-paths, in [10], the authors prove asymptotic stability of their CMPF scheme without requiring stabilizing terminal ingredients. In [4] and [12], the analysis based on N-paths is extended to DMPF with parallel optimization and double-integrator agent dynamics. There also exist different approaches for showing the stability of MPF [13], [14]. However, these approaches are not discussed in this paper.

At first glance, the N-path-based line of stability analysis for MPF appears elegant since, the way it is presented in [4], [10], and [12], it does not require the design of stabilizing terminal ingredients. Especially for nonlinear and distributed MPC, the design of these terminal ingredients can be challenging. Hence, the discussed approaches would provide a simple solution to this in general complex analysis problem [8], [15]. However, on closer scrutiny of the N-path-based MPF proofs, several errors in the reasoning become apparent, rendering the analysis results incorrect. As the papers using this N-path-based line of reasoning have been cited several hundred times, it is important to point out these errors in order to make the scientific community aware of the challenges and pitfalls in the stability analysis of MPF using N-paths.

In this paper, we point out and critically discuss errors in the stability analyses of MPF schemes using N-paths in [4], [10], and [12]. Mainly, the analysis results in the aforementioned publications are based on one particular assumption that can be shown to contain an incorrect statement. Moreover, the reasoning in the stability analysis requires information about the final setpoint. However, in the distributed schemes in [4] and [12], agents are not aware of the exact setpoint but obtain a local approximation. One can show that this approximation breaks the line of reasoning in the N-path-based analysis. Furthermore, in [4] and [12], the application to double-integrator agents requires an extension of the N-path properties stated in [11]. However, one can show that the proposed extension is incorrect. Throughout this paper, we provide several numerical examples and simulation results to illustrate our findings. The code for reproducing the simulation results is available in [16].

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The remainder of this paper is organized as follows: Section II provides the prerequisites for this paper. In Section III, the general setup for CMPF and DMPF as well as the line of reasoning for the stability analysis in the discussed papers are introduced. Section IV presents the critical discussion of the lines of reasoning for CMPF and DMPF analyses. Section V concludes this paper.

A. Notation

The symbols 0 and I denote zero and identity matrices with compatible dimensions, if not specified explicitly. Zero and one vectors of dimension $n \times 1$ are denoted by 0_n and 1_n, respectively. The prediction of variable x at time $k+i$ based on information at time k is denoted as $x(k+i|k)$. Additionally, the abbreviated notations x_{k+i} and $x_{k+i|k}$ are used. Furthermore, define $(x_1, x_2) := [x_1^\top x_2^\top]^\top$.

II. Prerequisites

In this section, the prerequisites for MPF and its stability analysis are presented. This includes the agent dynamics, graph theory, and a summary of the properties of N-paths.

A. Dynamics and State Prediction

Consider a group of N discrete-time single-integrator (si) or double-integrator (di) agents with sampling time τ. The dynamics of agent i are given by

$$q^{\mathrm{si}}_{i,k+1} = q^{\mathrm{si}}_{i,k} + \tau p^{\mathrm{si}}_{i,k},\ i = 1 \dots N, \tag{1}$$

for single-integrator agents and

$$\begin{aligned} q^{\mathrm{di}}_{i,k+1} &= q^{\mathrm{di}}_{i,k} + \tau p^{\mathrm{di}}_{i,k}, \\ p^{\mathrm{di}}_{i,k+1} &= p^{\mathrm{di}}_{i,k} + \tau u^{\mathrm{di}}_{i,k},\ i = 1 \dots N, \end{aligned} \tag{2}$$

for double-integrator agents. Here, $q_{i,k}$, $p_{i,k}$, $u_{i,k} \in \mathbb{R}^m$ denote the position, velocity, and input at time k, respectively. The corresponding state vectors are denoted by $x^{\mathrm{si}}_i = q^{\mathrm{si}}_i$ and $x^{\mathrm{di}}_i = (q^{\mathrm{di}}_i, p^{\mathrm{di}}_i)$. The superscripts si and di are dropped if the type of agent is clear from context or not important. The dynamics in (1) and (2) can also be written in state-space form as

$$x_{i,k+1} = A x_{i,k} + B u_{i,k}, \tag{3}$$

with

$$A^{\mathrm{si}} = I_m,\ B^{\mathrm{si}} = \tau I_m,\ A^{\mathrm{di}} = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \otimes I_m,\ B^{\mathrm{di}} = \begin{bmatrix} 0 \\ \tau \end{bmatrix} \otimes I_m.$$

Let H_p be the prediction horizon. Given $x_{i,k}$, the future H_p states of agent i can be computed according to

$$x_{i,k+\ell} = A^\ell x_{i,k} + \sum_{\lambda=0}^{\ell-1} B A^{\ell-\lambda-1} u_{i,k+\lambda},\ \ell = 1 \dots H_p. \tag{4}$$

Let capital letters denote signal values over the prediction horizon stacked into a column vector. For example, $X_{i,k+1} = (x_{i,k+1}, \dots, x_{i,k+H_p})$ and $U_{i,k} = (u_{i,k}, \dots, u_{i,k+H_p-1})$ denote the state and input sequences over the prediction horizon, respectively. Symbols of signals without an agent-specific index denote the corresponding signals of all agents stacked together. For example, $x_k = (x_{1,k}, \dots, x_{N,k})$ and $U_k = (U_{1,k}, \dots, U_{N,k})$ denote the stacked states and predicted inputs of all agents, respectively. The displacement vector between agents i and j is defined as $q_{ij} = q_j - q_i$.

B. Graph Theory

Agent interactions are modeled via a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with the elements of the vertex set $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$ representing the agents and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ representing the communication topology. A graph is said to be undirected if $(i,j) \in \mathcal{E} \Leftrightarrow (j,i) \in \mathcal{E}$. Furthermore, an undirected graph is said to be connected if there exists a path between every pair of vertices. Given the communication range $r_c > 0$ and the position vector q_k, the proximity graph $\mathcal{G}(q_k) = (\mathcal{V}, \mathcal{E}(q_k))$ is defined with $\mathcal{E}(q_k) = \{(i,j) \in \mathcal{V} \times \mathcal{V} : \|q_{ij,k}\| < r_c, j \neq i\}$. $\mathcal{G}(q_k)$ is said to be connected across the interval $[k, k+\kappa]$, $\kappa \geq 0$, if the collection of graphs $\{\mathcal{G}(q_k), \dots \mathcal{G}(q_{k+\kappa})\}$ is connected. The set of spatial neighbors of an agent i is defined as $\mathcal{N}_i = \left\{ j \in \mathcal{V} : \|q_{ij}\| < r_c, j \neq i \right\}$ with $\nu_i = |\mathcal{N}_i|$ denoting its cardinality.

C. N-Paths

The concept of N-paths was introduced in [11] to show stability for a distributed MPC consensus problem with single-integrator agents. In the following, the main definitions and theorems regarding N-paths are presented.

An N-path $T_A = \{A_1, A_2, \dots, A_N\} \in \mathbb{R}^n$ is an ordered sequence of N points. Given two points $A_i, A_j \in T_A$, let $\overline{A_i A_j}$ be the segment joining them. Furthermore, let $|\overline{A_i A_j}|$ denote the length of that segment. An N-path T_A is said to be non-increasing with respect to a point $O \in \mathbb{R}^n$ if $|\overline{A_{j+1} O}| \leq |\overline{A_j O}|$, $\forall j \in \{1, \dots, N-1\}$ [11, Definition 7]. Furthermore, an N-path is said to be pointing towards $O \in \mathbb{R}^n$ if it is non-increasing with respect to O and $A_j \in \overline{A_1 O}$, $\forall j \in \{1, \dots, N\}$ [11, Definition 8]. With these definitions, the following theorem can be stated:

Theorem 1: [11, Theorem 5] Let $T_A = \{A_1, \dots, A_N\} \in \mathbb{R}^n$ be an N-path. Given $O \in \mathbb{R}^n$, there always exists an N-path $T_B = \{B_1, \dots, B_N\} \in \mathbb{R}^n$ with $B_1 = A_1$, pointing towards O, and satisfying the following inequalities:

$$|\overline{B_j O}| \leq |\overline{A_j O}|,\ j = 1, \dots, N, \tag{5}$$

$$|\overline{B_j B_{j+1}}| \leq |\overline{A_j A_{j+1}}|,\ j = 1, \dots, (N-1). \tag{6}$$

The proof is provided in [11].

In order to apply the N-path-based analysis to double-integrator agents, [4] proposes an extension of Theorem 1.

Lemma 1: [4, Lemma 1] Let $T_A = \{A_1, \dots, A_N\} \in \mathbb{R}^n$ be an N-path. Given $O \in \mathbb{R}^n$, there always exists an N-path $T_B = \{B_1, \dots, B_N\} \in \mathbb{R}^n$ with $B_1 = A_1$, pointing towards O, and satisfying the following inequalities:

$$|\overline{B_j O}| \leq |\overline{A_j O}|,\ j = 1, \dots, N, \tag{7}$$

$$|\overline{B_j B_{j+1}}| \leq |\overline{A_j A_{j+1}}|,\ j = 1, \dots, (N-1), \tag{8}$$

$$|\overline{B_{j+1} B_{j+2}} - \overline{B_j B_{j+1}}| \leq |\overline{A_{j+1} A_{j+2}} - \overline{A_j A_{j+1}}|,\quad j = 1, \dots, (N-2), \tag{9}$$

with

$$\overline{B_{N-1} B_N} = \overline{B_{N-2} B_{N-1}}. \tag{10}$$

Note that Lemma 1 only differs from Theorem 1 by the additional constraints (9) and (10). This extension is essential for the analysis in [4] and [12] and is discussed in detail in Section IV-C.

III. Predictive Flocking

In this section, CMPF and DMPF are presented in a general notation. Furthermore, the general line of reasoning used for the stability analysis in all of the discussed approaches is introduced.

A. Centralized Predictive Flocking

In CMPF, an optimization problem is solved based on the global information of all agents. Given the desired inter-agent distance $d > 0$, the objective of the swarm to form a so-called α-lattice, defined by $\|q_{ij}\| = d,\ \forall (i,j) \in \mathcal{E}(q)$. In the discussed papers, the normed deviation from the desired configuration is denoted as

$$\begin{aligned}\|g(q_k)\|^n &= \sum_{(i,j)\in\mathcal{E}(q_k)} |\|q_{ij,k}\| - d|^n \\ &= \sum_{(i,j)\in\mathcal{E}(q_k)} |\|q_{ij,k}\| - d|^n \left\| \frac{q_{ij,k}}{\|q_{ij,k}\|} \right\|^n \\ &= \sum_{(i,j)\in\mathcal{E}(q_k)} \left\| q_{ij,k} - d\frac{q_{ij,k}}{\|q_{ij,k}\|} \right\|^n. \end{aligned} \tag{11}$$

The CMPF problem with the cost function J can then be stated as follows.

Problem 1: Centralized MPF

$$\min_{U_k} J(x_k, U_k) = \min_{U_k} \sum_{\ell=0}^{H_p-1} (\lambda \|u_{k+\ell}\|^2 + \|g(q_{k+\ell+1})\|^2)$$
$$\text{s.t.} \quad (4) \text{ for } i = 1 \dots N.$$

B. Distributed Predictive Flocking

In DMPF, each agent i solves an optimization problem based on local information. Let $j_{j_{i,1}}, \dots, j_{j_{i,v_i}}$ denote the elements of the neighbor set of agent i. The neighborhood state vector is then defined as $\bar{x}_{i,k} = (x_{j_{i,1}}, \dots, x_{j_{i,v_i}})$. Neighborhood position and velocity vectors $\bar{q}_{i,k}$ and $\bar{p}_{i,k}$ are defined analogously. The state vector for the i^{th}subsystem consisting of agent i and its neighbors is defined as $x_k^i = (x_{i,k}, \bar{x}_{i,k})$, with position and velocity vectors q_k^i and p_k^i defined analogously. Using these definitions, the deviations from the desired distances in (11) can also be stated subsystem-wise as

$$\|g_i(q_k^i)\|^n = \|g_i(q_{i,k}, \bar{q}_{i,k})\|^n = \sum_{j\in\mathcal{N}_i} \left\| q_{ij,k} - d\frac{q_{ij,k}}{\|q_{ij,k}\|} \right\|^n. \tag{12}$$

The DMPF problem with the local cost function J_i can then be stated as follows.

Problem 2: Distributed MPF

$$\begin{aligned}\min_{U_{i,k}} J_i(x_{i,k}, \bar{X}_{i,k}, U_{i,k}) = \min_{U_{i,k}} \sum_{\ell=0}^{H_p-1} &(\lambda \|u_{i,k+\ell}\|^2 \\ &+ \|g_i(q_{i,k+\ell+1}, \bar{q}_{i,k+\ell+1})\|^2) \end{aligned} \tag{13}$$
$$\text{s.t.} \quad (4).$$

When agents are solving their optimization problems in parallel, at time k, they are not aware of their neighbors' decisions at time k. Thus, each agent has to estimate its neighbors' decisions in order to evaluate its own cost function. In [4] and [12], the frameworks using parallel optimization, each agent i assumes that neighboring agents are moving with a constant velocity for the duration of the prediction horizon, i.e. $u_{j,k+\ell} = 0$ for $j \in \mathcal{N}_i$ and $\ell = 1 \dots H_p$. Assuming that agent i has access to $\bar{x}_{i,k}$, the estimated states can be calculated as

$$\hat{x}_{j,k+\ell|k} = A^\ell x_{j,k},\ \ \forall j \in \mathcal{N}_i,\ \ \ell = 1 \dots H_p. \tag{14}$$

The collective estimated states for the neighbors of agent i are then denoted as $\hat{\bar{x}}_{i,k+1|k}$. Using these approximated neighbor states, the cost function in Problem 2 is written as

$$\begin{aligned}J_i(x_{i,k}, \hat{\bar{X}}_{i,k}, U_{i,k}) = \sum_{\ell=0}^{H_p-1} &(\lambda \|u_{i,k+\ell}\|^2 \\ &+ \|g_i(q_{i,k+\ell+1}, \hat{\bar{q}}_{i,k+\ell+1|k})\|^2). \end{aligned}$$

C. Stability Analysis

In all of the MPF publications using the N-path line of reasoning, the following Assumptions 1 and 2 are used in the stability analysis.

Assumption 1: There exists a $\kappa \geq 0$ such that the MAS is connected across the interval $[k, k+\kappa]$ for all $k > 0$.

Assumption 2: Given an initial position state $q_0 \in \mathbb{R}^{Nm}$, there exists a nearest desired α-lattice state $q_\alpha^* \in \mathbb{R}^{Nm}$. Furthermore, it then holds $\forall q', q'' \in \mathbb{R}^{Nm}$ that

$$\|q' - q_\alpha^*\| \leq \|q'' - q_\alpha^*\| \Rightarrow \|g(q')\| \leq \|g(q'')\|. \tag{15}$$

The nearest desired α-lattice in Assumption 2 can be computed as

$$\begin{aligned} q_\alpha^* = \arg\min_q &\|q - q_0\| \\ \text{s.t. } &\|g(q)\| = 0. \end{aligned} \tag{16}$$

Based on these assumptions, the authors of [4], [10], and [12] then state theorems about the proposed schemes converging towards the closest α-lattice. We refer the reader to the cited papers for the exact statements of the stability theorems.

One of the key arguments in the lines of reasoning in [4], [10], and [12] is that the solutions of Problems 1 and 2 are necessarily pointing towards the desired α-lattice q_α^*. In the distributed schemes in [4] and [12], q_α^* is replaced with q_α^{i*} where q_α^{i*} is the closest desired α-lattice for the i$^{\text{th}}$ subsystem. In [4] and [12], q_α^{i*} is obtained by extracting the required states from q_α^*.

For CMPF, given any sequence of positions $T_{Q_k} = \{q_{k+1} \dots, q_{k+H_p}\}$ and the nearest desired α-lattice state q_α^*, by Theorem 1 (si agents), or Lemma 1 (di agents), with $O = q_\alpha^*$, the authors conclude that there always exists a sequence $T_{Q_k}^*$ pointing towards q_α^*. From the properties of N-paths, it then follows that $\|q_{k+\ell}^* - q_\alpha^*\| \leq \|q_{k+\ell} - q_\alpha^*\|$ for $\ell = 1 \dots H_p$, and consequently, with (15) in Assumption 2,

$$\|g(q_{k+\ell}^*)\| \leq \|g(q_{k+\ell})\|, \quad \ell = 1 \dots H_p. \tag{17}$$

Rewriting the dynamics in (1) and (2), the inputs at time k can be expressed as

$$u_k = (q_{k+1} - q_k)/\tau \tag{18}$$

for si-agents and

$$u_k = \big((q_{k+2} - q_{k+1}) - (q_{k+1} - q_k)\big)/\tau^2 \tag{19}$$

for di-agents. By Theorem 1 (si agents), or Lemma 1 (di agents), it follows that

$$\|u^*_{k+\ell}\| \leq \|u_{k+\ell}\|, \quad \ell = 0 \dots H_p - 1. \tag{20}$$

Consequently, $T^*_{Q_k}$, the path pointing towards q^*_α, results in a lower cost for Problem 1 than T_{Q_k}, i.e.

$$J(x_k, U^*_k) \leq J(x_k, U_k). \tag{21}$$

Based on this result, the authors then conclude that the state sequence resulting from the optimal solution of Problem 1 is necessarily pointing towards q^*_α. Moreover, since the optimal position sequence is pointing towards q^*_α, it follows that

$$\|g(q^*_{k+\ell+1})\| \leq \|g(q^*_{k+\ell})\|, \, \ell = 0 \dots H_p. \tag{22}$$

In the DMPF schemes, the same line of reasoning is used to argue that the optimal state sequence is pointing towards q^{i*}_α. The local counterparts of (17)-(22) are given by

$$\|g_i(q^{i*}_{k+\ell})\| \leq \|g_i(q^i_{k+\ell})\|, \quad \ell = 1 \dots H_p, \tag{23}$$

$$u_{i,k} = (q_{i,k+1} - q_{i,k})/\tau, \tag{24}$$

$$u_{i,k} = \big((q_{i,k+2} - q_{i,k+1}) - (q_{i,k+1} - q_{i,k})\big)/\tau^2, \tag{25}$$

$$\|u^*_{i,k+\ell}\| \leq \|u_{i,k+\ell}\|, \quad \ell = 0 \dots H_p - 1, \tag{26}$$

and consequently

$$J_i(x_{i,k}, U^*_{i,k}) \leq J_i(x_{i,k}, U_{i,k}), \tag{27}$$

$$\|g_i(q^{i*}_{k+\ell+1})\| \leq \|g_i(q^{i*}_{k+\ell})\|, \, \ell = 0 \dots H_p. \tag{28}$$

Based on these results, the authors then show asymptotic stability by proving that the global cost function, or in the case of DMPF, the local cost functions, are Lyapunov functions.

IV. Critical Discussion

In this section, the main errors in the line of reasoning of the N-path-based MPF analysis approaches are pointed out and discussed critically. For all of the discussed errors, the nature of the errors is presented first, followed by counterexamples. Finally, the consequences for the MPF analysis are discussed.

A. Assumption 2

One of the key ingredients in the CMPF and DMPF analyses is Assumption 2, as this assumption provides a relation between the distance from the desired α-lattice q^*_α and the corresponding flocking error. However, in the way this assumption is stated, it contains an incorrect statement. More precisely, condition (15) does not hold for all $q', q'' \in \mathbb{R}^{Nm}$, as stated in Assumption 2, but only for those configurations for which q^*_α is the nearest α-lattice. In the following, this is demonstrated by a simple counterexample.

1) Counterexample: Consider an α-lattice configuration $q^*_\alpha \in \mathbb{R}^{Nm}$ with $\|g(q^*_\alpha)\| = 0$. Since an α-lattice is defined based on the relative distances between agents, it follows from (11) that

$$\|g(q^*_\alpha + 1_{Nm})\| = 0$$

also holds. Next, define $\Delta = (1, 0_{Nm-1})$. By selecting

$$q' = q^*_\alpha + \varepsilon\Delta,$$
$$q'' = q^*_\alpha + \varepsilon 1_{Nm}, \;\; \varepsilon > 0,$$

one obtains

$$\|q' - q^*_\alpha\| < \|q'' - q^*_\alpha\|,$$

but

$$\|g(q')\| > \|g(q'')\| = 0.$$

This contradicts the statement of (15) in Assumption 2 and demonstrates that condition (15) does not hold for all $q', q'' \in \mathbb{R}^{Nm}$.

2) Implications for CMPF: With statement (15) in Assumption 2 not holding, one can no longer argue that the optimal solution of Problem 1 is pointing towards q^*_α since $\|q^*_{k+\ell} - q^*_\alpha\| \leq \|q_{k+\ell} - q^*_\alpha\|$ no longer implies $\|g(q^*_{k+\ell})\| \leq \|g(q_{k+\ell})\|$. Consequently, (22) also no longer holds. As a result, the optimal solution is not necessarily pointing towards q^*_α. However, as this is required in the stability analysis, the analysis no longer holds. A simulation scenario demonstrating that the optimal states for the CMPF algorithm in [10] are not pointing towards q^*_α but are in fact curved is provided in Appendix A.

Even though (15) is incorrect in its stated form, it can be enforced by constraining the admissible states of Problem 1 to those q with nearest α-lattice q^*_α. The set of these states, denoted by $\mathcal{Q}(q^*_\alpha)$, can be described by

$$\mathcal{Q}(q^*_\alpha) := \left\{ q \in \mathbb{R}^{Nm} : \begin{array}{l} \arg\min\limits_{\tilde{q}} \|\tilde{q} - q\| \\ \text{s.t. } \|g(\tilde{q})\| = 0 \end{array} = q^*_\alpha \right\}, \tag{29}$$

with q^*_α as the solution of (16). To fix Assumption 2, the constraint $q_{k+\ell} \in \mathcal{Q}(q^*_\alpha)$, $\ell = 1 \dots H_p$, then has to be added to Problem 1. However, even for small MAS moving only in one dimension, the resulting constraint is very restrictive. Consider the simple MAS with $m = 1$, $N = 2$, and $d = 2$. With $q_0 = (-2, 2)$, by (16), the closest desired α-lattice is $q^*_\alpha = (-1, 1)$. On can show that the set of admissible q is given by

$$\mathcal{Q}(q^*_\alpha) = \{q \in \mathbb{R}^2 : q = q_0 + \varepsilon(q_0 - q^*_\alpha), \varepsilon > -2\}.$$

By requiring the same ε for each agent, the MAS loses one degree of freedom. Due to the involved nonconvex constrained optimization problem, it is questionable whether $\mathcal{Q}(q^*_\alpha)$ can be predetermined analytically for large MASs with $m > 1$ and how much this constraint will reduce the performance.

3) Implications for DMPF: For the same reason as in CMPF, with (15) not holding, one can no longer argue that the optimal solution to Problem 2 is pointing towards q_α^{i*}. However, contrary to the centralized approach, the statement of Assumption 2 cannot be corrected. In CMPF, (15) can be enforced by constraining the admissible global configurations q_k to $\mathcal{Q}(q_\alpha^*)$ in (29). In DMPF on the other hand, individual agents are not aware of the global state q_k. As the satisfaction of this constraint requires a coordinated (centralized) decision of the MAS, the additional constraint cannot be enforced in distributed schemes. Consequently, with (15) not holding, the optimal state sequence is not necessarily pointing towards q_α^{i*}, breaking the line of reasoning in the stability analysis.

B. Distributed Computation of the Desired Lattice

As presented in Section III-C, in the distributed schemes in [4] and [12], the authors claim that the optimal solution is pointing towards q_α^{i*}, where q_α^{i*} is the closest desired α-lattice for the i$^{\text{th}}$ subsystem. However, due to each agent only having access to the information of their spatial neighbors, this is not true in general, as demonstrated with the following counterexample.

1) Counterexample: Consider an MAS with $N=3$, $m=1$, $d=1$, $r=1.2$, and initial state $q_0=(-0.22,0,1.03)$. Note that agents 1 and 3 are not considered neighbors in this scenario ($\|q_{13}\|>r$). According to (16), the nearest desired α-lattice is

$$q_\alpha^* = (-0.73, 0.27, 1.27).$$

Extracting the desired configurations for the subsystems, we have

$$q_\alpha^{1*} = \begin{bmatrix} -0.73 \\ 0.27 \end{bmatrix}, \quad q_\alpha^{2*} = \begin{bmatrix} 0.27 \\ -0.73 \\ 1.27 \end{bmatrix}, \quad q_\alpha^{3*} = \begin{bmatrix} 1.27 \\ 0.27 \end{bmatrix}.$$

Note that the computation of q_α^{1*}, q_α^{2*}, and q_α^{3*} requires the information of the global configuration q as they are extracted from q_α^*. However, when using (16) to compute $\hat{q}_\alpha^{1*}$ and $\hat{q}_\alpha^{3*}$ based only on the information available to each agent, the resulting desired states are

$$\hat{q}_\alpha^{1*} = \begin{bmatrix} -0.61 \\ 0.39 \end{bmatrix} \neq q_\alpha^{1*}, \quad \hat{q}_\alpha^{2*} = q_\alpha^{2*}, \quad \hat{q}_\alpha^{3*} = \begin{bmatrix} 1.015 \\ 0.015 \end{bmatrix} \neq q_\alpha^{3*}.$$

In this example, only agent 2 is connected to all other agents and therefore has access to the global configuration q. Hence, it follows that $\hat{q}_\alpha^{2*} = q_\alpha^{2*}$. Since each agent i solves its optimization problem with locally available information, the optimal solution will point towards $\hat{q}_\alpha^{i*}$, not q_α^{i*}. In general, for graphs that are not complete, $\hat{q}_\alpha^{i*}$ and q_α^{i*} will not coincide.

2) Implications for DMPF: With the optimal solution not necessarily pointing towards q_α^{i*}, (28) and (27) no longer hold. As these equations are required for the DMPF stability analysis in [4] and [12], the mismatch of $\hat{q}_\alpha^{i*}$ and q_α^{i*} breaks the line of reasoning in the DMPF stability proofs.

One possible solution for this problem would be to include the distributed computation of (16) in the algorithms in [4] and [12]. However, due to the nonlinear, coupled equality constraints in (16), the distributed solution of this problem is not straightforward [17].

C. Extension of N-path Theorem

One of the key ingredients in the analysis for double-integrator agents in [4] and [12] is the extension of Theorem 1. The proposed extension and its proof were proposed in [4] in the form of Lemma 1. In the proof, the authors claim that one can always find an N-path satisfying (7)-(9). This is, however, not true, as demonstrated by the following counterexample.

1) Counterexample: Consider the one-dimensional N-path $T_A=\{2,0.5,-1\}$ with $O=0$. Next, T_B is constructed according to Lemma 1. The sequences T_A and T_B are depicted in Fig. 1. By definition, it follows that

$$B_1 = A_1 = 2.$$

From (7) and (8), the only possible choice for B_2 is

$$B_2 = A_2 = 0.5.$$

Since T_B has to point towards $O=0$, one must select $B_3 \in [0,0.5]$. This, however, results in

$$|\overline{B_2B_3} - \overline{B_1B_2}| = |(B_3-0.5)-(-1.5)| \geq 1,$$

and

$$|\overline{B_2B_3} - \overline{B_1B_2}| > |\overline{A_2A_3} - \overline{A_1A_2}| = |(0.5-2)-(-1-0.5)| = 0.$$

This contradicts (9) and therefore disproves Lemma 1. Code for generating counterexamples with longer paths and higher dimensions is provided in [16].

In the proof of Lemma 1 provided in [4], the authors prove the additional property (9) by adding additional points to the N-path T_A, effectively making it an N'-path with $N' \geq N$. However, this clearly contradicts the statement of Lemma 1, which explicitly requires T_A and T_B to be paths of length N. For the exact formulation of the proof, we refer the reader to [4].

2) Implications for DMPF: For single-integrator agents, (6) in Theorem 1 is used to show that (20) and (26) follow from (18) and (24). Lemma 1 fulfills the same purpose for di agents. More precisely, (9) in Lemma 1 is used to show that (20) and (26) follow from (19) and (25). With Lemma 1 not holding, (20) and (26), and consequently (21) and (27), do not hold. As a result, one can no longer argue that the optimal solution is necessarily pointing towards q_α^* or q_α^{i*} since the N-path is not the minimum-cost sequence and hence not the optimal solution. However, as discussed in Section III-C, this is required in the stability analysis.

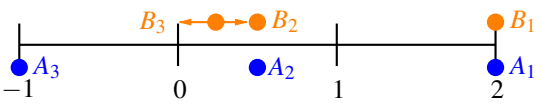

Fig. 1: Counterexample for Lemma 1 in [4].

V. Conclusions and Future Work

In this paper, we discussed the stability analyses of MPF based on N-paths, as presented in [4], [10], and [12]. Despite all of the investigated schemes displaying good performance in simulation, a thorough investigation of the stated assumptions, theorems, and lines of reasoning reveals that the proposed stability results are incorrect in their stated forms.

While the incorrect statement in the essential Assumption 2 can be corrected for the single-integrator CMPF analysis, the required modification constrains the set of admissible states. Especially for larger MAS moving in two- or three-dimensional spaces, these constraints are very restrictive, putting a question mark over the practical relevance of the proposed scheme. For the DMPF schemes, the incorrect statement in Assumption 2 cannot be corrected, breaking the N-path-based lines of reasoning.

Additionally, the DMPF analysis requires the solutions of the optimization problems to point towards the desired global configuration. This is, however, not the case in the DMPF problems, as agents are only aware of the states of their spatial neighbors.

Furthermore, the application of the N-path MPF analysis to double-integrator agents requires an extension of the N-path properties stated in [11]. However, as the extension used in [4] and [12] is incorrect, the N-path line of reasoning cannot be applied to double-integrator agents.

In conclusion, it is highly questionable whether the framework of N-paths is suited to analyze the stability of MPF schemes, as it is only applicable to centralized schemes for agents with single-integrator dynamics. Even for those limited use cases, restrictive constraints on the admissible states are required. Therefore, our future work will focus on formulating a stability analysis for MPF based on carefully designed terminal ingredients, as proposed in [8], [15].

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Appendix

The code for reproducing the following simulation example is provided in [16]. The simulation results were generated using the open-source Matlab simulation library WiMAS, presented in [18].

A. Simulation Example CMPF

Consider a MAS governed by (1) with $N = 7$, $m = 2$, $\tau = 0.2$, $r = 8.4$, $d = 7$, $H_p = 5$, and $\lambda = 0.1$, using the CMPF algorithm in [10]. Agents are initialized close to an α-lattice configuration. The agents' trajectories are depicted in Fig. 2 with markers for the initial positions. The desired α-lattice is mapped to the origin. The resulting trajectories are curved and not pointing towards the origin (which in this case corresponds to the desired α-lattice). This confirms the results of the discussion in Section IV-A.

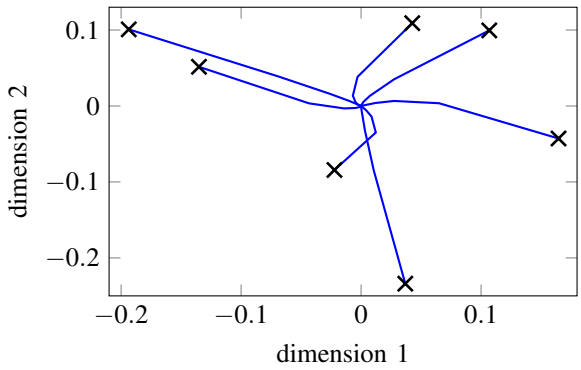

Fig. 2: CMPF trajectories with markers for the initial positions and the desired α-lattice mapped to the origin.