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Data-Driven Model Predictive Control for Continuous-Time Systems

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***Abstract*— We present some preliminary ideas on a data-driven Model Predictive Control framework for continuous-time systems. We use Chebyshev polynomial orthogonal bases to represent system trajectories and subsequently develop a data-driven continuous-time version of the classical Model Predictive Control algorithm. We investigate the effects of the parameters in our framework with two numerical examples and draw comparison to model-driven MPC schemes.**

I. Introduction

Deviation from familiar control schemes that utilise mathematical models is becoming more apparent in research. The key principles of classical control schemes are being converted into algorithms that can be driven only by system *data*. Model Predictive Control (MPC) is one example. Its success in the model-driven, discrete-time domain has been well-documented since its formulation and many developments of MPC have followed (see e.g. [1]). The majority of these innovations have targeted discrete-time systems.

Previous extension of MPC into the continuous-time domain was conducted such that the control inputs were continuous-time signals that could drive the state of the real-world system continuously and continuous-time mathematical models of systems could be used directly. Hence, information about the system dynamics remained that would otherwise have been removed by discretisation techniques. This was first accomplished by means of so-called *emulators* and subsequently series expansions to approximate the trajectories of the system (see e.g. [2], [3]). Techniques that were evolving in the discrete-time case were simultaneously adopted in the continuous-time domain to improve designs such as the replacement of transfer functions with state space models (see e.g. [4], [5]). More recent advancements have seen the employment of orthonormal functions in order to facilitate the design of MPC schemes in continuous-time with more desirable tuning capabilities (see [6]). Still however, such control schemes rely on an accurate system representation by a mathematical model.

Data-driven approaches to discrete-time MPC and other control schemes such as Iterative Learning Control (ILC) have been presented (see e.g. [7] and [8] respectively). More recent developments in data-driven approaches have followed from advances in the fundamental ideas of data-driven control in the continuous-time domain (see e.g. [9]–[11]). Subsequently, the last two authors of the present paper introduced in [12] a Chebyshev basis approach for applying data-driven ILC ideas to continuous-time systems. However, no MPC algorithm currently exists for controlling continuous-time problems without any mathematical model of the system.

In this paper, we present some preliminary ideas on the use of Chebyshev polynomials for the application of data-driven MPC in the continuous-time domain. We investigate the key parameters involved in the described control scheme in simulations and show that this framework provides equivalent results to schemes that use system models.

The remainder of this paper is structured as follows. In Section II, we recall the basic definitions and concepts of MPC (Section II-A) and of Chebyshev Polynomial Orthogonal Bases (CPOB) (Section II-B). Section III contains the statement of a fundamental result relating CPOB representations of 'sufficiently informative' input-output data produced by a continuous-time linear time-invariant (LTI) system, and the CPOB representations of *all* system trajectories. Before this result is exploited, in Section IV we consider an important condition in the applicability of MPC in continuous-time. In Section V, we reformulate the classical MPC problem into a data-driven problem defined on an infinite-dimensional space of coefficient sequences. An implementable version of the framework is analysed in Section VI. We investigate key parameters and characteristics in three simulations when our framework is applied to a simple numerical example and a problem from a model-driven approach to MPC in continuous-time to draw comparison. Section VII includes detailed discussions of the results of the investigations; we conclude the paper and discuss future work in Section VIII.

Notation

We denote by $\mathbb{N}$ and $\mathbb{R}$ respectively the set of natural and real numbers. $\mathbb{R}^n$ denotes the space of n-dimensional vectors with real entries. $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ matrices with real entries; $\mathbb{R}^{n\times\infty}$ the set of real matrices with n rows and an infinite number of columns, and $\mathbb{R}^{\infty\times\infty}$ the set of real matrices with an infinite number of rows and columns. The transpose of a matrix M is denoted by $M^\top$. The i^{th} row of a matrix M is denoted by $M_{i,:}$, and its i^{th} column is denoted by $M_{:,i}$. If A and B are two matrices with the same number of columns, we define $\mathrm{col}(A,B) := \begin{bmatrix} A^\top & B^\top \end{bmatrix}^\top$. For a continuous-time trajectory $u(\cdot)$, we denote its i^{th} derivative with respect to time t by $u^{(i)}$ and its restriction over $\mathbb{I} := [t_0, t_1]$ as $u_{|[t_0,t_1]}$ or $u_{|\mathbb{I}}$. For a linear operator L, its range is denoted by $\mathcal{R}[L]$. We denote by $\mathcal{L}_2(\mathbb{I},\mathbb{R})$ the space of square-integrable real-valued functions defined on a finite interval

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$\mathbb{I} := [t_0, t_1] \subset \mathbb{R}$ and by $\ell_2(\mathbb{N}, \mathbb{R})$ the space of real square-summable sequences. We denote by $u(t^-)$ and $u(t^+)$ the left, respectively the right limit of the function.

II. Background

A. Model Predictive Control

We formalise one version of the continuous-time MPC design problem as follows. Consider the linear time-invariant continuous-time system

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t),\ x(0) = x_0\ , \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ denote the state, input and output at time t; A, B, C, D are system matrices with appropriate dimensions. Let $T_s \in \mathbb{R}_{>0}$ denote some fixed *sampling time* and $T_p \in \mathbb{R}_{\geq T_s}$ some fixed *prediction horizon*. We denote by $\gamma \in \mathbb{N}$ the *time-step index* and define the time interval of a prediction horizon by $\mathbb{I}_{\gamma,p} := [\gamma T_s, \gamma T_s + T_p]$ and of a sample by $\mathbb{I}_{\gamma,s} := [\gamma T_s, (\gamma+1)T_s] \subset \mathbb{I}_{\gamma,p}$. The length of a time-step $\gamma T_s - (\gamma-1)T_s = T_s$. The objective of one version of MPC is to compute a control input $u_{|\mathbb{I}_{\gamma,p}}$ such that the system output $y_{|\mathbb{I}_{\gamma,p}}$ tracks a given reference $r_{|\mathbb{I}_{\gamma,p}}$. At each time-step, the input $u_\gamma := u_{|\mathbb{I}_{\gamma,s}}$ is applied to the system resulting in the state $x_\gamma := x_{|\mathbb{I}_{\gamma,s}}$ and output $y_\gamma := y_{|\mathbb{I}_{\gamma,s}}$. The initial condition of each time-step is given by the value of the state at the end of the previous time-step such that the state is continuous. This objective holds for every time-step until a final time-step γ_F; we define the overall operating time interval of the system by $\mathbb{I}_F := [0, (\gamma_F + 1)T_s]$.

A squared 2-norm is defined by the following, with $Q \succ 0$ and $R \succ 0$ having compatible dimensions, in the output and input spaces $y \in \mathcal{L}_2(\mathbb{I}_{\gamma,p}, \mathbb{R}^p)$ and $u \in \mathcal{L}_2(\mathbb{I}_{\gamma,p}, \mathbb{R}^m)$ respectively

$$\|y\|_Q^2 := \int_{\mathbb{I}_{\gamma,p}} y^\top(t)Qy(t)dt\ ;\ \|u\|_R^2 := \int_{\mathbb{I}_{\gamma,p}} u^\top(t)Ru(t)dt\ .$$

The control input for the current time-step is the following

$$\begin{aligned} u_\gamma = \text{argmin}_{u_\gamma} &\{\|r_{|\mathbb{I}_{\gamma,p}} - y_{|\mathbb{I}_{\gamma,p}}\|_Q^2 + \|u^{(1)}_{|\mathbb{I}_{\gamma,p}}\|_R^2\} \\ \text{s.t.}\ \ &\frac{d}{dt}x(t) = Ax(t) + Bu(t) \\ &y(t) = Cx(t) + Du(t), \\ &x_\gamma(\gamma T_s) = x_{\gamma-1}(\gamma T_s)\ . \end{aligned} \tag{2}$$

The full control input u over the whole interval $\mathbb{I}_F = [0, (\gamma_F+1)T_s]$ is the concatenation of all trajectories u_γ; similarly for the full system output y.

Clearly, this version of MPC in continuous-time relies on the mathematical model for the system (1). We wish to develop a framework that uses only system *data* and requires no mathematical model of the system.

B. Chebyshev polynomial orthogonal bases

The *Chebyshev polynomials* are defined by $C_i(t) := \cos(i \arccos(t))$, $i \in \mathbb{N}$, and thus exist within the interval $\mathbb{I} := [-1, 1]$, see e.g. Chapter 3 p. 14 in [13]. By linear translation and scaling, the interval $[-1, 1]$ can be transformed into any MPC prediction horizon $\mathbb{I}_{\gamma,p} = [\gamma T_s, \gamma T_s + T_p]$. We define the following variables

$$\alpha_\gamma := \frac{T_p}{2} + \gamma T_s\ ; \quad \beta := \frac{2}{T_p}\ , \tag{3}$$

such that the Chebyshev polynomials on $\mathbb{I}_{\gamma,p}$ are defined by

$$C_{\gamma,n}(t) := \cos(n \arccos(\beta(t - \alpha_\gamma))), n \in \mathbb{N}\ . \tag{4}$$

Thus, $C_{\gamma,0}(t) = 1$, $C_{\gamma,1}(t) = \beta(t - \alpha_\gamma)$, and $C_{\gamma,n+1}(t) = 2\beta(t-\alpha_\gamma)C_{\gamma,n}(t) - C_{\gamma,n-1}(t)$, $n \geq 1$.

We define the *weight function* $w(\cdot)$ at time t by

$$w_\gamma(t) := \frac{1}{\sqrt{1 - (\beta(t - \alpha_\gamma))^2}}\ ,\ t \in \mathbb{I}_{\gamma,p}\ .$$

The *Chebyshev polynomials* on $\mathbb{I}_{\gamma,p}$ are orthogonal to each other with respect to the inner product defined by $\langle f, g\rangle_{w_\gamma} := \int_{\mathbb{I}_{\gamma,p}} f(t)g(t)w_\gamma(t)dt$, and they form a complete basis for $\mathcal{L}_2(\mathbb{I}_{\gamma,p}, \mathbb{R})$, i.e. their linear span is dense in $\mathcal{L}_2(\mathbb{I}_{\gamma,p}, \mathbb{R})$. Given this density, every $f \in \mathcal{L}_2(\mathbb{I}_{\gamma,p}, \mathbb{R})$ can be written as a series $\left\{\sum_{k=0}^\infty \widetilde{f}_k C_{\gamma,k}\right\}_{k\in\mathbb{N}}$ where $\widetilde{f}_k := \langle f, C_{\gamma,k}\rangle_{w_\gamma}$. The coefficients $\widetilde{f}_k$ can be computed effectively and accurately using a sampling and interpolation procedure rather than evaluating the aforementioned inner product, see Chapter 2 p. 7 in [13]. The convergence of this series to f depends on the smoothness of f, see Remark 3 p. 3 in [11]. The sequence of coefficients $\{\widetilde{f}_k\}_{k\in\mathbb{N}}$ is square-summable, see Theorem 23 p. 23 in [14]. We define the infinite vector of Chebyshev polynomials $\mathfrak{C}_\gamma := \begin{bmatrix} C_{\gamma,0} & C_{\gamma,1} & \ldots \end{bmatrix}^\top$ and the infinite vector of coefficients $\widetilde{f} := \begin{bmatrix} \widetilde{f}_0 & \widetilde{f}_1 & \ldots \end{bmatrix}$. Hence, we write

$$f = \sum_{k=0}^\infty \widetilde{f}_k C_{\gamma,k} = \widetilde{f}\mathfrak{C}_\gamma\ . \tag{5}$$

We call the right-hand side of (5) the *polynomial transform* of f (see Section 2.2.2 p. 69 of [15]).

The notation of Section II.B p. 4628 in [12] is utilised for *vector* functions. We call $\Pi_N(f)$ the N^{th} degree *truncation* of the *projection* of f on the space of Chebyshev coefficient sequences, and $f - \Pi_N(f)$ the *approximation error*. It can be shown that the approximation error decays with N.

If $f \in \mathcal{L}_2(\mathbb{I}, \mathbb{R})$ is differentiable and $f^{(1)} \in \mathcal{L}_2(\mathbb{I}, \mathbb{R})$, then the *differentiation in the transform space* equality holds:

$$f^{(1)} = \sum_{k=0}^\infty \widetilde{f}_k^{(1)} C_{\gamma,k} \tag{6}$$

with $\begin{bmatrix} \widetilde{f}_0^{(1)} & \ldots & \widetilde{f}_k^{(1)} & \ldots \end{bmatrix} = \begin{bmatrix} \widetilde{f}_0 & \ldots & \widetilde{f}_k & \ldots \end{bmatrix} \mathcal{D}_\mathbb{I}$, where $\mathcal{D}_\mathbb{I} \in \mathbb{R}^{\infty\times\infty}$ is a fixed *differentiation matrix* for vector functions on $\mathbb{I} = [-1, 1]$ whose entries are computed according to standard formulas, see Example 4 p. 4 in [11]. For vector functions that are not on $\mathbb{I}$ but rather

$\mathbb{I}_{\gamma,p} = [\gamma T_s, \gamma T_s + T_p]$, we denote by $\mathcal{D}_\beta$ the matrix $\mathcal{D}_\mathbb{I}$ scaled by β (see (3)), i.e. $\mathcal{D}_\beta := \beta\mathcal{D}_\mathbb{I}$. It can be shown that $\mathcal{D}_\beta$ is a differentiation matrix for vector functions on $\mathbb{I}_{\gamma,p}$. We therefore state that for any $f \in \mathcal{L}_2(\mathbb{I}_{\gamma,p}, \mathbb{R})$,

$$f^{(1)} = \widetilde{f}^{(1)}\mathfrak{C}_\gamma = \widetilde{f}\beta\mathcal{D}_\mathbb{I}\mathfrak{C}_\gamma = \widetilde{f}\mathcal{D}_\beta\mathfrak{C}_\gamma \ . \tag{7}$$

The derivative of any $\mathcal{L}_2(\mathbb{I}_{\gamma,p}, \mathbb{R})$ function can be computed with CPOB and linear operations with the matrix $\mathcal{D}_\beta$.

The truncation of the derivative of the Chebyshev series for f can be computed by partitioning the matrix $\mathcal{D}_\beta$, see Section II.D p. 4 in [11].

III. All system trajectories from one

In this section, we state the fundamental result that enables the CPOB representation of *all* system trajectories from one 'sufficiently informative' input-output trajectory produced by a continuous-time LTI system.

The following is Definition 1 p. 589 in [9] for functions on $\mathbb{I}_{\gamma,p} = [\gamma T_s, \gamma T_s + T_p]$.

Definition 1: Let $\mathbb{I}_{\gamma,p} = [\gamma T_s, \gamma T_s + T_p]$. $f : \mathbb{I}_{\gamma,p} \to \mathbb{R}^m$ is *persistently exciting of order* k on $\mathbb{I}_{\gamma,p}$ if:

a) f is $(k-1)$-times continuously differentiable in $\mathbb{I}_{\gamma,p}$;
b) For every $v := \begin{bmatrix} v_0 & \dots & v_{k-1} \end{bmatrix} \in \mathbb{R}^{1\times km}$ it holds that

$$v \begin{bmatrix} f(t) \\ f^{(1)}(t) \\ \vdots \\ f^{(k-1)}(t) \end{bmatrix} = 0 \ \forall \ t \in \mathbb{I}_{\gamma,p} \Longrightarrow v_0, \dots, v_{k-1} = 0 \ . \tag{8}$$

We define the *system lag* ℓ of the input-state-output representation (1) as in Section 3 p. 3 in [10]. Associate to (1) its *input-output* behaviour on $\mathbb{I}_{\gamma,p}$, defined by

$$\begin{aligned} &\mathfrak{B}_{\mathbb{I}_{\gamma,p}} := \{\mathrm{col}(u, y) \in \mathcal{C}^\infty(\mathbb{I}_{\gamma,p}, \mathbb{R}^{m+p}) \mid \\ &\exists \ x \in \mathcal{C}^\infty(\mathbb{I}_{\gamma,p}, \mathbb{R}^n) \ \text{ s.t. } \mathrm{col}(u, x, y) \text{ satisfies (1)}\} \ . \end{aligned} \tag{9}$$

The linearity of the system (1) ensures that all possible trajectories of $\mathfrak{B}_{\mathbb{I}_{\gamma,p}}$ belong to $\mathcal{L}_2(\mathbb{I}_{\gamma,p}, \mathbb{R}^{m+p})$ and thus can be represented using CPOB. The projection of $\mathfrak{B}_{\mathbb{I}_{\gamma,p}}$ on the space of Chebyshev coefficient sequences is defined by

$$\begin{aligned} &\Pi(\mathfrak{B}_{\mathbb{I}_{\gamma,p}}) := \{\mathrm{col}(\widetilde{u}, \widetilde{y}) \in \ell_2(\mathbb{N}, \mathbb{R}^{m+p}) \mid \\ &\ \exists \ \mathrm{col}(u, y) \in \mathfrak{B}_{\mathbb{I}_{\gamma,p}} \ \text{ s.t. } \mathrm{col}(\widetilde{u}, \widetilde{y}) = \Pi(\mathrm{col}(u, y))\} \ . \end{aligned} \tag{10}$$

Given the CPOB representation $\widetilde{f}$ of $f \in \mathcal{L}_2(\mathbb{I}_{\gamma,p}, \mathbb{R}^r)$, we denote by $\mathcal{W}_L(\widetilde{f})$ the $Lr \times \infty$ matrix

$$\mathcal{W}_L(\widetilde{f}) := \mathrm{col}\left(\widetilde{f}\mathcal{D}_\beta^j\right)_{j=0,\dots,L-1} \ . \tag{11}$$

The following is Theorem 4 p. 6 of [10].

Theorem 1: Define $\mathfrak{B}_{\mathbb{I}_{\gamma,p}}$ by (9), and let $\mathrm{col}\,(u, y) \in \mathfrak{B}_{\mathbb{I}_{\gamma,p}}$. Assume that (A, B) is controllable, and that u is persistently exciting of order $L > \ell + n$. Define $\mathcal{W}_L(\widetilde{u})$, $\mathcal{W}_L(\widetilde{y})$ by (11). Then $\dim \mathcal{R}\,[\mathrm{col}(\mathcal{W}_L(\widetilde{u}), \mathcal{W}_L(\widetilde{y}))] = Lm + n =: d$.

Let $V_u \in \mathbb{R}^{Lm\times d}$ and $V_y \in \mathbb{R}^{Lp\times d}$ be such that $\mathrm{col}(V_u, V_y)$ is a basis matrix for $\mathcal{R}\,[\mathrm{col}(\mathcal{W}_L(\widetilde{u}), \mathcal{W}_L(\widetilde{y}))]$; note that V_u, V_y depend on $\mathbb{I}_{\gamma,p}$. Define $\Pi(\mathfrak{B}_{\mathbb{I}_{\gamma,p}})$ by (10). The following are equivalent:

1) $\mathrm{col}\,(\widetilde{u}', \widetilde{y}') \in \Pi(\mathfrak{B}_{\mathbb{I}_{\gamma,p}})$;
2) There exists $G \in \mathbb{R}^{d\times\infty}$ such that

$$\begin{bmatrix} \mathcal{W}(\widetilde{u}') \\ \mathcal{W}(\widetilde{y}') \end{bmatrix} = \begin{bmatrix} V_u \\ V_y \end{bmatrix} G \ , \tag{12}$$

3) There exists $G \in \mathbb{R}^{d\times\infty}$ such that

$$\begin{aligned} (V_u G)_{1,:} &= \widetilde{u}' \\ (V_y G)_{1,:} &= \widetilde{y}' \\ (V_u G)_{1,:}\mathcal{D}_\beta^i - (V_u G)_{i+1,:} &= 0 \\ (V_y G)_{1,:}\mathcal{D}_\beta^i - (V_y G)_{i+1,:} &= 0 \ , \end{aligned} \tag{13}$$

$i = 1, \dots, L-1$.

The Chebyshev representation of *all* possible system trajectories can be derived directly from the Chebyshev representation of a single 'sufficiently informative' trajectory, as shown by Theorem 1.

IV. Applicability of MPC in Continuous-Time

Bumpless transfer in control improves tracking performance by preventing impulsive changes in the control input signal that cannot be accurately applied to the input of a system (see e.g. [16]). The following result ensures a continuous state, i.e. a bumpless transfer, between time-steps for systems operating in continuous-time.

We first define the vector of system trajectory derivatives

$$w_e := \begin{bmatrix} u^\top & y^\top & \cdots & (u^{(L-1)})^\top & (y^{(L-1)})^\top \end{bmatrix}^\top \ . \tag{14}$$

Proposition 1: If every u_γ is persistently exciting of order L in $[\gamma T_s, (\gamma+1)T_s]$ and if $w_e(\gamma T_s^-) = w_e(\gamma T_s^+)$ then u is persistently exciting of order L over the whole interval $\mathbb{I}_F = [0, (\gamma_F+1)T_s]$ and the underlying state of the system is continuous at every time γT_s and over $\mathbb{I}_F = [0, (\gamma_F+1)T_s]$.

Proof: It can be shown (see [17]) that every state variable x of an LTI system is computed as $x = X(\frac{d}{dt})\binom{u}{y}$ where $X(\frac{d}{dt}) = X_0 + X_1\frac{d}{dt} + ... + X_{L-1}\frac{d^{L-1}}{dt^{L-1}}$ is a *state map* and $X_i \in \mathbb{R}^{n\times(m+p)}, i = 0, ..., L-1$.

If $w_e(\gamma T_s^-) = w_e(\gamma T_s^+)$, then

$$\begin{aligned} \lim_{t\to\gamma T_s^-} x(t) &= \lim_{t\to\gamma T_s^-} \begin{bmatrix} X_0 & X_1 & \dots & X_{L-1} \end{bmatrix} w_e(t) \\ &= \begin{bmatrix} X_0 & X_1 & \dots & X_{L-1} \end{bmatrix} \lim_{t\to\gamma T_s^-} w_e(t) \\ &= \begin{bmatrix} X_0 & X_1 & \dots & X_{L-1} \end{bmatrix} w_e(\gamma T_s^-) \\ &= \begin{bmatrix} X_0 & X_1 & \dots & X_{L-1} \end{bmatrix} w_e(\gamma T_s^+) = \lim_{t\to\gamma T_s^+} x(t) \ , \end{aligned}$$

and x is continuous at γT_s.

To prove persistency of excitation over the whole interval $\mathbb{I}_F$, observe that u_γ is persistently exciting on $\mathbb{I}_{\gamma,s} = [\gamma T_s, (\gamma + 1)T_s] \subset \mathbb{I}_F$ so, from Definition 1, the only thing to prove is that u is $(L-1)$-times differentiable at γT_s. This follows from $w_e(\gamma T_s^-) = w_e(\gamma T_s^+)$. ■

A control input trajectory that is the concatenation of persistently exciting control input trajectories is therefore itself persistently exciting and moreover gives rise to a continuous state given that the conditions of Proposition 1 hold true.

V. A CPOB BASED MPC DESIGN FOR CONTINUOUS-TIME SYSTEMS

In this section, we reformulate the MPC design problem (2) using only 'sufficiently informative' input-output data. We first give a data-based characterisation of all solutions of (2) and then define a CPOB formulation of the result of Proposition 1 such that the state of the system is guaranteed to be continuous across $\mathbb{I}_F = [0, (\gamma_F + 1)T_s]$.

Proposition 2: Under the assumptions of Theorem 1, u_γ solves the MPC design problem (2) on time-step γ, with some reference $r_{|\mathbb{I}_{\gamma,p}}$, if and only if there exists $G_\gamma \in \mathbb{R}^{d\times\infty}$ that solves the following optimisation problem

$$\begin{aligned}
&\min_{G_\gamma} \{\|r_{|\mathbb{I}_{\gamma,p}} - (V_y G_\gamma)_{1,:}\mathfrak{C}_\gamma\|_Q^2 + \|(V_u G_\gamma)_{1,:}\mathcal{D}_\beta \mathfrak{C}_\gamma\|_R^2\} \\
&\text{s.t. } (V_u G_\gamma)_{1,:}\mathcal{D}_\beta^i - (V_u G_\gamma)_{i+1,:} = 0 \\
&\quad (V_y G_\gamma)_{1,:}\mathcal{D}_\beta^i - (V_y G_\gamma)_{i+1,:} = 0 \\
&\quad i = 1, \ldots, L-1, \\
&\quad (V_u G_\gamma)_{1,:}\mathcal{D}_\beta^i \mathfrak{C}_\gamma(\gamma T_s) = u_{\gamma-1}^{(i)}(\gamma T_s) \\
&\quad (V_y G_\gamma)_{1,:}\mathcal{D}_\beta^i \mathfrak{C}_\gamma(\gamma T_s) = y_{\gamma-1}^{(i)}(\gamma T_s) \\
&\quad i = 0, \ldots, L-1.
\end{aligned} \tag{15}$$

Moreover, the optimal input u_γ on time-step γ is

$$u_\gamma = ((V_u G_\gamma)_{1,:}\mathfrak{C}_\gamma)_{|[\gamma T_s, (\gamma+1)T_s]} \; .$$

Proof: The first two constraints appearing in (2) are equivalent to requiring that $\mathrm{col}(u_\gamma, y_\gamma)$ is an input-output trajectory of (1). Using Theorem 1 and the assumption of persistent excitation of u, $\mathrm{col}(u_\gamma, y_\gamma)$ is a system trajectory if and only if it satisfies (13), i.e. the first two constraints in (15). The last constraint of (2) corresponds to requiring that the underlying state of the system x is continuous, equivalently the last two constraints in (15), the CPOB representation of Proposition 1, must hold true. ■

Remark 1: The proposed design framework involves the use of (high-order) derivatives of the input and output signals. However, there is no need to measure them (which can be difficult in practice) as they can be directly computed using CPOB representations and (7).

Remark 2: In Proposition 2, we proposed a data-driven MPC framework in continuous-time (15) that involves working on G_γ in an infinite dimensional space (i.e. an infinite number of Chebyshev coefficients). In practice, for computational purposes, one would truncate the framework to use a finite number N of Chebyshev coefficients. The following is a truncated, implementable CPOB-based MPC approximate design procedure that uses only a finite number $N \in \mathbb{N}$ of Chebyshev coefficients. Denote by $\mathfrak{C}_{\gamma,N}$ the truncation of the basis vector $\mathfrak{C}_\gamma$ to the first $N+1$ entries, by $\mathcal{D}_{\beta,N}$ the $(N+1)\times(N+1)$ principal submatrix of $\mathcal{D}_\beta$, and by r_C the truncated CPOB representation of the reference $r_{|\mathbb{I}_{\gamma,p}}$. We reformulate the optimisation problem (15) for truncated signals as follows. We solve for $G_{\gamma,N} \in \mathbb{R}^{d\times(N+1)}$ the following finite-dimensional version of (15)

$$\begin{aligned}
&\min_{G_{\gamma,N}} \{\|r_C - (V_y G_{\gamma,N})_{1,:}\mathfrak{C}_{\gamma,N}\|_Q^2 \\
&\qquad + \|(V_u G_{\gamma,N})_{1,:}\mathcal{D}_{\beta,N}\mathfrak{C}_{\gamma,N}\|_R^2\} \\
&s.t.\ (V_u G_{\gamma,N})_{1,:}\mathcal{D}_{\beta,N}^i - (V_u G_{\gamma,N})_{i+1,:} = 0 \\
&\quad (V_y G_{\gamma,N})_{1,:}\mathcal{D}_{\beta,N}^i - (V_y G_{\gamma,N})_{i+1,:} = 0 \\
&\quad i = 1, \ldots, L-1, \\
&\quad (V_u G_{\gamma,N})_{1,:}\mathcal{D}_{\beta,N}^i \mathfrak{C}_{\gamma,N}(\gamma T_s) = u_{\gamma-1}^{(i)}(\gamma T_s) \\
&\quad (V_y G_{\gamma,N})_{1,:}\mathcal{D}_{\beta,N}^i \mathfrak{C}_{\gamma,N}(\gamma T_s) = y_{\gamma-1}^{(i)}(\gamma T_s) \\
&\quad i = 0, \ldots, L-1.
\end{aligned} \tag{16}$$

We then compute the input u_γ for the current time-step as

$$u_\gamma = ((V_u G_{\gamma,N})_{1,:}\mathfrak{C}_{\gamma,N})_{|[\gamma T_s, (\gamma+1)T_s]} \; .$$

VI. SIMULATIONS

A. A first-order example

Consider the dynamical system (adapted from [10])

$$\frac{d}{dt}x = -x + u \; , \; y = x \; . \tag{17}$$

The system is operating over $\mathbb{I}_F = [0, 8]$ with initial condition $x(0) = 0$; the output is required to track a constant reference trajectory with magnitude one. The system has a lag of $\ell = 1$ and order $n = 1$. It can be shown that the input signal whose value at t is $u(t) = -3e^{-4t} - 2e^{-3t} - e^{-2t} \in \mathcal{L}_2(\mathbb{I}_{\gamma,p}, \mathbb{R})$ for any finite γ and T_p and is persistently exciting of order $L = 3 > \ell + n$, and gives rise to the output $y(t) = e^{-4t} + e^{-3t} + e^{-2t} + e^{-t}$. Any input signal that belongs to $\mathcal{L}_2(\mathbb{I}_{\gamma,p}, \mathbb{R}^m)$ and is persistently exciting of order at least 3 could have been selected. The derivatives of these signals can be computed effectively using the CPOB representations of the signals and (7). It can be shown that the following is a basis of $\mathcal{R}[\mathrm{col}(\mathcal{W}_L(\widetilde{u}), \mathcal{W}_L(\widetilde{y}))]$ (as defined in Theorem 1) and can be used to characterise all system trajectories, see (12),

$$\begin{bmatrix} V_u \\ V_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^\top .$$

1) The effects of the prediction horizon: We first investigated the effects of the prediction horizon with the values of the parameters in (16) as $T_p = 1,\ 2,\ 4,\ 8s$; $T_s = 0.1s$; $R = 0.01$; $Q = 1$ and $N = 20$. The output and input trajectories are depicted in Figure 1.

2) The effects of the truncation index: The truncation index of Chebyshev representations was studied with the values $N = 8,\ 10,\ 15,\ 20,\ 40$; $T_p = 4s$; $T_s = 0.1s$; $R = 0.01$ and $Q = 1$. Figure 2 depicts the output and input trajectories.

The tracking error is defined by $e := r - y$. The finite-horizon cost of each input-output trajectory pair u, y over $\mathbb{I}_F = [0, (\gamma_F + 1)T_s]$ is defined by $J := \|e\|_Q^2 + \|u^{(1)}\|_R^2$ in which a squared 2-norm is defined by the following, with $Q \succ 0$ and $R \succ 0$ having compatible dimensions, in the

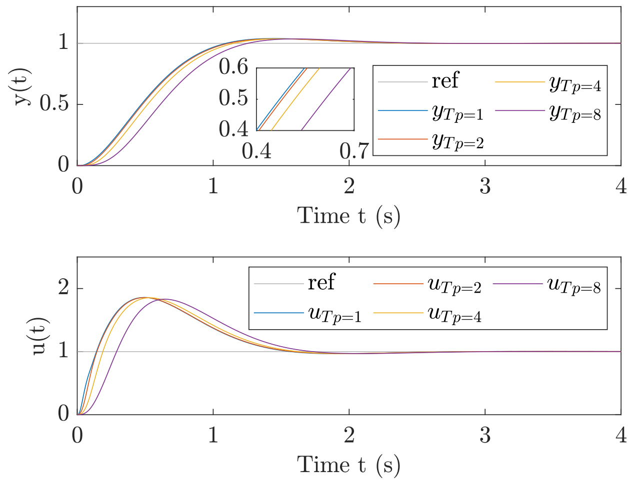

Fig. 1. Output and input signals y, u with different values of T_p.

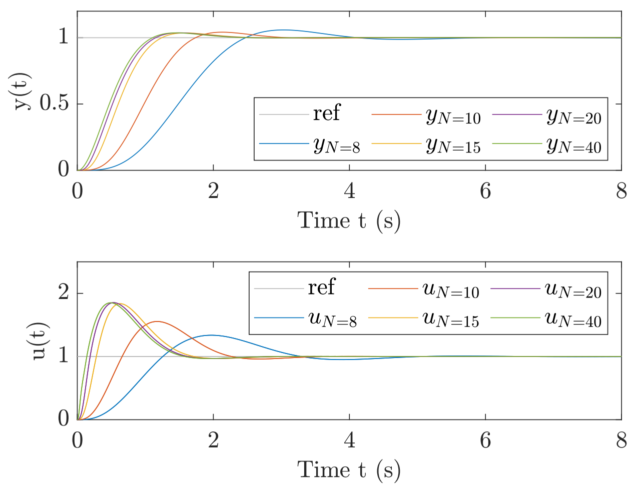

Fig. 2. Output and input signals y, u with different values of N.

output and input spaces $y \in \mathcal{L}_2(\mathbb{I}_F, \mathbb{R}^p)$ and $u \in \mathcal{L}_2(\mathbb{I}_F, \mathbb{R}^m)$ respectively

$$\|y\|_Q^2 := \int_{\mathbb{I}_F} y^\top(t) Q y(t) dt \; ; \; \|u\|_R^2 := \int_{\mathbb{I}_F} u^\top(t) R u(t) dt \, .$$

Figure 3 shows the finite-horizon costs of the simulation with the case of varying N as the sum of the tracking error and control input trajectory derivative costs. The colours of the bars correspond to the colours of the trajectories in Figure 2.

B. A non-minimum phase system

The following transfer function is from Example 1 p. 1596 in [6]

$$\frac{(0.1s+1)(-s+1)}{(s+1)^2(s+2)} \, . \tag{18}$$

The system is operating over $\mathbb{I}_F = [0, 15]$ with initial condition $x(0) = 0$ and the output trajectory is required to track a constant reference trajectory with magnitude one. The system has a lag of $\ell = 3$ and order $n = 3$. It can be shown that the input signal whose value at t is

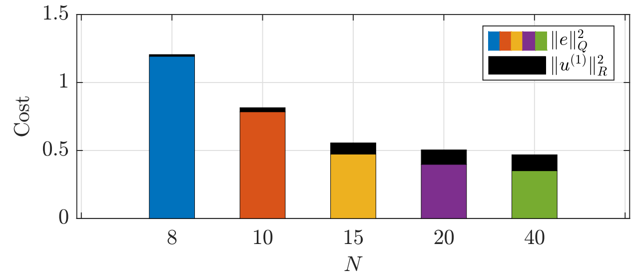

Fig. 3. Tracking error cost (colour); control input derivative cost (black), and finite-horizon cost (overall height of bar) of the signals y, u with different values of N in Figure 2.

$u(t) = -7e^{-8t} - 6e^{-7t} - 5e^{-6t} - 4e^{-5t} - 3e^{-4t} - 2e^{-3t} - e^{-2t} \in \mathcal{L}_2(\mathbb{I}_{\gamma,p}, \mathbb{R})$ for any finite γ and T_p and is persistently exciting of order $L = 7 > \ell + n$. It can be shown that a basis of $\mathcal{R}\left[\mathrm{col}(\mathcal{W}_L(\tilde{u}), \mathcal{W}_L(\tilde{y}))\right]$ (as defined in Theorem 1) is given by

$$\begin{bmatrix} V_u \\ \hline V_y \end{bmatrix} = \left[\begin{array}{cccccc} 2 & 5 & 4 & 1 & 0 & \dots \\ 0 & 2 & 5 & 4 & 1 & \dots \\ 0 & 0 & 2 & 5 & 4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \hline 1 & -0.9 & -0.1 & 0 & 0 & \dots \\ 0 & 1 & -0.9 & -0.1 & 0 & \dots \\ 0 & 0 & 1 & -0.9 & -0.1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right]$$

where $V_u \in \mathbb{R}^{7\times 10}$ and $V_y \in \mathbb{R}^{7\times 10}$.

The optimisation problem (16) was simulated with $N = 18, 20$; $R = 0.00001$; $Q = 1$; $T_p = 10s$ and $T_s = 0.1s$. The output and input trajectories are depicted in Figure 4; the signals that are labelled '*cmpc*' are from Example 1 p. 1596 in [6] with different numbers of orthogonal polynomials, $N = 1, 2$, and *pole location* of the Laguerre model for the future control signal $p = 0.2$.

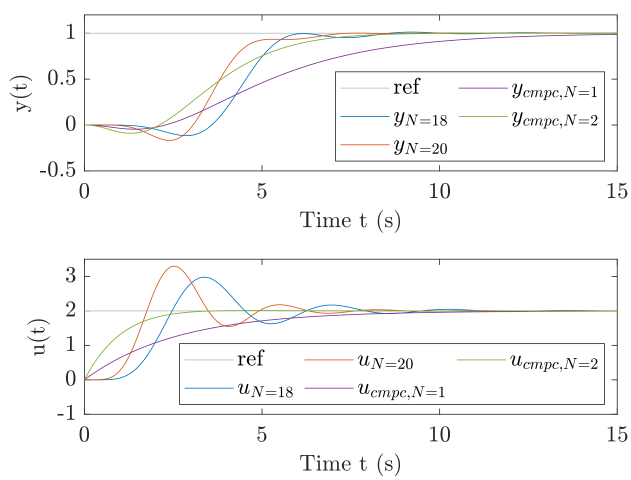

Fig. 4. Output and input signals y, u for the non-minimum phase case with different values of N in our framework and the continuous-time MPC framework presented in [6].

VII. Discussion

We can draw from the examples in the previous section some indications about the validity of our approach and some clues about pressing issues for further research.

Comment 1 (Prediction horizon). The parameter T_p in our framework plays a similar role to the prediction horizons T_2 and N_2 in the classical model-driven MPC schemes in continuous-time, see [3], and discrete-time, see [18], respectively. It is stated in Section 4.2 p. 65 in [3] and shown in Figure 5 p. 67 therein that larger values of T_2 increase the rise time of the system output trajectory. Accordantly, in the discrete-time case, it is shown in Figure 5 p. 156 in [18] that the rise time of the system output increases with larger values of N_2. We therefore expect that with increasing values of T_p, we observe larger rise times. This trend is depicted in Figure 1 with our framework.

Comment 2 (Chebyshev truncation index). The truncation index N in our framework plays a similar role to the *control order* N_u in [3]. N_u is used to tune the accuracy of the computed predicted input to the actual optimal predicted input (see Section 2.5 p. 60 and Section 4.4 p. 66 in [3]) and makes the algorithm computationally tractable. Thus, the computational complexity of the algorithm is tuned implicitly. It is shown in Figure 6 p. 68 in [3] that the rise time decreases with larger values of N_u. We depict the same trend in Figure 2 with N in our framework. Furthermore, Figure 3 shows that the tracking error and finite-horizon costs decrease and the control input derivative cost increases with larger values of N.

Comment 3 (Comparison to model-driven MPC in continuous-time using orthonormal functions). From Example 1 p. 1596 in [6], we expect that the output trajectory settles to the reference value within 15s and that an increase in the number of orthonormal functions leads to a shorter rise time in the control input. Figure 4 depicts that with our framework, the output trajectory successfully tracks and is within 2% of the reference in approximately $8.2s$ and $6.9s$ with $N = 18$ and $N = 20$ respectively. We also observe that the rise time of the control input with $N = 20$ is shorter than that with $N = 18$. The algorithm presented in [6] requires an augmented state space model of the system involving the derivative of the control input in order to implement integral action (see Section 3.1 p. 1590 in [6]). The dimensions of the matrices in the mathematical model are therefore increased and the computation of the control input at each time step requires an extra process (see Section 4.2 p. 1592 in [6]); thus the computational complexity of the algorithm increases. Not only does our framework remove the need for any mathematical model of the system but the derivatives of the system trajectories can be computed effectively and accurately using linear operations with the differentiation matrix introduced in Section II-B.

VIII. Conclusions

We presented some preliminary ideas on the development of a data-driven MPC algorithm for continuous-time systems using Chebyshev polynomial bases. Our framework provides a purely data-driven solution to the MPC problem in continuous-time. An implementable and computationally tractable version of the framework was simulated in order to investigate the influence of the parameters involved in our framework and draw comparison to model-driven MPC algorithms in both discrete-time and continuous-time.

Future work includes the formulation of a sound framework and rigorous guidelines for the design of data-driven MPC control schemes in continuous-time using orthogonal bases. In this paper, the fixed values of the design parameters (e.g. the truncation index N) are chosen in advance; the use of time-varying parameters is an area for future research. Adaptations to our framework for other classifications of systems and applications beyond LTI tracking problems should be considered - such as linear *time-varying* systems and *repetitive control*. A thorough analysis of the computational complexity and theoretical guarantees (such as stability) of our framework shall be conducted.

References

[1] D. Q. Mayne, "Model predictive control: Recent developments and future promise," *Automatica*, vol. 50, no. 12, pp. 2967–2986, 2014.

[2] P. Gawthrop, "Continuous-time self-tuning control - a unified approach," *IFAC Proceedings Volumes*, vol. 20, no. 2, pp. 19–24, 1987.

[3] H. Demircioğlu and P. Gawthrop, "Continuous-time generalized predictive control (CGPC)," *Automatica*, vol. 27, no. 1, pp. 55–74, 1991.

[4] P. J. Gawthrop, H. Demircioğlu, and I. I. Siller-Alcalá, "Multivariable continuous-time generalised predictive control: A state-space approach to linear and nonlinear systems," *IEE Proceedings: Control Theory and Applications*, vol. 145, no. 3, pp. 241–250, 1998.

[5] B. Kouvaritakis, "Recent developments in generalized predictive control for continuous-time systems," *International Journal of Control*, vol. 72, no. 2, pp. 164–173, 1999.

[6] L. Wang, "Continuous time model predictive control design using orthonormal functions," *International Journal of Control*, vol. 74, no. 16, pp. 1588–1600, 2001.

[7] J. Coulson, J. Lygeros, and F. Dörfler, "Data-enabled predictive control: In the shallows of the DeePC," in *2019 18th European Control Conference (ECC)*, 2019, pp. 307–312.

[8] Z. Jiang and B. Chu, "Norm optimal iterative learning control: A data-driven approach," *IFAC-PapersOnLine*, vol. 55, no. 12, pp. 482–487, 2022.

[9] P. Rapisarda, M. K. Camlibel, and H. J. van Waarde, "A persistency of excitation condition for continuous-time systems," *IEEE Control Systems Letters*, vol. 7, pp. 589–594, 2023.

[10] P. Rapisarda, H. J. van Waarde, and M. K. Camlibel, "A "fundamental lemma" for continuous-time systems, with applications to data-driven simulation," *Systems & Control Letters*, vol. 179, pp. 1–9, 2023.

[11] P. Rapisarda, H. van Waarde, and M. Camlibel, "Orthogonal polynomial bases for data-driven analysis and control of continuous-time systems," *IEEE Transactions on Automatic Control*, pp. 1–12, 2023.

[12] B. Chu and P. Rapisarda, "Data-driven iterative learning control for continuous-time systems," in *2023 62nd IEEE Conference on Decision and Control (CDC)*, 2023, pp. 4626–4631.

[13] L. N. Trefethen, *Approximation Theory and Approximation Practice, Extended Edition*. SIAM, 2019.

[14] G. Sansone, *Orthogonal functions*. Interscience Publishers, 1959, vol. 9.

[15] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods: Fundamentals in Single Domains*. Springer-Verlag, 2006.

[16] Y. Peng, D. Vrancic, and R. Hanus, "Anti-windup, bumpless, and conditioned transfer techniques for PID controllers," *IEEE Control Systems Magazine*, vol. 16, no. 4, pp. 48–57, 1996.

[17] P. Rapisarda and J. C. Willems, "State maps for linear systems," *SIAM Journal on Control and Optimization*, vol. 35, no. 3, pp. 1053–1091, 1997.

[18] D. Clarke, C. Mohtadi, and P. Tuffs, "Generalized predictive control—part ii extensions and interpretations," *Automatica*, vol. 23, no. 2, pp. 149–160, 1987.