2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

ISS of rapidly time-varying systems via a novel presentation and delay-free transformation

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***Abstract*— We treat input-to-state stability (ISS) of linear continuous-time systems with multiple time-scales. These systems contain rapidly-varying, piecewise continuous and almost periodic coefficients with small parameters (time-scales). Our method relies on a novel delay-free system transformation in conjunction with a new system presentation, where the rapidly-varying coefficients are scalars that have zero average. We employ time-varying Lyapunov functions for ISS analysis. The analysis yields LMI conditions for ISS, leading to explicit bounds on the small parameters, decay rate and ISS gains. The novel system presentation plays a crucial role in the ISS analysis by allowing for essentially less conservative upper bounds on terms containing the small parameters. The derived LMIs are accompanied by suitable feasibility guarantees. Numerical examples demonstrate the efficacy of the proposed approach in comparison to existing methods.**

***Index Terms*— stability, averaging, time-varying systems, ISS, Lyapunov methods.**

I. INTRODUCTION

Systems with almost periodic signals and/or excitations are central to physics and engineering. Applications of such systems include vibrational control [5], power systems [18] and time-delay systems [21] (see also references therein). Such systems often include componenets evolving over multiple time-scales (see e.g. [9] for applications to systems biology). Hence, it is not surprising that perturbation theory has played an essential part in the analysis of systems with rapidly time-varying coefficients and led to important results [2], [11], [12], [20], [16]. However, most of the existing results are qualitative in nature.

The method of averaging is an important perturbation-based technique for the study of stability of systems with oscillatory control inputs [3], [13], [15]. The fundamental idea behind asymptotic averaging is that stability of the first-order averaged system guarantees stability of the original rapidly-varying system for small enough values of the time-scale parameter (see e.g. [17, Chapter 8]). However, it is often the case that asymptotic averaging provides only an existence result, without an efficient and explicit bound on the small parameter for which the stability of the original system is preserved. For singularly perturbed systems, such bounds were derived in, e.g., [12] and [6] via a direct Lyapunov approach.

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Supported by Israel Science Foundation (grant no. 673/19) and by Chana and Heinrich Manderman Chair at Tel Aviv University.

Recently, the first efficient quantitative methods for stability by averaging were suggested. A constructive time-delay approach to periodic averaging of a system with a single fast time-scale was suggested in [8]. The approch relies on backward integration of the system, which yields a neutral-type system presentation where the delay magnitude is equal to to the time-scale parameter. The stability and ISS of the delayed system were shown to guarantee the stability and ISS of the original system. Stability of the delayed system was analyzed via a direct Lyapunov method, leading to LMI conditions which yield an efficient upper bound on the small parameter which preserves the stability of the original system. This method is also well suited for averaging of systems with time-varying delays, where the delay magnitude is of equal order to the time-scale parameter. These results were extended to L_2-gain analysis for periodic averaging and to stochastic systems in [22]. Second, [10] presented a complementary method for averaging-based ISS and stability in the presence of constant delays and multiple time-scales. Differently from [8], this method employs a non-delayed system transformation, leading to a new system whose ISS guarantees the ISS of the original system. ISS analysis of the transformed system was performed via a direct Lyapunov method leading to simple LMIs which provide quantitative estimates on the small parameters, internal decay rate and ISS gains. The approach in [10] was further extended to rapidly time-varying systems with constant delay, where the novel transformation decoupled the effects of the delay and time-scale parameter on stability, thereby leading to stability results for non-small delay (relative to the small parameter). However, in most of the numerical examples the results of [10] were more conservative than the results via the time-delay approach [8], which motivated the present work.

In this paper we study ISS of rapidly time-varying systems with multiple time-scales. We employ a novel presentation of the system, in conjunction with a delay-free tranformation suggested in [10]. The new presentation relies on two key ingredients: first, inpired by a similar presentation for systems with distributed delays and variable kernels [19], we present the rapidly-varying system matrices as linear combinations of constant matrices with rapidly-varying *scalar* coefficients. Second, we force the latter coefficients to have zero averages. We then employ the transformation from [10], thereby obtaining a transformed system whose ISS guarantees the ISS of the original system. ISS of the transformed system is studied by employing time-varying Lyapunov functions and tight bounds on the scalar time-varying coefficients (which are less conservative than the bounds on the time-varying matrices in [10], which were obtained using Jensen's inequalities [7]). The resulting LMIs are backed by theoretical feasibility guarantees. Extensive numerical examples show that, compared to the existing results, our approach essentially improves the small parameter bounds for which ISS of the original system is preserved.

The article is organized as follows. Section 2 presents ISS results for rapidly time-varying systems. Numerical examples are given in Section 3. Conclusions are drawn in Section 4.

Notations: Throughout the paper $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices with the induced matrix norm $\|\cdot\|$. We also denote $\mathbb{Z}_+=\{0,1,2,\dots\}$ and $\mathbb{R}_{\geq 0}=[0,\infty)$. The superscript $\top$ denotes matrix transposition, and the notation $P>0$, for $P\in\mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix are denoted by $*$. For $0<P\in\mathbb{R}^{n\times n}$ and $x\in\mathbb{R}^n$, we write $|x|_P^2=x^\top Px$. $\otimes$ denotes the Kronecker product. The standard lexicographic order on $\mathbb{R}^n$ is denoted by $\leq_{\text{lex}}$.

II. ISS-LIKE ESTIMATES OF RAPIDLY TIME-VARYING SYSTEMS

A. Problem formulation

The recent work [8] considered the fast-varying system

$$\dot{x}(t)=A\left(\tfrac{t}{\epsilon}\right)x(t)+B\left(\tfrac{t}{\epsilon}\right)d(t),\ t\geq 0 \tag{1}$$

where $x(t)\in\mathbb{R}^n$ for $t\geq 0$, $\epsilon>0$ is a small parameter defining a fast time-scale, d is a piecewise continuous disturbance and $A:\mathbb{R}\to\mathbb{R}^{n\times n}$ and $B:\mathbb{R}\to\mathbb{R}^{n\times n_d}$ are piecewise continuous matrix functions, which are norm-bounded uniformly for $t\in[0,\infty)$. Under the assumption that there exist $0<T$ and matrices A_{av},B_{av}, such that

$$T^{-1}\int_t^{t+T}N(\tau)d\tau=N_{av}+\Delta N(t),\ \forall t\in\mathbb{R},\ N\in\{A,B\} \tag{2}$$

with A_{av} is Hurwitz and $\Delta A,\Delta B:\mathbb{R}\to\mathbb{R}^{n\times n}$ sufficiently small in norm, [8] proposed a novel time-delay transformation, leading to quantitative estimate on ϵ for which ISS of (1) is preserved.

In this work we consider a more general system in the following presentation with scalar time-varying coefficients that have zero average (see Assumption 1 below):

$$\begin{aligned}\dot{x}(t)\ &=\left[A_{av}+\textstyle\sum_{i=1}^N a_i\left(\tfrac{t}{\epsilon_i}\right)A_i\right]x(t)\\ &+\left[B_{av}+\textstyle\sum_{i=1}^{N_d} b_i\left(\tfrac{t}{\epsilon_{d,i}}\right)B_i\right]d(t),\quad t\geq 0\end{aligned} \tag{3}$$

where $x(t)\in\mathbb{R}^n$ for $t\geq 0$, $d\in C^1([0,\infty))$, $N,N_d\in\mathbb{N}$, $\{\epsilon_i\}_{i=1}^N$ and $\{\epsilon_{d,i}\}_{i=1}^{N_d}$ are positive small parameters, $\{A_i\}_{i=1}^N\subseteq\mathbb{R}^{n\times n}$, $\{B_i\}_{i=1}^{N_d}\subseteq\mathbb{R}^{n\times n_d}$ are constant matrices, and $\{a_i\}_{i=1}^N$, $\{b_i\}_{i=1}^{N_d}$ are piecewise continuous scalar functions which are uniformly bounded on $[0,\infty)$. We allow the arguments of the scalar functions to depend on *different and independent* time-scales.

Remark 1: System (1) can be presented as (3) by fixing $\epsilon_i\equiv\epsilon_{d,j}\equiv\epsilon$ and presenting $A\left(\frac{t}{\epsilon}\right)$, $B\left(\frac{t}{\epsilon}\right)$ as linear combinations of constant matrices with time-varying coefficients. In this case, $N,N_d\leq\max(n^2,nn_d)$.

For simplicity of presentation we proceed with the case $N=N_d=2$. The general case follows similar arguments. We make the following assumption:

Assumption 1: The matrix A_{av} is Hurwitz, whereas $\{a_i\}_{i=1}^2$ and $\{b_j\}_{j=1}^2$ are almost periodic. I.e., there exist positive constants $\{T_i\}_{i=1}^2$, $\{T_{d,j}\}_{j=1}^2$ such that

$$\begin{aligned}&T_i^{-1}\int_t^{t+T_i}a_i(\tau)d\tau=:\Delta a_i(t),\\ &T_{d,j}^{-1}\int_t^{t+T_{d,j}}b_j(\tau)d\tau=:\Delta b_j(t),\quad\forall t\in\mathbb{R}\end{aligned} \tag{4}$$

with $\{\Delta a_i\}_{i=1}^2,\{\Delta b_j\}_{j=1}^2$ satisfying

$$\sup_{\tau\in\mathbb{R}}\|\Delta n_i(\tau)\|^2\leq\Delta_{n_i,M},\quad 1\leq i\leq 2,\ n\in\{a,b\} \tag{5}$$

for positive constants $\{\Delta_{a_i,M}\}_{i=1}^2,\{\Delta_{b_j,M}\}_{j=1}^2$.

We aim to derive efficient and constructive conditions which guarantee ISS-like estimates for (3), with respect to d and $\dot{d}$ (see Theorem 1).

B. System transformation and Lyapunov analysis

Inspired by [14], for $t\geq 0$, $1\leq i,j\leq 2$ we introduce

$$\begin{aligned}&\varrho_{\epsilon,i}(t)=-\tfrac{1}{\epsilon_iT_i}\int_t^{t+\epsilon_iT_i}g_i(\tau)a_i\left(\tfrac{\tau}{\epsilon_i}\right)d\tau,\\ &\omega_{\epsilon_{d,j}}(t)=-\tfrac{1}{\epsilon_{d,j}T_{d,j}}\int_t^{t+\epsilon_{d,j}T_{d,j}}g_{d,j}(\tau)b_j\left(\tfrac{\tau}{\epsilon_{d,j}}\right)d\tau,\\ &g_i(\tau)=t+\epsilon_iT_i-\tau,\ g_{d,j}(\tau)=t+\epsilon_{d,j}T_{d,j}-\tau\end{aligned} \tag{6}$$

for which a simple computation yields

$$\begin{aligned}&\sup_{t\in\mathbb{R}}|\varrho_{\epsilon,i}(t)|\leq\epsilon_iT_i\sup_{t\in\mathbb{R}}|a_i(t)|,\\ &\sup_{t\in\mathbb{R}}|\omega_{\epsilon_{d,j}}(t)|\leq\epsilon_{d,j}T_{d,j}\sup_{t\in\mathbb{R}}|b_j(t)|.\end{aligned} \tag{7}$$

Differentiating (6), we have for $t\geq 0$

$$\begin{aligned}&\dot{\varrho}_{\epsilon,i}(t)=a_i\left(\tfrac{t}{\epsilon_i}\right)-\Delta a_i\left(\tfrac{t}{\epsilon_i}\right),\\ &\dot{\omega}_{\epsilon_{d,j}}(t)=b_j\left(\tfrac{t}{\epsilon_{d,j}}\right)-\Delta b_j\left(\tfrac{t}{\epsilon_{d,j}}\right).\end{aligned} \tag{8}$$

We introduce the following transformation

$$z(t)=x(t)-\sum_{i=1}^2\varrho_{\epsilon,i}(t)A_ix(t)-\sum_{i=1}^2\omega_{\epsilon_{d,i}}(t)B_id(t). \tag{9}$$

and the following assumption:

Assumption 2: We assume that $I_n-\sum_{i=1}^2\varrho_{\epsilon,i}(t)A_i$ is invertible for all $t\geq 0$ with

$$\sup_{t\geq 0}\left\|\left(I_n-\sum_{i=1}^2\varrho_{\epsilon,i}(t)A_i\right)^{-1}\right\|\leq\delta_{1,x}<\infty.$$

By (7), a sufficient condition for Assumption 2 to hold is $\sum_{i=1}^2\epsilon_iT_ia_{i,M}\|A_i\|<2$, where $a_{i,M}:=\sup_{\tau\in\mathbb{R}}|a_i(\tau)|$. Indeed, in this case we have

$$\sup_{t\geq 0}\left\|\textstyle\sum_{i=1}^2\varrho_{\epsilon,i}(t)A_i\right\|\leq\tfrac{\sum_{i=1}^2\epsilon_iT_ia_{i,M}\|A_i\|}{2}=:\delta_{2,x}<1. \tag{10}$$

Using a Neumann series, the latter implies that we can take

$$\delta_{1,x} = \frac{2}{2 - \sum_{i=1}^{2} \epsilon_i T_i a_{i,M} \|A_i\|} = \frac{1}{1-\delta_{2,x}}. \quad (11)$$

We will further employ the notation

$$\delta_d := \sup_{t\geq 0} \|\textstyle\sum_{i=1}^{n} \omega_{\epsilon_d,i}(t) B_i\|. \quad (12)$$

Analogously to (10), we have

$$\delta_d \leq \frac{1}{2}\sum_{i=1}^{2} \epsilon_i T_i b_{i,M} \|B_i\|, \quad b_{i,M} := \sup_{\tau\in\mathbb{R}} |b_i(\tau)|. \quad (13)$$

Remark 2: For (1) with a single time-scale, the time-delay transformation employed in [8] has the form

$$\begin{array}{l} z(t) = x(t) - G(t), \quad g(\tau) = \tau - t + \epsilon T, \\ G(t) = \frac{1}{\epsilon T}\int_{t-\epsilon T}^{t} g(\tau)\left[A(\tau)x(\epsilon\tau) + B(\tau)d(\epsilon\tau)\right] d\tau, \end{array}$$

which leads to a neutral-type system. This transformation allows for ISS analysis based on averaging of $B\left(\frac{t}{\epsilon}\right)$ and measurable functions d, whereas (9) allows for non differentiable d *without averaging* of $B\left(\frac{t}{\epsilon}\right)$ only, which may be restrictive. Compared to [8], we consider multiple fast time-scales and unify the transformation in [10] with a novel system presentation. The non-delayed transformation (9) simplifies the Lyaponov-based analysis whereas the new system presentation (3) improves the results in the numerical examples (see Section III below).

Since $d \in C^1([0,\infty))$, $z(t)$ is continuously differentiable. By (3) we have the following expression for $\dot{z}(t),\ t\geq 0$:

$$\begin{array}{l} \dot{z}(t) = A_{av}\left[z(t) + \sum_{i=1}^{2}\varrho_{\epsilon,i}(t)A_i x(t) + \sum_{i=1}^{2}\omega_{\epsilon_{d,i}}(t)B_i \right. \\ \left. \times d(t)\right] + B_{av}d(t) - \sum_{i=1}^{2}\omega_{\epsilon_{d,i}}(t)B_i\dot{d}(t) \\ + \sum_{i=1}^{2}\left[\Delta a_i\left(\frac{t}{\epsilon_i}\right)A_i x(t) + \Delta b_i\left(\frac{t}{\epsilon_{d,i}}\right)B_i d(t)\right] \\ - \sum_{i=1}^{2}\varrho_{\epsilon,i}(t)A_i\left[A_{av} + \sum_{j=1}^{2} a_j\left(\frac{t}{\epsilon_j}\right)A_j\right]x(t) \\ - \sum_{i=1}^{2}\varrho_{\epsilon,i}(t)A_i\left[B_{av} + \sum_{j=1}^{2} b_j\left(\frac{t}{\epsilon_{d,j}}\right)B_j\right]d(t). \end{array} \quad (14)$$

Next, we aim to vectorize (14). For that purpose, let $\leq_{\text{lex}}$ be the lexicographic order on $\mathbb{R}^n$ ($(i,j)\leq_{\text{lex}}(k,l)$ iff $i<k$ or $i=k,\ j\leq l$). Introduce

$$\begin{array}{l} \Upsilon_\varrho(t) = \text{col}\left\{\varrho_{\epsilon,i}(t)x(t)\right\}_{i=1}^{2}, \\ \mathcal{Z}_\omega(t) = \text{col}\left\{\omega_{\epsilon_{d,j}}(t)d(t)\right\}_{j=1}^{2}, \\ \mathcal{Z}_\varrho(t) = \text{col}\left\{\varrho_{\epsilon,i}(t)d(t)\right\}_{i=1}^{2}, \\ \Xi_\omega(t) = \text{col}\left\{\omega_{\epsilon_{d,j}}(t)\dot{d}(t)\right\}_{j=1}^{2}, \\ \Upsilon_{\varrho,a}(t) = \text{col}\left\{\varrho_{\epsilon,i}(t)a_k\left(\frac{t}{\epsilon_k}\right)x(t)\right\}_{\{(i,k)\}_{\leq_{\text{lex}}}}, \\ \mathcal{Z}_{\varrho,b}(t) = \text{col}\left\{\varrho_{\epsilon,i}(t)b_j\left(\frac{t}{\epsilon_{d,j}}\right)d(t)\right\}_{\{(i,j)\}_{\leq_{\text{lex}}}}, \\ \Upsilon_{\Delta a}(t) = \text{col}\left\{\Delta a_i\left(\frac{t}{\epsilon_i}\right)x(t)\right\}_{i=1}^{2}, \\ \mathcal{Z}_{\Delta b}(t) = \text{col}\left\{\Delta b_j\left(\frac{t}{\epsilon_{d,j}}\right)d(t)\right\}_{j=1}^{2}, \\ \mathbb{A} = \begin{bmatrix}A_1 & A_2\end{bmatrix},\ \mathbb{A}_1 = [A_1^2\ A_1A_2\ A_2A_1\ A_2^2], \\ \mathbb{A}_2 = [A_1B_1\ A_1B_2\ A_2B_1\ A_2B_2],\ \mathbb{B} = \begin{bmatrix}B_1 & B_2\end{bmatrix}, \\ \mathbb{W} = \begin{bmatrix}W_1 & W_2\end{bmatrix},\ W_i = A_{av}A_i - A_iA_{av},\ 1\leq i\leq 2. \end{array} \quad (15)$$

By (9), (14) and (15), we have the following for $\dot{z}(t),\ t\geq 0$:

$$\begin{array}{rl} \dot{z}(t) & = A_{av}z(t) + B_{av}d(t) + \mathbb{A}\Upsilon_{\Delta a}(t) + \mathbb{B}\mathcal{Z}_{\Delta b}(t) \\ & -\mathbb{A}\left(I_2\otimes B_{av}\right)\mathcal{Z}_\varrho(t) + \mathbb{W}\Upsilon_\varrho(t) - \mathbb{B}\Xi_\omega(t) \\ & +A_{av}\mathbb{B}\mathcal{Z}_\omega(t) - \mathbb{A}_1\Upsilon_{\varrho,a}(t) - \mathbb{A}_2\mathcal{Z}_{\varrho,b}(t). \end{array} \quad (16)$$

For ISS-like estimates of (3), let $\alpha>0$ be a desired decay rate and $0<P\in\mathbb{R}^{n\times n}$. Introduce the Lyapunov function

$$V(t) = |z(t)|_P^2 \quad (17)$$

and the notation

$$Q_\alpha := PA_{av} + A_{av}^\top P + 2\alpha P \quad (18)$$

Differentiating V along the solution to (16), we obtain

$$\begin{array}{l} \dot{V} + 2\alpha V = |z(t)|_{Q_\alpha}^2 + 2z^\top(t)PB_{av}d(t) \\ +2z^\top(t)P\left[\mathbb{A}\Upsilon_{\Delta a}(t) + \mathbb{B}\mathcal{Z}_{\Delta b}(t) + \mathbb{W}\Upsilon_\varrho(t) + A_{av}\right. \\ \left.\times\mathbb{B}\mathcal{Z}_\omega(t)\right] - 2z^\top(t)P\left[\mathbb{A}\left(I_2\otimes B_{av}\right)\mathcal{Z}_\varrho(t) + \mathbb{B}\Xi_\omega(t)\right] \\ -2z^\top(t)P\left[\mathbb{A}_1\Upsilon_{\varrho,a}(t) + \mathbb{A}_2\mathcal{Z}_{\varrho,b}(t)\right]. \end{array} \quad (19)$$

Substituting (9) and recalling (15), we have

$$\begin{array}{l} |z(t)|_{Q_\alpha}^2 = |x(t) - \mathbb{A}\Upsilon_\varrho(t) - \mathbb{B}\mathcal{Z}_\omega(t)|_{Q_\alpha}^2 = |x(t)|_{Q_\alpha}^2 \\ + |\Upsilon_\varrho(t)|_{\mathbb{A}^\top Q_\alpha\mathbb{A}}^2 + |\mathcal{Z}_\omega(t)|_{\mathbb{B}^\top Q_\alpha\mathbb{B}}^2 - 2x^\top(t)Q_\alpha\mathbb{A}\Upsilon_\varrho(t) \\ -2x^\top(t)Q_\alpha\mathbb{B}\mathcal{Z}_\omega(t) + 2\Upsilon_\varrho^\top(t)\mathbb{A}^\top Q_\alpha\mathbb{B}\mathcal{Z}_\omega(t). \end{array} \quad (20)$$

Similarly,

$$\begin{array}{l} z^\top(t)PB_{av}d(t) = \left[x(t) - \mathbb{A}\Upsilon_\varrho(t) - \mathbb{B}\mathcal{Z}_\omega(t)\right]^\top PB_{av}d(t), \\ z^\top(t)P\left[\mathbb{A}\Upsilon_{\Delta a}(t) + \mathbb{B}\mathcal{Z}_{\Delta b}(t) + \mathbb{W}\Upsilon_\varrho(t) + A_{av}\mathbb{B}\mathcal{Z}_\omega(t)\right. \\ -\mathbb{A}\left(I_2\otimes B_{av}\right)\mathcal{Z}_\varrho(t) - \mathbb{B}\Xi_\omega(t) - \mathbb{A}_1\Upsilon_{\varrho,a}(t) \\ \left.-\mathbb{A}_2\mathcal{Z}_{\varrho,b}(t)\right] = \left[x(t) - \mathbb{A}\Upsilon_\varrho(t) - \mathbb{B}\mathcal{Z}_\omega(t)\right]^\top P\left[\mathbb{A}\Upsilon_{\Delta a}(t)\right. \\ +\mathbb{B}\mathcal{Z}_{\Delta b}(t) + A_{av}\mathbb{B}\mathcal{Z}_\omega(t) - \mathbb{A}\left(I_2\otimes B_{av}\right)\mathcal{Z}_\varrho(t) \\ \left.-\mathbb{B}\Xi_\omega(t) - \mathbb{A}_1\Upsilon_{\varrho,a}(t) - \mathbb{A}_2\mathcal{Z}_{\varrho,b}(t) + \mathbb{W}\Upsilon_\varrho(t)\right]. \end{array} \quad (21)$$

To compensate $\Upsilon_\varrho(t), \mathcal{Z}_\varrho(t), \mathcal{Z}_\omega(t), \Xi_\omega(t), \Upsilon_{\varrho,a}(t), \mathcal{Z}_{\varrho,b}(t)$, $\Upsilon_{\Delta a}(t)$ and $\mathcal{Z}_{\Delta b}(t)$, we employ the S-procedure [7]. Let

$$\begin{array}{l} H_\varrho = \text{col}\left\{\mathfrak{h}_\varrho^{(i)}\right\}_{i=1}^{2},\ H_\omega = \text{col}\left\{\mathfrak{h}_\omega^{(j)}\right\}_{j=1}^{2}, \\ H_{\varrho,a} = \text{col}\left\{\mathfrak{h}_{\varrho,a}^{(i,k)}\right\}_{\leq_{\text{lex}}},\ H_{\varrho,b} = \text{col}\left\{\mathfrak{h}_{\varrho,b}^{(i,j)}\right\}_{\leq_{\text{lex}}} \end{array} \quad (22)$$

be vectors with *nonnegative entries* such that for any $i,k,j=1,2$ and $t\geq 0$ the following hold:

$$\begin{array}{l} I)\ \varrho_{\epsilon,i}^2(t)\leq\mathfrak{h}_\varrho^{(i)},\ II)\ \omega_{\epsilon_{d,j}}^2(t)\leq\mathfrak{h}_\omega^{(j)}, \\ III)\ \varrho_{\epsilon,i}^2(t)a_k^2\left(\frac{t}{\epsilon_k}\right)\leq\mathfrak{h}_{\varrho,a}^{(i,k)},\ IV)\ \varrho_{\epsilon,i}^2(t)b_j^2\left(\frac{t}{\epsilon_{d,j}}\right)\leq\mathfrak{h}_{\varrho,b}^{(i,j)}. \end{array} \quad (23)$$

Let $\Lambda_{\Upsilon_\varrho}, \Lambda_{\mathcal{Z}_\varrho}, \Lambda_{\Upsilon_{\Delta a}}\in\mathbb{R}^{2\times 2}$, $\Lambda_{\mathcal{Z}_\omega}, \Lambda_{\Xi_\omega}, \Lambda_{\mathcal{Z}_{\Delta b}}\in\mathbb{R}^{2\times 2}$ and $\Lambda_{\Upsilon_{\varrho,a}}, \Lambda_{\mathcal{Z}_{\varrho,b}}\in\mathbb{R}^{4\times 4}$ be *diagonal* matrices with *positive* diagonal entries and recall (15). By (5) and (23) we have

$$\begin{array}{l} \Upsilon_\varrho^\top(t)\left(\Lambda_{\Upsilon_\varrho}\otimes I_n\right)\Upsilon_\varrho(t)\leq\left|\Lambda_{\Upsilon_\varrho}H_\varrho\right|_1|x(t)|^2, \\ \mathcal{Z}_\varrho^\top(t)\left(\Lambda_{\mathcal{Z}_\varrho}\otimes I_{n_d}\right)\mathcal{Z}_\varrho(t)\leq\left|\Lambda_{\mathcal{Z}_\varrho}H_\varrho\right|_1|d(t)|^2, \\ \mathcal{Z}_\omega^\top(t)\left(\Lambda_{\mathcal{Z}_\omega}\otimes I_{n_d}\right)\mathcal{Z}_\omega(t)\leq\left|\Lambda_{\mathcal{Z}_\omega}H_\omega\right|_1|d(t)|^2, \\ \Xi_\omega^\top(t)\left(\Lambda_{\Xi_\omega}\otimes I_{n_d}\right)\Xi_\omega(t)\leq\left|\Lambda_{\Xi_\omega}H_\omega\right|_1\left|\dot{d}(t)\right|^2, \end{array}$$

$$
\begin{aligned}
\Upsilon_{\varrho,a}^{\top}(t)\left(\Lambda_{\Upsilon_{\varrho,a}}\otimes I_n\right)\Upsilon_{\varrho,a}(t) &\le \left|\Lambda_{\Upsilon_{\varrho,a}}H_{\varrho,a}\right|_1 |x(t)|^2,\\
\mathcal{Z}_{\varrho,b}^{\top}(t)\left(\Lambda_{\mathcal{Z}_{\varrho,b}}\otimes I_{n_d}\right)\mathcal{Z}_{\varrho,b}(t) &\le \left|\Lambda_{\mathcal{Z}_{\varrho,b}}H_{\varrho,b}\right|_1 |d(t)|^2,\\
\Upsilon_{\Delta a}^{\top}(t)\left(\Lambda_{\Upsilon_{\Delta a}}\otimes I_n\right)\Upsilon_{\Delta a}(t) &\le \left|\Lambda_{\Upsilon_{\Delta a}}\Delta_{a,M}\right|_1 |x(t)|^2,\\
\mathcal{Z}_{\Delta b}^{\top}(t)\left(\Lambda_{\mathcal{Z}_{\Delta b}}\otimes I_{n_d}\right)\mathcal{Z}_{\Delta b}(t) &\le \left|\Lambda_{\mathcal{Z}_{\Delta b}}\Delta_{b,M}\right|_1 |d(t)|^2
\end{aligned} \tag{24}
$$

where

$$
\Delta_{a,M}=\operatorname{col}\left\{\Delta_{a_i,M}\right\}_{i=1}^{2},\ \Delta_{b,M}=\operatorname{col}\left\{\Delta_{b_j,M}\right\}_{j=1}^{2}. \tag{25}
$$

Remark 3: The assumption that the averages of a_i, b_i are zero is central to our approach. By assuming this, $\{a_i,\varrho_{\epsilon,i}\}_{i=1}^2$ and $\{b_j,\omega_{\epsilon_d,j}\}_{j=1}^2$ have smaller L^∞ norms, whence the upper bounds in (23) and (24) will be of smaller magnitude. This fact yields less conservative LMIs in the Lyapunov analysis. Finally, this assumption poses no loss of generality since we can substract the averages from the corresponding functions, while retaining Δa_i and Δb_j in (5) and modifying A_{av} and B_{av}.

Introducing

$$
\begin{aligned}
\eta(t) \quad =\operatorname{col}\Big\{&x(t),d(t),\dot d(t),\Upsilon_\varrho(t),\Upsilon_{\varrho,a}(t),\Upsilon_{\Delta a}(t)\\
&\mathcal{Z}_\varrho(t),\mathcal{Z}_{\varrho,b}(t),\mathcal{Z}_{\Delta b}(t),\mathcal{Z}_\omega(t),\Xi_\omega(t)\Big\}
\end{aligned}, \tag{26}
$$

(24) implies that

$$
\begin{aligned}
&0\le W=\eta^\top(t)\left[\Lambda_0-\Lambda_1\right]\eta(t),\\
&\Lambda_0=\operatorname{diag}\left\{\Lambda_0^{(1)},\Lambda_0^{(2)},\Lambda_0^{(3)},0,0,0,0,0,0,0,0\right\},\\
&\Lambda_1=\operatorname{diag}\left\{0,0,0,\Lambda_1^{(1)}\right\},\ \Lambda_0^{(3)}=\left|\Lambda_{\Xi_\omega}H_\omega\right|_1 I_{n_d},\\
&\Lambda_0^{(1)}=\left(\left|\Lambda_{\Upsilon_\varrho}H_\varrho\right|_1+\left|\Lambda_{\Upsilon_{\varrho,a}}H_{\varrho,a}\right|_1+\left|\Lambda_{\Upsilon_{\Delta a}}\Delta_{a,M}\right|_1\right)I_n,\\
&\Lambda_0^{(2)}=\left(\left|\Lambda_{\mathcal{Z}_\varrho}H_\varrho\right|_1+\left|\Lambda_{\mathcal{Z}_\omega}H_\omega\right|_1+\left|\Lambda_{\mathcal{Z}_{\varrho,b}}H_{\varrho,b}\right|_1\right.\\
&\qquad\left.+\left|\Lambda_{\mathcal{Z}_{\Delta b}}\Delta_{b,M}\right|_1\right)I_{n_d},\\
&\Lambda_1^{(1)}=\operatorname{diag}\left\{\Lambda_{\Upsilon_\varrho}\otimes I_n,\Lambda_{\Upsilon_{\varrho,a}}\otimes I_n,\Lambda_{\Upsilon_{\Delta a}}\otimes I_n,\Lambda_{\mathcal{Z}_\varrho}\right.\\
&\quad\left.\otimes I_{n_d},\Lambda_{\mathcal{Z}_{\varrho,b}}\otimes I_{n_d},\Lambda_{\mathcal{Z}_{\Delta b}}\otimes I_{n_d},\Lambda_{\mathcal{Z}_\omega}\otimes I_{n_d},\Lambda_{\Xi_\omega}\otimes I_{n_d}\right\}
\end{aligned} \tag{27}
$$

Let $\gamma_i>0,\ i=1,2$. By (24)-(27) and the S-procedure

$$
\begin{aligned}
\dot V+2\alpha V-\gamma_1^2|d(t)|^2-\gamma_2^2\left|\dot d(t)\right|^2+W&\\
\le \eta^\top(t)\Psi_{\epsilon,\epsilon_d}\eta(t)\le 0,&
\end{aligned} \tag{28}
$$

provided

$$
\Psi_{\epsilon,\epsilon_d}=\left[\begin{array}{c|c|c|c}
\Psi_{\epsilon,\epsilon_d}^{(1)} & \Psi_{\epsilon,\epsilon_d}^{(2)} & \Psi_{\epsilon,\epsilon_d}^{(3)} & \Psi_{\epsilon,\epsilon_d}^{(4)}\\ \hline
* & \Psi_{\epsilon,\epsilon_d}^{(5)} & \Psi_{\epsilon,\epsilon_d}^{(6)} & \Psi_{\epsilon,\epsilon_d}^{(7)}\\ \hline
* & * & \Psi_{\epsilon,\epsilon_d}^{(8)} & \Psi_{\epsilon,\epsilon_d}^{(9)}\\ \hline
* & * & * & \Psi_{\epsilon,\epsilon_d}^{(10)}
\end{array}\right]<0 \tag{29}
$$

with

$$
\Psi_{\epsilon,\epsilon_d}^{(1)}=\begin{bmatrix}Q_\alpha+\Lambda_0^{(1)} & PB_{av} & 0\\ * & -\gamma_1^2 I_{n_d}+\Lambda_0^{(2)} & 0\\ * & * & -\gamma_2^2 I_{n_d}+\Lambda_0^{(3)}\end{bmatrix},
$$

$$
\Psi_{\epsilon,\epsilon_d}^{(2)}=\begin{bmatrix}-Q_\alpha\mathbb{A}+P\mathbb{W} & -P\mathbb{A}_1 & P\mathbb{A}\\ -B_{av}^\top P\mathbb{A} & 0 & 0\\ 0 & 0 & 0\end{bmatrix},
$$

$$
\Psi_{\epsilon,\epsilon_d}^{(1)}=\begin{bmatrix}Q_\alpha+\Lambda_0^{(1)} & PB_{av} & 0\\ * & -\gamma_1^2 I_{n_d}+\Lambda_0^{(2)} & 0\\ * & * & -\gamma_2^2 I_{n_d}+\Lambda_0^{(3)}\end{bmatrix},
$$

$$
\Psi_{\epsilon,\epsilon_d}^{(2)}=\begin{bmatrix}-Q_\alpha\mathbb{A}+P\mathbb{W} & -P\mathbb{A}_1 & P\mathbb{A}\\ -B_{av}^\top P\mathbb{A} & 0 & 0\\ 0 & 0 & 0\end{bmatrix},
$$

$$
\Psi_{\epsilon,\epsilon_d}^{(3)}=\begin{bmatrix}-P\mathbb{A}\left(I_2\otimes B_{av}\right) & -P\mathbb{A}_2 & P\mathbb{B}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},
$$

$$
\Psi_{\epsilon,\epsilon_d}^{(4)}=\begin{bmatrix}-Q_\alpha\mathbb{B}+PA_{av}\mathbb{B} & -P\mathbb{B}\\ -B_{av}^\top P\mathbb{B} & 0\\ 0 & 0\end{bmatrix},
$$

$$
\Psi_{\epsilon,\epsilon_d}^{(5)}=\begin{bmatrix}\psi_{\epsilon,\epsilon_d}^{(1)} & \mathbb{A}^\top P\mathbb{A}_1 & -\mathbb{A}^\top P\mathbb{A}\\ * & -\left(\Lambda_{\Upsilon_{\varrho,a}}\otimes I_n\right) & 0\\ * & * & -\left(\Lambda_{\Upsilon_{\Delta a}}\otimes I_n\right)\end{bmatrix},
$$

$$
\Psi_{\epsilon,\epsilon_d}^{(6)}=\begin{bmatrix}\psi_{\epsilon,\epsilon_d}^{(4)} & \mathbb{A}^\top P\mathbb{A}_2 & -\mathbb{A}^\top P\mathbb{B}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},
$$

$$
\Psi_{\epsilon,\epsilon_d}^{(7)}=\begin{bmatrix}\psi_{\epsilon,\epsilon_d}^{(2)} & \mathbb{A}^\top P\mathbb{B}\\ \mathbb{A}_1^\top P\mathbb{B} & 0\\ -\mathbb{A}^\top P\mathbb{B} & 0\end{bmatrix},\ \Psi_{\epsilon,\epsilon_d}^{(9)}=\begin{bmatrix}\psi_{\epsilon,\epsilon_d}^{(3)} & 0\\ \mathbb{A}_2^\top P\mathbb{B} & 0\\ -\mathbb{B}^\top P\mathbb{B} & 0\end{bmatrix},
$$

$$
\Psi_{\epsilon,\epsilon_d}^{(8)}=-\operatorname{diag}\left\{\Lambda_{\mathcal{Z}_\varrho},\Lambda_{\mathcal{Z}_{\varrho,b}},\Lambda_{\mathcal{Z}_{\Delta b}}\right\}\otimes I_n,
$$

$$
\Psi_{\epsilon,\epsilon_d}^{(10)}=\begin{bmatrix}-\left(\Lambda_{\mathcal{Z}_\omega}\otimes I_n\right)+2\alpha\mathbb{B}^\top P\mathbb{B} & \mathbb{B}^\top P\mathbb{B}\\ * & -\left(\Lambda_{\Xi_\omega}\otimes I_n\right)\end{bmatrix},
$$

$$
\begin{aligned}
\psi_{\epsilon,\epsilon_d}^{(1)}&=-\left(\Lambda_{\Upsilon_\varrho}\otimes I_n\right)+\mathbb{A}^\top Q_\alpha\mathbb{A}-\mathbb{A}^\top P\mathbb{W}-\mathbb{W}^\top P\mathbb{A},\\
\psi_{\epsilon,\epsilon_d}^{(2)}&=\mathbb{A}^\top Q_\alpha\mathbb{B}-\mathbb{W}^\top P\mathbb{B}-\mathbb{A}^\top PA_{av}\mathbb{B},\\
\psi_{\epsilon,\epsilon_d}^{(3)}&=\left(I_2\otimes B_{av}\right)^\top\mathbb{A}^\top P\mathbb{B},\ \psi_{\epsilon,\epsilon_d}^{(4)}=\mathbb{A}^\top P\mathbb{A}\left(I_2\otimes B_{av}\right).
\end{aligned}
$$

Summarizing, we arrive at:

Theorem 1: Consider (3) under Assumptions 1-2. Let $H_\varrho, H_\omega, H_{\varrho,a}, H_{\varrho,b}$ be given by (22) and (23). Given positive tuning parameters $\alpha,\{\epsilon_i^*\}_{i=1}^2,\left\{\epsilon_{d,j}^*\right\}_{j=1}^2$, $\left\{\Delta_{a_i,M}\right\}_{i=1}^2,\left\{\Delta_{b_j,M}\right\}_{j=1}^2$, let there exist $0<P\in\mathbb{R}^{n\times n}$, positive diagonal matrices $\Lambda_{\Upsilon_\varrho},\Lambda_{\mathcal{Z}_\varrho},\Lambda_{\Upsilon_{\Delta a}}\in\mathbb{R}^{2\times 2}$, $\Lambda_{\mathcal{Z}_\omega},\Lambda_{\Xi_\omega},\Lambda_{\mathcal{Z}_{\Delta b}}\in\mathbb{R}^{2\times 2}$ and $\Lambda_{\Upsilon_{\varrho,a}},\Lambda_{\mathcal{Z}_{\varrho,b}}\in\mathbb{R}^{4\times 4}$, and positive scalars γ_1^2,γ_2^2 such that $\Psi_{\epsilon^*,\epsilon_d^*}<0$, with $\Psi_{\epsilon,\epsilon_d}$ given by (29). Then (3) satisfies the ISS-like estimate

$$
\begin{aligned}
|x(t)|^2 \quad &\le \beta_1^2 e^{-2\alpha t}|x(0)|^2+\beta_2^2\max_{s\in[0,t]}|d(s)|^2\\
&+\beta_3^2\max_{s\in[0,t]}\left|\dot d(s)\right|^2,\quad t\ge 0
\end{aligned} \tag{30}
$$

for some $\beta_i>0,\ i=1,2,3$. The LMI $\Psi_{\epsilon,\epsilon_d}<0$ is feasible for small enough $\alpha,\{\epsilon_i\}_{i=1}^2,\{\epsilon_{d,j}\}_{j=1}^2$, $\left\{\Delta_{a_i,M}\right\}_{i=1}^2,\left\{\Delta_{b_j,M}\right\}_{j=1}^2$ and large $\gamma_i^2,\ i=1,2$.

III. Numerical Examples

A. Example 3.1: Stabilization by fast switching I

We consider stabilization by fast switching of a linear system (see [8, Example 2.2]). Let $\epsilon>0$ and

$$
\bar A_1=\begin{bmatrix}0.1 & 0.3\\ 0.6 & -0.2\end{bmatrix},\ \bar A_2=\begin{bmatrix}-0.13 & -0.16\\ -0.33 & 0.03\end{bmatrix}
$$

and for $\tau\in[k,k+1)$, set

$$
\bar A(\tau)=\chi_{[k,k+0.4)}(\tau)A_1+\left[1-\chi_{[k+0.4,k+1)}(\tau)\right]A_2, \tag{31}
$$

where $\chi_{[k,k+0.4)}$ is the indicator function of the interval $[k,k+0.4)$. Note that $\bar A(\tau)$ is 1-periodic. We present the system $\dot x(t)=\bar A\left(\frac{t}{\epsilon}\right)x(t)$ as (3) with $\epsilon_i=\epsilon, T_i=1,\ i=$

$1, 2,\ B_{av} = B_1 = B_2 = 0$,

$$A_{av} = \begin{bmatrix} -0.038 & 0.024 \\ 0.042 & -0.062 \end{bmatrix}, \tag{32}$$

and

$$a_1(\tau) = \begin{cases} 0.6, \ \tau \in [k, k+0.4), \ k \in \mathbb{Z} \\ -0.4, \ \tau \in [k+0.4, k+1), \ k \in \mathbb{Z} \end{cases},$$
$$a_2(\tau) = -a_1(\tau).$$

Note that the latter functions are 1-periodic, meaning that $\Delta_{a_i,M} = 0,\ i = 1, 2$ in (25). Let $t \in [m\epsilon, (m+1)\epsilon),\ m \in \mathbb{Z}_+$ and denote $w = t - m\epsilon \in [0, \epsilon),\ m \in \mathbb{Z}_+$. An explicit computation of $\varrho_{\epsilon,i}(t),\ i = 1, 2$ in (6) yields the bounds $\varrho^2_{\epsilon,1}(t) \le 0.0144\epsilon^2$ and $\varrho^2_{\epsilon,2}(t) \le 0.0144\epsilon^2$. We then use the fact that $a_1(\tau)$, $a_2(\tau)$ are indicator functions to separate the analysis into two cases

$$a_1\left(\tfrac{t}{\epsilon}\right) \varrho_{\epsilon_j}(t) = \begin{cases} 0.6\varrho_{\epsilon,j}(t), & w \in [0, 0.4\epsilon) \\ -0.4\varrho_{\epsilon,j}(t), & w \in [0.4\epsilon, \epsilon) \end{cases}$$
$$a_2\left(\tfrac{t}{\epsilon}\right) \varrho_{\epsilon,j}(t) = -a_1\left(\tfrac{t}{\epsilon}\right) \varrho_{\epsilon_j}(t)$$

and obtain tight upper bounds in (23) for each of the cases. Thus, we separate the analysis into the two subintervals $0 \le w < 0.4\epsilon$ and $0.4\epsilon \le w < \epsilon$. For each subinterval (and its corresponding bounds (23)) we obtain an LMI of the form (29). We verify feasibility for both LMIs with the same α and P. We consider $\alpha \in \{0, 0.005, 0.01\}$ and verify the LMIs of Theorem 1 to obtain the maximal value ϵ^* which preserves feasibility of the LMIs. Note that ϵ^* guarantees internal exponential stability (and thus the ISS-like bounds) of (3). The values of ϵ^* are given in Table I, where we further compare our results to the bounds in the recent work [22]. It is seen that our results essentially improve the results of [22] with a value of ϵ^* larger by more than 2.5 times. Next, we

	$\alpha = 0$	$\alpha = 0.005$	$\alpha = 0.01$
Zhang & Fridman	0.1920	0.1306	Unchecked
Thm. 1	0.4332	0.3013	0.1662

TABLE I
SWITCHED SYSTEM I - MAXIMUM VALUE ϵ^* PRESERVING LMI FEASIBILITY.

set $B_{av} = [0\ 1]^\top$ and $B_1 = B_2 = 0_{2\times 1}$ and verify feasibility of (29) in order to guarantee (30). Note that in this case the transformation (9) will not result in terms involving $\dot{d}$. Hence, we obtain classical ISS estimates (i.e., we have $\gamma_2 = 0$ in (28) $\beta_3 = 0$ in (30)). Table II presents several pairs (β_1, β_2) (see proof of Theorem 1) for different choices of α and ϵ. Here $\delta_{1,x}$ and $\delta_{2,x}$ were computed using (10) and (11).

	$\epsilon = 0.002$	$\epsilon = 0.16$
$\alpha = 0.005$	$(0.0054, 73.503)$	$(0.5147, 99.266)$
$\alpha = 0.01$	$(0.006, 76.48)$	$(0.7126, 389.89)$

TABLE II
SWITCHED SYSTEM I - ISS GAINS: (β_1, β_2).

B. Example 3.2: Stabilization by fast switching II

We consider stabilization by fast switching of a linear system with three functioning modes (see [1] and [4]) . Let $\epsilon > 0$ and

$$A_1 = \begin{bmatrix} 0 & 0.5 \\ 0 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.1 & 0 \\ -1 & -1 \end{bmatrix},\ A_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{33}$$

and set

$$\bar{A}(\tau) = \begin{cases} A_1, \ \tau \in [k, k+0.4), \ k \in \mathbb{Z}_+ \\ A_2, \ \tau \in [k+0.4, k+0.87), \ k \in \mathbb{Z}_+ \\ A_3, \ \tau \in [k+0.87, k+1), \ k \in \mathbb{Z}_+ \end{cases}. \tag{34}$$

Note that $\bar{A}(\tau)$ is 1-periodic and can be presented as a linear combination of $\bar{A}_i,\ i = 1, 2, 3$ with indicator coefficients. We present the system $\dot{x}(t) = \bar{A}\left(\frac{t}{\epsilon}\right) x(t)$ as (3) with $\epsilon_i = \epsilon, T_i = 1,\ i = 1, 2, 3,\ B_{av} = B_1 = B_2 = B_3 = 0$,

$$A_{av} = \begin{bmatrix} 0.047 & 0.33 \\ -0.6 & -0.87 \end{bmatrix} \tag{35}$$

and, for $k \in \mathbb{Z}_+$,

$$a_1(\tau) = \chi_{[k,k+0.4)}(\tau) - 0.4,$$
$$a_2(\tau) = \chi_{[k+0.4,k+0.87)}(\tau) - 0.47,$$
$$a_3(\tau) = \chi_{[k+0.87,k+1)}(\tau) - 0.13.$$

Note that the latter functions are 1-periodic, meaning that $\Delta_{a_i,M} = 0,\ i = 1, 2, 3$ in (25). Similarly to Example 3.2.1, an explicit computation of $\varrho_{\epsilon,i}(t),\ i = 1, 2$ in (6) yields the bounds $\varrho^2_{\epsilon,1}(t) \le 0.0144\epsilon^2$, $\varrho^2_{\epsilon,2}(t) \le 0.0155127\epsilon^2$ and $\varrho^2_{\epsilon,3}(t) \le 0.0031979\epsilon^2$. We then use the fact that $a_1(\tau)$, $a_2(\tau)$ and $a_3(\tau)$ are indicator functions to separate the analysis into three cases, corresponding to the subintervals in (34). For each subinterval (and corresponding bounds (23)) we obtain an LMI of the form (29). We verify feasibility for both LMIs with the same α and P. We consider $\alpha \in \{0, 0.005, 0.25\}$ and verify the LMIs of Theorem 1 to obtain the maximal value ϵ^* which preserves feasibility of the LMI. Note that ϵ^* guarantees internal exponential stability (and ISS-like bounds) of (3). The values of ϵ^* are given in Table III.

	$\alpha = 0$	$\alpha = 0.005$	$\alpha = 0.25$
Thm. 1	0.4341	0.4177	0.0591

TABLE III
SWITCHED SYSTEM II - MAXIMUM VALUE ϵ^* PRESERVING LMI FEASIBILITY.

C. Example 3.3: Control of a pendulum

We consider a suspended pendulum with the suspension point that is subject to vertical vibrations of small amplitude and high frequency (see [11, Example 10.10] and [8, Example 2.1]). Let $\epsilon > 0$ and

$$\bar{A}(\tau) = \begin{bmatrix} \cos(\tau) & 1 \\ 0.04 - \cos^2(\tau) & -0.2 - \cos(\tau) \end{bmatrix}. \tag{36}$$

Note that $\bar{A}(\tau)$ is 2π periodic.

Employing the trigonometric identity $2\cos^2(\tau) = 1 + \cos(2\tau)$, we present the system $\dot{x}(t) = \bar{A}\left(\frac{t}{\epsilon}\right)x(t)$ as (3) with $\epsilon_i = \epsilon, T_i = 2\pi,\ i = 1, 2,\ B_{av} = B_1 = B_2 = 0$ and

$$A_{av} = \begin{bmatrix} 0 & 1 \\ -0.46 & -0.2 \end{bmatrix},\ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0 & 0 \\ -0.5 & 0 \end{bmatrix},\ a_1(\tau) = \cos(\tau),\ a_2(\tau) = \cos(2\tau).$$

Note that $a_i(\tau),\ i = 1, 2$ are 2π-periodic, whence $\Delta_{a_i,M} = 0,\ i = 1, 2$ in (25). An explicit computation of $\varrho_{\epsilon,i}(t),\ i = 1, 2$ in (6) yields

$$\begin{aligned}
&\varrho_{\epsilon,1}(t) = \epsilon\sin(\tau),\ a_2(\tau)\varrho_{\epsilon,2}(t) = \tfrac{\epsilon}{4}\sin(4\tau), \\
&\varrho_{\epsilon,2}(t) = a_1(\tau)\varrho_{\epsilon,1}(t) = \epsilon\cos(\tau)\sin(\tau), \\
&a_2(\tau)\varrho_{\epsilon,1}(t) = (2\cos^2(\tau) - 1)\varrho_{\epsilon,1}(t), \\
&a_1(\tau)\varrho_{\epsilon,2}(t) = \cos^2(\tau)\varrho_{\epsilon,1}(t),\ \tau = \tfrac{t}{\epsilon}
\end{aligned}$$

which are used to derive the upper bounds in (23). Differently from the previous examples, here we obtain only one LMI of the form (29).

We consider $\alpha \in \left\{0, \frac{1}{10\pi}\right\}$ and verify the LMIs of Theorem 1 to obtain the maximal value ϵ^* which preserves feasibility of the LMI. Note that ϵ^* guarantees internal exponential stability (and thus the ISS-like bounds) of (3). The values of ϵ^* are given in Table IV, where we further compare our results to the bounds in the recent work [22].

	$\alpha = 0$	$\alpha = (10\pi)^{-1}$
Zhang & Fridman	0.0074	0.005
Thm. 1	0.0457	0.0321

TABLE IV

PENDULUM - MAXIMUM VALUE ϵ^* PRESERVING LMI FEASIBILITY.

Finally, we consider this example subject to uncertainty. For that purpose, we replace $a_2(\tau) = \cos(2\tau)$ with $a_2(\tau) = \cos(2\tau) + 0.4g(\tau)$, where $\|g\|_\infty \leq 0.1$. In this case we obtain a nonzero $\Delta a_2(t)$ in (4), satisfying $\|\Delta a_2\|_\infty \leq 0.04 =: \Delta_{a_2,M}$. We consider $\alpha \in \left\{0, \frac{1}{10\pi}\right\}$ and verify the LMIs of Theorem 1 to obtain the maximal value ϵ^* which preserves feasibility of the LMI. We further compare our results with [22, Example 4.1]. The results are given in Table V. Our results are essentially better than the results of [22].

	$\alpha = 0$	$\alpha = (10\pi)^{-1}$
Fridman & Zhang	0.0058	0.0034
Thm. 1	0.0204	0.0146

TABLE V

PENDULUM WITH UNCERTAINTY - MAXIMUM VALUE ϵ^* PRESERVING LMI FEASIBILITY.

IV. CONCLUSION

We introduced a novel quantitative methodology for deriving ISS-like estimates for linear continuous-time systems. The presented methodology relies on a new system presentation, in conjunction with a delay-free system transformation. Compared to the time-delay approach to averaging, the presented method is based on a simpler Lyapunov analysis of non-delayed transformed systems, and achieves essentially less conservative LMI conditions for ISS-like estimates. However, time-delay approach is applicable not just to classical averaging as considered in the present paper, but also to Lie-brackets-based averaging [23] where application of the non-delay transformation seems to be questionable. Future work may include extension of the method to systems with delays and applications to control problems that employ averaging.

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