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Equivariant filter for feature-based homography estimation for general camera motion

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***Abstract*— Recent advances in nonlinear observer design for homography estimation have exploited the Lie group structure of $\mathbf{SL}(3)$. Existing work requires the group (homography) velocity input, while the available measurements are typically only the camera velocity. Consequently, prior contributions exploiting the $\mathbf{SL}(3)$ geometry either reconstruct the group velocity or restrict the camera motion to allow adaptive estimation of the group velocity online. This paper presents a novel symmetry-based approach to observer design for the more general problem of estimating both the homography and the structure parameters of a planar scene, allowing homography estimation for arbitrary trajectories using only camera velocity measurements and direct point-feature correspondences between images. A new Lie group is introduced for the homography and structure parameters, whose symmetry structure is exploited to establish the system and output equivariance properties. We show that the system kinematics admit an equivariant lift, and the proposed observer is then designed based on the recently developed Equivariant Filter framework. Simulation results demonstrate the performance and consistency of the proposed approach.**

I. Introduction

Homographies are invertible mathematical transformations that relate 3D point projections from multiple views of a planar structure [8]. This projective mapping encodes the camera's relative position and orientation between two views of the scene, the distance between the camera and the scene, and its normal vector into a single matrix. Homographies are widely used for various computer vision applications such as 3D reconstruction [5]. They have also found significant applications in the field of robotics [14], [17] and are particularly suitable in scenarios where the primary features of the surrounding environment are planar surfaces [16].

Computing homographies has been extensively studied over the past two decades. Traditional estimation methods in the computer vision literature rely on algebraic algorithms that compute the homography on a frame-by-frame basis. These algorithms solve algebraic constraints related to image feature correspondences [1], [8] resulting in high-quality homography estimates. In robotic systems, measurements provided by the camera are a continuous temporal sequence of images that depict a time-varying homography due to the robot's motion in the environment. Algebraic algorithms proved ineffective in such scenarios since they compute individual homographies for each image and cannot improve the homography estimate over time. To address this impediment, significant work has been devoted during the last fifteen years to designing nonlinear observers for homography estimation [11], [6] that exploit kinematic models and velocity information to provide improved homography estimates. Recent successful approaches have exploited the structure of the Special Linear group $\mathbf{SL}(3)$, isomorphic to homographies [2]. These observers have provided powerful stability guarantees and have proven to be highly effective when it comes to robustness and practical implementation [9]. A drawback of these algorithms is that they rely on the homography group velocity measurements to propagate state estimates over time. The group velocity is induced by the rigid-body motion of the camera but also depends on the unknown *structure parameters* of the scene and cannot be directly constructed from camera velocity measurements. Recent work has attempted to overcome this limitation by assuming that the group velocity is either unknown [15] or only partially known by fusing angular velocity information [6] and proposed an integral extension of the observer to obtain estimates for both the homography and the velocity under certain motion assumptions. In [6] and [9], the authors proposed an observer under the assumption that the camera motion is constant or slowly time-varying. A solution to estimate the group velocity for periodic and nearly periodic rigid-body motions was later introduced in [4]. These assumptions remain constraining, and the considered motions are not intuitive for a rigid-body robotic system. The authors presented another approach to address the problem in [3], where the observer was designed to estimate the camera pose and structure parameters, and the homography structure was modeled in the state measurement process.

In this paper, we consider the problem of estimating the homography along with the structure parameters using direct feature point correspondences between a pair of images and the camera velocity measurements. The proposed solution, unlike previous work [6], [9] makes no restrictive assumptions on the group velocity. The homography kinematics are instead explicitly modeled in terms of the structure parameters and the rigid-body velocities. We introduce an appropriate symmetry group that acts on the total state space consisting of the homography and the structure parameters. We present a set of Lie group symmetries, and we show that the system and the output are equivariant under these actions [12], [13]. We then introduce a novel equivariant lift of the system kinematics onto the symmetry group and derive the

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proposed equivariant filter on the lifted system [19], [18]. The performance of the proposed observer is demonstrated through simulation experiments.

II. Preliminaries

For a thorough introduction to smooth manifolds and Lie groups, the authors recommend [10] and [20].

A. Smooth Manifolds

Given a smooth manifold $\mathcal{M}$, $\mathrm{T}_\xi\mathcal{M}$ denotes the tangent space at $\xi \in \mathcal{M}$ and $\mathfrak{X}(\mathcal{M})$ denotes the set of all smooth vector fields on $\mathcal{M}$. For a differentiable function between smooth manifolds $h : \mathcal{M} \to \mathcal{N}$, we denote its differential at ξ_0 by $\mathrm{D}_\xi|_{\xi_0} h(\xi) : \mathrm{T}_{\xi_0}\mathcal{M} \to \mathrm{T}_{\xi_0}\mathcal{N}$.

Let $f : \mathcal{M} \to \mathcal{N}$ and $g : \mathcal{N} \to \mathcal{N}'$ be linear maps. The composition of f and g is denoted $f \cdot g$.

Denote the 2-sphere as $\mathcal{S}^2 := \{v \in \mathbb{R}^3 \mid |v| = 1\}$, the 2-sphere projection $\pi_{\mathcal{S}^2} : \mathbb{R}^3 \setminus \{0\} \to \mathcal{S}^2$ is defined as

$$\pi_{\mathcal{S}^2}(v) := \frac{v}{|v|}.$$

The projection onto the tangent space of the 2-sphere at the point $v \in \mathcal{S}^2$ is denoted $\Pi_v := (I_3 - vv^\top)$.

B. Lie Groups

Denote a general Lie group $\mathbf{G}$, and its Lie algebra $\mathfrak{g}$. The identity is denoted $\mathrm{id} \in \mathbf{G}$, left and right translation are written $L_X(Y) := XY$ and $R_X(Y) := YX$, respectively, which induce the corresponding mappings on $\mathfrak{g}$, $\mathrm{d}L_X : \mathfrak{g} \to \mathrm{T}_X\mathbf{G}$ and $\mathrm{d}R_X : \mathfrak{g} \to \mathrm{T}_X\mathbf{G}$, respectively.

The Lie algebra $\mathfrak{g}$ is isomorphic to a vector space $\mathbb{R}^{\dim \mathfrak{g}}$. Define *wedge* $(\cdot)^\wedge : \mathbb{R}^{\dim \mathfrak{g}} \to \mathfrak{g}$ and its inverse *vee* $(\cdot)^\vee : \mathfrak{g} \to \mathbb{R}^{\dim \mathfrak{g}}$, as linear isomorphisms satisfying $(u^\vee)^\wedge = u$ for all $u \in \mathfrak{g}$. The exponential map $\exp : \mathfrak{g} \to \mathbf{G}$ defines a local diffeomorphism from a neighbourhood of $0 \in \mathfrak{g}$ to a neighbourhood of $\mathrm{id} \in \mathbf{G}$, and its inverse (when defined) is the logarithmic map $\log : \mathbf{G} \to \mathfrak{g}$. The Adjoint map $\mathrm{Ad} : \mathbf{G} \times \mathfrak{g} \to \mathfrak{g}$ is defined by $\mathrm{Ad}_X(U) := \mathrm{D}L_X \mathrm{D}R_{X^{-1}} U$. If $\mathbf{G}$ is a matrix Lie group, then $\mathrm{Ad}_X(U) := XUX^{-1}$.

The $\mathbf{G}$-torsor, denoted $\mathcal{G}$, is defined as the set of elements of $\mathbf{G}$ (underlying manifold), but without the group structure.

A right action is a smooth function $\phi : \mathbf{G} \times \mathcal{M} \to \mathcal{M}$ satisfying the identity and compatibility properties:

$$\phi(\mathrm{id}, \xi) = \xi, \qquad \phi\left(Y, \phi(X, \xi)\right) = \phi\left(XY, \xi\right),$$

for all $\xi \in \mathcal{M}$ and $X \in \mathbf{G}$. For any $\xi \in \mathcal{M}$ the partial map $\phi_X : \mathcal{M} \to \mathcal{M}$ is defined to be $\phi_X(\xi) = \phi(X, \xi)$. Similarly, for any $X \in \mathbf{G}$, the partial map $\phi_\xi : \mathbf{G} \to \mathcal{M}$ is defined to be $\phi_\xi(X) = \phi(X, \xi)$.

A group action is called *transitive* if for all $\xi_1, \xi_2 \in \mathcal{M}$, there exists $X \in \mathbf{G}$ such that $\phi(X, \xi_1) = \xi_2$. If a manifold $\mathcal{M}$ admits a transitive group action then it is called a *homogeneous space*. The group $\mathbf{G}$ acting on $\mathcal{M}$ is called a *symmetry* of $\mathcal{M}$.

A product Lie group is formed by combining multiple existing Lie groups. If $\mathbf{G}_1, \dots, \mathbf{G}_n$ are Lie groups, then their product Lie group is

$$\mathbf{G}_1 \times \dots \times \mathbf{G}_n := \{(X_1, \dots, X_n) | X_i \in \mathbf{G}_i\},$$

with identity $\mathrm{id}_{\mathbf{G}_1 \times \dots \times \mathbf{G}_n} := (\mathrm{id}_{\mathbf{G}_1}, \dots, \mathrm{id}_{\mathbf{G}_n})$, the group multiplication is a direct product with the standard inverse.

C. Matrix Lie Groups

The special orthogonal group $\mathbf{SO}(3)$ of 3D rotations and its Lie algebra $\mathfrak{so}(3)$ are defined by

$$\mathbf{SO}(3) := \{R \in \mathbb{R}^{3\times 3} \mid RR^\top = R^\top R = I_3, \det(R) = 1\},$$
$$\mathfrak{so}(3) := \{a^\times \mid a \in \mathbb{R}^3\},\ a^\times := \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix},$$

respectively, where $a^\times$ denotes the skew-symmetric matrix associated to the cross product, satisfying $a^\times b = a \times b$ for all $b \in \mathbb{R}^3$. For any $a \in \mathbb{R}^3$, $a^\wedge_{\mathfrak{so}(3)} = a^\times$.

The special linear group $\mathbf{SL}(3)$ and its Lie algebra $\mathfrak{sl}(3)$ are defined by

$$\mathbf{SL}(3) := \{H \in \mathbb{R}^{3\times 3} \mid \det(H) = 1\},$$
$$\mathfrak{sl}(3) := \{u^\wedge_{\mathfrak{sl}(3)} \mid u \in \mathbb{R}^8\},$$
$$u^\wedge_{\mathfrak{sl}(3)} := \begin{bmatrix} u_1 & u_4 & u_7 \\ u_2 & u_5 & u_8 \\ u_3 & u_6 & -u_1 - u_5 \end{bmatrix},$$

respectively. The $\mathbf{SL}(3)$-torsor is denoted $\mathcal{SL}(3)$.

D. Homographies

Consider the scenario of a moving camera observing a static planar scene. Let $\{\mathring{\mathcal{C}}\}$ be the fixed reference frame and $\{\mathcal{C}_t\}$ the current camera frame at time t. Let $R \in \mathbf{SO}(3)$ and $x \in \mathbb{R}^3$ denote the orientation and position, respectively, of the camera frame $\{\mathcal{C}_t\}$ with respect to the reference frame $\{\mathring{\mathcal{C}}\}$. Given a set of n points that belong to the scene, the Euclidean coordinates $\mathring{P}_i \in \{\mathring{\mathcal{C}}\}$ and $P_i \in \{\mathcal{C}_t\}$, $i \in \{1, \dots, N\}$, of the same i-th point are related by

$$\mathring{P}_i = RP_i + x. \tag{1}$$

Let $\lambda \in \mathbb{R}_+$ denote the distance from the origin of $\{\mathcal{C}_t\}$ to the planar scene and $\eta \in \mathcal{S}^2$ denote the normal vector pointing towards the scene expressed in $\{\mathcal{C}_t\}$. Inserting the planar constraint $\frac{\eta^\top P_i}{\lambda} = 1$ in (1) yields

$$\mathring{P}_i = \left(R + \frac{x\eta^\top}{\lambda}\right) P_i, \tag{2}$$

$H := R + \frac{x\eta^\top}{\lambda}$ denotes the Euclidean homography that maps the scene points' Euclidean coordinates from $\{\mathcal{C}_t\}$ to $\{\mathring{\mathcal{C}}\}$.

For any homography matrix H, there exists a unique corresponding matrix $H' := \det(H)^{-\frac{1}{3}} H \in \mathcal{SL}(3)$. All homographies are appropriately scaled to satisfy $H \in \mathcal{SL}(3)$ to ensure consistency throughout this work.

Using the pinhole camera model as described in [8], the homogeneous coordinates for the image point $\overline{\zeta_i^\circ} \in \{\mathring{\mathcal{C}}\}$ and $\overline{\zeta_i} \in \{\mathcal{C}_t\}$ corresponding to $\mathring{P}_i$ and P_i, respectively, are

$$\overline{\zeta_i^\circ} := K\mathring{P}_i, \qquad \overline{\zeta_i} := KP_i,$$

where K denotes the invertible 3×3 camera matrix. Provided the camera is well-calibrated, the image coordinates can be projected onto the 2-sphere as

$$\mathring{p}_i = \pi_{\mathcal{S}^2}(K^{-1}\overline{\zeta_i^{\circ}}), \qquad p_i = \pi_{\mathcal{S}^2}(K^{-1}\overline{\zeta_i}),$$

and the projected vectors satisfy the homography constraint

$$p_i = \pi_{\mathcal{S}^2}\left(H^{-1}\mathring{p}_i\right). \tag{3}$$

The measurements are represented, in this work, directly on the 2-sphere rather than on the image plane. This offers several benefits as it provides a more natural representation and can model a wide range of monocular cameras.

The parameters η and λ are termed *structure parameters* because they describe the scene's structure. They correspond to the reference structure parameters $\mathring{\eta} = R\eta$ and $\mathring{\lambda} = \lambda + \mathring{\eta}^\top x$, which remain constant for a static scene.

E. Problem formulation

The rigid-body angular and linear velocities of the camera with respect to frame $\{\mathring{\mathcal{C}}\}$ expressed in $\{\mathcal{C}_t\}$ are denoted Ω and v respectively.

The objective is to design an observer for the homography H and structure parameters (η, λ) using the available camera velocity measurements (Ω, v) and the set of n feature point correspondences between the frames $\{\mathcal{C}_t\}$ and $\{\mathring{\mathcal{C}}\}$.

Define the total *state space*

$$\mathcal{M} := \mathcal{SL}(3) \times \mathcal{S}^2 \times \mathbb{R}_+,$$

with elements $\xi = (H, \eta, \lambda) \in \mathcal{M}$. The *input space* is defined as $\mathbb{V} := \mathbb{R}^3 \times \mathbb{R}^3$ with elements $u = (\Omega, v) \in \mathbb{V}$.

The measurements are represented by n bearings of the current image, each with an output space $\mathcal{N}_i := \mathcal{S}^2$. The total *output space* is defined as $\mathcal{N} := \prod_{i=1}^n \mathcal{N}_i = (\mathcal{S}^2)^n$. Similarly, the reference points define the parameter spaces $\mathcal{P}_i := \mathcal{S}^2$, and the overall *parameter space* is $\mathcal{P} := (\mathcal{S}^2)^n$.

The system's output is defined as $y = [p_1, \ldots, p_n]^\top$. Therefore, from equation (3), we define the output function $h : \mathcal{M} \times \mathcal{P} \to \mathcal{N}$ as

$$h\left(\xi; (\mathring{p}_1, \ldots, \mathring{p}_n)\right) := \left[h^1(\xi; \mathring{p}_1), \ldots, h^n(\xi; \mathring{p}_n)\right], \tag{4}$$

$$h^i(\xi; \mathring{p}_i) := \pi_{\mathcal{S}^2}\left(H^{-1}\mathring{p}_i\right). \tag{5}$$

Note that all $\mathring{p}_i \in \mathcal{S}^2$, $i = \{1, \ldots, n\}$ are constant vectors.

III. System kinematics

This section will derive the kinematics for an element of the state space $\xi = (H, \eta, \lambda)$.

The kinematics of the pose of the camera are given by $\dot{R} = R\Omega^\times$ and $\dot{x} = Rv$ for orientation and position, respectively. However, these kinematics are not directly used in the development.

Recall that $\eta(t) = R^\top \mathring{\eta}$ and $\mathring{\eta}$ is a constant, then

$$\dot{\eta} = -\Omega^\times R^\top \mathring{\eta} = -\Omega^\times \eta. \tag{6}$$

For λ, recall that $\eta(t)^\top P_i(t) - \lambda(t) = 0$, $P_i(t) = R^\top(\mathring{P}_i - x)$ and $\mathring{\lambda}$ and $\mathring{P}_i$ are constants, then

$$\dot{\lambda} = -\eta^\top v. \tag{7}$$

According to [11, Lemma 5.3], the kinematics of the homography $H \in \mathbf{SL}(3)$ are given by

$$\dot{H} = H\left(\Omega^\times + \frac{v\eta^\top}{\lambda} - \frac{\eta^\top v}{3\lambda} I_3\right). \tag{8}$$

The full kinematics of the system state considered can be expressed as a system that takes the form $\dot{\xi} = f(\xi, u)$, $f : \mathbb{V} \to \mathfrak{X}(\mathcal{M})$, with the velocity $u = (\Omega, v) \in \mathbb{V}$ as input

$$\dot{\xi} = \begin{cases} \dot{H} = H\left(\Omega^\times + \frac{v\eta^\top}{\lambda} - \frac{\eta^\top v}{3\lambda} I_3\right), \\ \dot{\eta} = -\Omega^\times \eta, \\ \dot{\lambda} = -\eta^\top v. \end{cases} \tag{9}$$

Note that we explicitly model the structure of $\dot{H}$ in terms of the state (H, η, λ) and velocities (Ω, v).

IV. Group actions and equivariance

We introduce a novel *symmetry group*

$$\mathbf{G} := \mathbf{SL}(3) \times \mathbf{SO}(3) \times \mathbf{MR}(1) \tag{10}$$

where $\mathbf{MR}(1)$ denotes the multiplicative group of positive (non-zero) reals, $\mathfrak{mr}(1)$ denotes the associated Lie algebra. We will write the elements of $\mathbf{G}$ as $X = (P, Q, r) \in \mathbf{G}$, $P \in \mathbf{SL}(3)$, $Q \in \mathbf{SO}(3)$, $r \in \mathbf{MR}(1)$. The group identity is $\mathrm{id}_{\mathbf{G}} = (I_3, I_3, 1)$ and the associated Lie algebra is denoted $\mathfrak{g} = \mathfrak{sl}(3) \times \mathfrak{so}(3) \times \mathfrak{mr}(1)$.

In the following, we provide the key symmetry properties of the group $\mathbf{G}$ that will be exploited to design an equivariant observer for the kinematic system described in (9).

A. Symmetry actions

We define four key actions of the symmetry group $\mathbf{G}$: ϕ, the action of $\mathbf{G}$ on the state space $\mathcal{M}$; ψ, its action on the input space $\mathbb{V}$; ρ, its action on the output spaces $\mathcal{N}_i$; and θ, a novelty of this work, its action on the parameter spaces $\mathcal{P}_i$.

The proofs of the following lemmas are provided in the Appendix A.

Lemma 1 (*Symmetry of the total space*)**.** *Define the mapping* $\phi : \mathbf{G} \times \mathcal{M} \to \mathcal{M}$ *as*

$$\phi((P, Q, r), (H, \eta, \lambda)) := \left(P^{-1}HQ, Q^\top \eta, \frac{\lambda}{r}\right). \tag{11}$$

Then, ϕ is a right group action of $\mathbf{G}$ on $\mathcal{M}$.

Note that ϕ is transitive on the state space $\mathcal{M}$.

Lemma 2 (*Symmetry of the velocity space*)**.** *The mapping* $\psi : \mathbf{G} \times \mathbb{V} \to \mathbb{V}$ *defined as*

$$\psi((P, Q, r), (\Omega, v)) := \left(Q^\top \Omega, \frac{Q^\top v}{r}\right) \tag{12}$$

is a right group action of $\mathbf{G}$ on the velocity space $\mathbb{V}$.

Lemma 3 (*Symmetry of the output space*)**.** *The function* $\rho : \mathbf{G} \times \mathcal{N}_i \to \mathcal{N}_i$ *defined by*

$$\rho((P, Q, r), p_i) := Q^\top p_i \tag{13}$$

is a right group action of $\mathbf{G}$ on $\mathcal{N}_i$.

Lemma 4 (*Symmetry of the parameter space*)**.** *The function* $\theta : \mathbf{G} \times \mathcal{P}_i \to \mathcal{P}_i$ *defined by*

$$\theta((P, Q, r), \mathring{p}_i) := \pi_{\mathcal{S}^2}\left(P^{-1}\mathring{p}_i\right) \tag{14}$$

is a right group action of $\mathbf{G}$ *on* $\mathcal{P}_i$.

B. Equivariance

The *equivariance* of the system and the output under the group actions defined in IV-A are established below.

Lemma 5 (*System equivariance*)**.** *The system kinematics given in equation* (9) *are equivariant under the group actions* ϕ *and* ψ. *That is,*

$$\mathrm{D}\phi_X f(\xi, u) = f(\phi(X, \xi), \psi(X, u)), \tag{15}$$

for any $X \in \mathbf{G}$, $\xi \in \mathcal{M}$ *and* $u \in \mathbb{V}$.

Although the present formulation does not satisfy the equivariant output property as defined in [19], an alternative approach to attaining output equivariance involves using the symmetry action θ, as outlined in the following lemma.

Lemma 6 (*Output equivariance*)**.** *The output configurations defined in equation* (5) *are equivariant with respect to actions* ϕ, ρ *and* θ. *That is, for any* $X \in \mathbf{G}$, *any* $\xi \in \mathcal{M}$, *and any* $\mathring{p}_i \in \mathcal{P}_i$,

$$\begin{aligned}&\rho((P, Q, r), h^i((H, \eta, \lambda); \mathring{p}_i))\\ &= h^i(\phi((P, Q, r), (H, \eta, \lambda)); \theta((P, Q, r), \mathring{p}_i)).\end{aligned} \tag{16}$$

V. Lifting the kinematics to the Lie algebra

To consider the system on the symmetry group, a *lift* of the kinematics from the state space onto the group is necessary. An equivariant lift is guaranteed since the group $\mathbf{G}$ acts transitively on the state space $\mathcal{M}$, and the system is equivariant [12].

Lemma 7 (*Equivariant lift*)**.** *The smooth map* $\Lambda : \mathcal{M} \times \mathbb{V} \to \mathfrak{g}$ *defined as*

$$\begin{aligned}&\Lambda((H, \eta, \lambda),(\Omega, v))\\ &:= \left(-\operatorname{Ad}_H\left(\frac{v\eta^\top}{\lambda} - \frac{\eta^\top v}{3\lambda} I_3\right), \Omega^\times, \frac{\eta^\top v}{\lambda}\right)\end{aligned} \tag{17}$$

is a lift for the system (9). *That is,* Λ *satisfies*

$$\mathrm{D}_{X|\mathrm{id}}\phi_\xi \Lambda(\xi, u) = f(\xi, u). \tag{18}$$

In addition, the lift Λ *is equivariant with respect to the symmetry group* ϕ, *i.e.*

$$Ad_{X^{-1}}(\Lambda(\xi, u)) = \Lambda(\phi_X(\xi), \psi_X(u)),$$

for all $X \in \mathbf{G}$, $\xi \in \mathcal{M}$ *and* $u \in \mathbb{V}$.

The lift defined in (17) allows the construction of a lifted system on the symmetry group. This requires choosing a global state origin $\xi^\circ \in \mathcal{M}$ for a global coordinate parametrization of $\mathcal{M}$ by the group $\mathbf{G}$ given by the projection $\phi_{\xi^\circ} : \mathbf{G} \to \mathcal{M}$. The lifted system is expressed as

$$\dot{X} = \mathrm{d}L_X \Lambda\left(\phi(X, \xi^\circ), u\right). \tag{19}$$

VI. Observer design

This section presents an *equivariant observer* designed on the symmetry group $\mathbf{G}$ and uses the lifted system (19) as its internal model. It follows the equivariant filter (EqF) design approach, as presented in [12], [19] and [18].

A. Equivariant observer

Let $\hat{X} = (\hat{P}, \hat{Q}, \hat{r}) \in \mathbf{G}$ be the observer state. The filter error on the group $E = (E_P, E_Q, E_r) \in \mathbf{G}$ is defined as

$$E := X\hat{X}^{-1} = \left(P\hat{P}^{-1}, Q\hat{Q}^\top, r\hat{r}^{-1}\right). \tag{20}$$

To compute the linearized error dynamics, we need to fix a state origin. Let $\xi^\circ = (I_3, e_3, 1) \in \mathcal{M}$ be the chosen origin, the observer state estimate at any time t is given by

$$\hat{\xi}(t) = \phi(\hat{X}(t), \xi^\circ). \tag{21}$$

The global state error $e = (e_H, e_\eta, e_\lambda) := \phi_{\hat{X}^{-1}}(\xi) \in \mathcal{M}$ is written as

$$e = \phi(E, \xi^\circ) = \left(E_P^{-1}E_Q, E_Q^\top e_3, E_r^{-1}\right). \tag{22}$$

The goal of the equivariant observer is to drive the state error $e \to \xi^\circ$ to ensure that the state estimate $\hat{\xi} = \phi_{\hat{X}}(\xi^\circ)$ converges to the true state ξ.

Let $\varepsilon = (\varepsilon_H, \varepsilon_\eta, \varepsilon_\lambda) := \vartheta(e) \in \mathbb{R}^{11}$ represent local coordinates on the state around the origin ξ°, defined as

$$\vartheta(e) := \left(\log_{\mathbf{SL}(3)}(e_H)^\vee, \frac{e_{3\times}e_\eta}{|e_{3\times}e_\eta|}\sin^{-1}(|e_{3\times}e_\eta|), \log\left(e_\lambda\right)\right), \tag{23}$$

with inverse given by

$$\begin{aligned}&\vartheta^{-1}(\varepsilon) :=\\ &\left(\exp_{\mathbf{SL}(3)}\left((\varepsilon_H)^\wedge_{\mathfrak{sl}(3)}\right), \exp_{\mathbf{SO}(3)}\left(\begin{bmatrix}\varepsilon_\eta\\ 0\end{bmatrix}^\times\right)e_3, \exp(\varepsilon_\lambda)\right).\end{aligned} \tag{24}$$

Note that the equivariant observer provides a minimal (11-dimensional) representation of the observer error.

A key contribution of this work is allowing both the reference points and the current points to be acted on by the group, as discussed in IV-A. Then, the innovation is defined as $\delta = \left[\delta_1^\top \dots \delta_n^\top\right]^\top$, where each element δ_i is given by

$$\delta_i = \rho_{\hat{X}^{-1}}(p_i) - h^i(\xi^\circ; \theta_{\hat{X}^{-1}}(\mathring{p}_i)). \tag{25}$$

Using the output equivariance from Lemma 6 yields

$$\rho_{\hat{X}^{-1}}(p_i) = \rho_{\hat{X}^{-1}}\left(h^i(\xi; \mathring{p}_i)\right) = h^i(e; \theta_{\hat{X}^{-1}}(\mathring{p}_i)).$$

From there, the derivation of the linearised error dynamics, as well as linearised output are defined according to [18], with state matrix $\mathring{A}_t$ and output matrix C_t given by

$$\mathring{A}_t = \mathrm{D}_{e|\xi^\circ}\vartheta(e) \cdot \mathrm{D}_{E|\mathrm{id}}\phi_{\xi^\circ}(E) \cdot \mathrm{D}_{e|\xi^\circ}\Lambda(e, \mathring{u}) \cdot \mathrm{D}_{\varepsilon|0}\vartheta^{-1}(\varepsilon), \tag{26}$$

$$C_t = \mathrm{D}_{e|\xi^\circ}h(e; \theta_{\hat{X}^{-1}}(\mathring{p}_i)) \cdot \mathrm{D}_{\varepsilon|0}\vartheta^{-1}(\varepsilon), \tag{27}$$

where $\mathring{u}(t) = \psi_{\hat{X}^{-1}}(u(t))$ denotes the origin velocity.

Remark 1. *The state and output matrices of the linearised system are similar to those in the classical EqF design [18], with one key difference. In the current design, the output matrix employs $\theta_{\hat{X}^{-1}}(\mathring{p}_i)$ instead of just $\mathring{p}_i$, which, unlike earlier work, leads to a time-varying output matrix C_t.*

Then the observer is given by the solution of

$$\dot{\hat{X}} := \mathrm{d}L_{\hat{X}}\Lambda(\phi(\hat{X}, \xi^\circ), u) + \mathrm{d}R_{\hat{X}}\Delta, \tag{28}$$

$$\Delta := \mathrm{D}_{E|\mathrm{id}}\phi_{\xi^\circ}(E)^\dagger \mathrm{d}\vartheta^{-1}\Sigma C_t^\top N_t^{-1}\delta, \tag{29}$$

$$\dot{\Sigma} := \mathring{A}_t\Sigma + \Sigma\mathring{A}_t^\top + M_t - \Sigma C_t^\top N_t^{-1}C_t\Sigma,\ \Sigma(0) = \Sigma_0, \tag{30}$$

where $\Delta \in \mathfrak{g}$ is the equivariant observer correction term, $\Sigma \in \mathbb{S}_+(11)$ is the Riccati gain, $M_t \in \mathbb{S}_+(11)$ and $N_t \in \mathbb{S}_+(3n)$ are the covariances of the state and output, respectively. $\mathbb{S}_+(k)$ denotes the set of positive definite $k \times k$ matrices. $\mathrm{D}_{E|\mathrm{id}}\phi_{\xi^\circ}(E)^\dagger$ is a fixed right-inverse of $\mathrm{D}_{E|\mathrm{id}}\phi_{\xi^\circ}(E)$; that is $\mathrm{D}_{E|\mathrm{id}}\phi_{\xi^\circ}(E) \cdot \mathrm{D}_{E|\mathrm{id}}\phi_{\xi^\circ}(E)^\dagger = \mathrm{id}$.

Lemma 8. *If the pair $(\mathring{A}_t, C_t)$ of the linearised system is uniformly observable in the sense of [7, Theorem 3.1]. Then, $\Sigma(t)$ and $\Sigma^{-1}(t)$ are uniformly bounded and the origin $e(t) \to \xi^\circ$ is locally exponentially stable.*

The Riccati matrix $\Sigma(t)$ of the EqF can be seen as the covariance of the linearised error $\varepsilon = \vartheta\left(\phi_{\hat{X}^{-1}}(\xi)\right)$, such that $\varepsilon \sim \mathbf{N}(0, \Sigma)$. The initial Riccati matrix Σ_0 is modeled as being on the state ξ and needs to be transformed to an uncertainty on the group. This is done using the relation

$$\Sigma_{\mathrm{EqF}} = \mathrm{D}_{e|\xi^\circ}\vartheta(e)\mathrm{D}_{\xi|\xi^\circ}\phi_{\hat{X}^{-1}}(\xi)\Sigma_0\mathrm{D}_{\xi|\xi^\circ}\phi_{\hat{X}^{-1}}(\xi)\mathrm{D}_{e|\xi^\circ}\vartheta(e).$$

The noise is modeled as being injected into the velocity $u = (\Omega, v)$, s.t. $u \sim \mathbf{N}(u_t^m, V_t)$, where u_t^m is the measured velocity and V_t is the velocity noise covariance. Thus the process noise covariance is expressed as

$$M_t = M_\varepsilon + B_t V_t B_t^\top, \tag{31}$$

where M_ε is the state noise covariance and the linearised input matrix B_t is given by

$$B_t = \mathrm{D}_{e|\xi^\circ}\vartheta(e) \cdot \mathrm{D}_{E|\mathrm{id}}\phi_{\xi^\circ}(E) \cdot \mathrm{D}_{u|0}\Lambda(\xi^\circ, u) \cdot \mathrm{D}_{u|0}\psi_{\hat{X}^{-1}}(u).$$

The measurement noise covariance also needs to be transformed, to reflect the fact that the measurements p_i are transformed by $\rho_{\hat{X}^{-1}}$, as illustrated in equation (25). Let N_s be the measurement noise covariance, then the transformed noise covariance for a given point p_i is

$$N_t^i = \mathrm{D}_{z|p_i}\rho_{\hat{X}^{-1}}(z)\, N_s \left(\mathrm{D}_{z|p_i}\rho_{\hat{X}^{-1}}(z)\right)^\top. \tag{32}$$

The components of matrices $\mathring{A}_t, C_t, B_t$, $\mathrm{D}_{E|\mathrm{id}}\phi_{\xi^\circ}(E)^\dagger$, $\mathrm{d}\vartheta^{-1}$ and $\mathrm{D}_{z|\mathring{p}_i}\rho_{\hat{X}^{-1}}(z)$ are derived in Appendix B.

VII. Simulation Results

In the following section, we discuss the performance of the proposed observer through simulation results. The simulation scenario involves a planar target located at the stationary horizontal plane and a moving camera following a Lissajous trajectory with known linear and angular velocities.

We conduct a Monte-Carlo simulation with $R_{MC} = 100$ runs. The initial position estimates are distributed normally around zero with a standard deviation of 1m per axis and the initial orientation estimates are distributed normally around zero with a standard deviation of $30°$ per axis. Similarly, the initial normal vector and distance to the scene estimates are randomly generated using Gaussian distributions, with a standard deviation of $22.5°$ around $\hat{\eta}(0) = e_3$ and a standard deviation of 1m around $\hat{\lambda}(0) = 7$m, respectively. The observer is initialized with the Riccati matrix $\Sigma_0 = \mathrm{diag}(0.1)$ and the velocity and output measurement noise covariances $V_k = \mathrm{diag}(0.01)$ and $N_k = \mathrm{diag}(0.01)$, respectively.

While the EqF is derived using a continuous algebraic Riccati equation (30), its practical implementation involves using the corresponding discrete algebraic Riccati equation. The update is performed using the matrix exponential map to ensure that the estimates remain in the Lie group.

To assess the consistency of the equivariant filter, we compute the Average Normalized Estimation Error Squared (ANEES), also known as the average energy, given by

$$\mathrm{ANEES} = \frac{1}{R_{MC}}\sum_{k=1}^{R_{MC}}\left(\frac{1}{m}\varepsilon_k^\top\Sigma_k^{-1}\varepsilon_k\right),$$

where ε_k denotes the filter state error for the k-th simulation run and $m = 11$ is the dimension of the state space $\mathcal{M}$.

A. Results and discussion

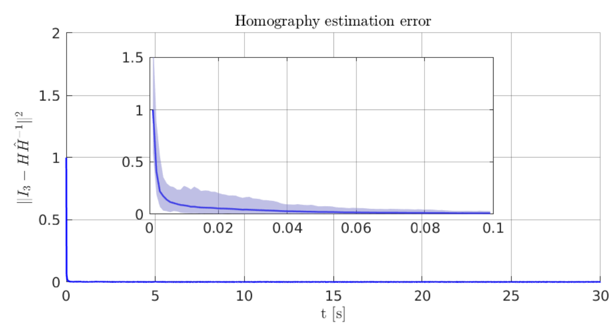

Fig. 1. Average homography error.

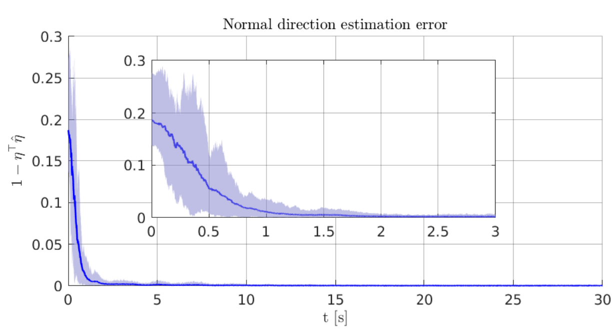

Fig. 2. Average normal direction error.

Figures 1, 2 and 3 illustrate the average estimation errors $\epsilon_H = \|I_3 - H\hat{H}^{-1}\|^2$, $\epsilon_\eta = (1 - \eta^\top\hat{\eta})$ and $\epsilon_\lambda = |\lambda - \hat{\lambda}|^2$, respectively. The shaded areas show the 5th to 95th percentile

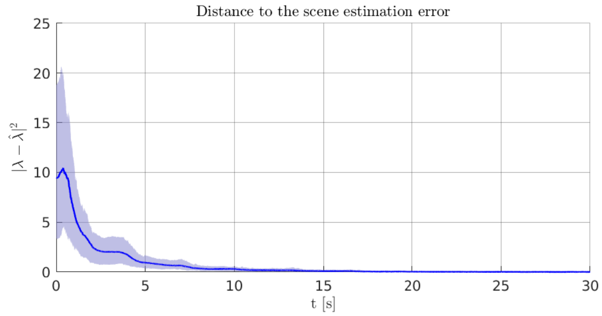

Fig. 3. Average distance to the scene error.

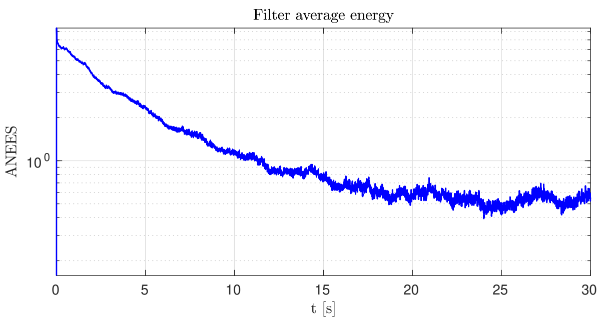

Fig. 4. Log plot of the average NEES.

of error. The errors gradually decrease and converge to zero over time, with a fast transient response of the homography and normal direction. Figure 4 shows the log of the average energy of the filter which settles slightly below the ideal value of 1. These results confirm that the proposed equivariant observer provides accurate and consistent estimates of the true state of the system.

VIII. Conclusions

In this paper, we introduced a novel symmetry-based observer design to estimate the homography and structure parameters of a planar scene. The proposed approach is founded on the recently proposed Equivariant Filter (EqF) design methodology and relies solely on camera velocity measurements and direct feature point correspondences between a pair of images. The unknown homography velocity is retrieved, by explicitly expressing it in terms of structure parameters and rigid-body velocities, allowing the homography to be estimated for arbitrary camera motion. A new symmetry group was introduced, incorporating the Special Linear group $\mathbf{SL}(3)$ and operating on the total state space. The symmetry structure of this group was exploited to establish system and output equivariance properties. It was shown that the system kinematics admit an equivariant lift, providing the basis for the proposed equivariant observer designed on the resulting lifted system. We validated the theoretical findings through simulation, demonstrating the performance and consistency of the proposed observer solution.

Acknowledgment

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Appendix

A. Proofs

Proof of Lemma 1. Trivially $\phi((I_3, I_3, 1), (H, \eta, \lambda)) = (H, \eta, \lambda)$ for any $(H, \eta, \lambda) \in \mathcal{M}$. Let $(H, \eta, \lambda) \in \mathcal{M}$ and $(P_1, Q_1, r_1), (P_2, Q_2, r_2) \in \mathbf{G}$ be arbitrary. Then

$$\begin{aligned}
&\phi((P_1, Q_1, r_1), \phi((P_2, Q_2, r_2), (H, \eta, \lambda))),\\
&= \phi\left((P_1, Q_1, r_1), \left(P_2^{-1} H Q_2, Q_2^\top \eta, \frac{\lambda}{r_2}\right)\right),\\
&= \left(P_1^{-1} P_2^{-1} H Q_2 Q_1, Q_1^\top Q_2^\top \eta, \frac{\lambda}{r_1 r_2}\right),\\
&= \phi((P_2 P_1, Q_2 Q_1, r_1 r_2), (H, \eta, \lambda)),\\
&= \phi((P_2, Q_2, r_2)(P_1, Q_1, r_1), (H, \eta, \lambda)).
\end{aligned}$$

so it satisfies compatibility. This demonstrates that ϕ is a right action as required. □

Proof of Lemma 2. It is straightforward to verify that ψ is a right group action. Let $(\Omega, v) \in \mathbb{V}$ and $(P_1, Q_1, r_1), (P_2, Q_2, r_2) \in \mathbf{G}$ be arbitrary. Then

$$\begin{aligned}
&\psi((P_1, Q_1, r_1), \psi((P_2, Q_2, r_2), (\Omega, v))),\\
&= \psi\left((P_1, Q_1, r_1), \left(Q_2^\top \Omega, \frac{Q_2^\top v}{r_2}\right)\right),\\
&= \left(Q_1^\top Q_2^\top \Omega, \frac{Q_1^\top Q_2^\top v}{r_1 r_2}\right),\\
&= \psi((P_2 P_1, Q_2 Q_1, r_2 r_1), (\Omega, v)),\\
&= \psi((P_2, Q_2, r_2)(P_1, Q_1, r_1), (\Omega, v)),
\end{aligned}$$

ψ satisfies compatibility, and $\psi((I_3, I_3, 1), (\Omega, v)) = (\Omega, v)$, so ψ is a right group action as required. □

Proof of Lemma 3. It is straightforward to verify that $\rho((I_3, I_3, 1), p_i) = p_i$ for all $p_i \in \mathcal{N}_i$. Let $p_i \in \mathcal{N}_i$ and $(P_1, Q_1, r_1), (P_2, Q_2, r_2) \in \mathbf{G}$ be arbitrary. Then,

$$\begin{aligned}
&\rho((P_1, Q_1, r_1), \rho((P_2, Q_2, r_2), p_i)),\\
&= \rho((P_1, Q_1, r_1), Q_2^\top p_i),\\
&= (Q_1^\top Q_2^\top p_i),\\
&= \rho((P_2 P_1, Q_2 Q_1, r_1 r_2), p_i),\\
&= \rho((P_2, Q_2, r_2)(P_1, Q_1, r_1), p_i).
\end{aligned}$$

As required. □

Proof of Lemma 4. It is straightforward to verify that $\theta((I_3, I_3, 1), \mathring{p}_i) = \mathring{p}_i$ for any $\mathring{p}_i \in \mathcal{P}_i$. Let $\mathring{p}_i \in \mathcal{P}_i$ and $(P_1, Q_1, r_1), (P_2, Q_2, r_2) \in \mathbf{G}$ be arbitrary. Then,

$$\begin{aligned}
&\theta((P_1, Q_1, r_1), \theta((P_2, Q_2, r_2), \mathring{p}_i)),\\
&= \theta\left((P_1, Q_1, r_1), \pi_{\mathcal{S}^2}(P_2^{-1} \mathring{p}_i)\right),\\
&= \left(\pi_{\mathcal{S}^2}\left((P_2 P_1)^{-1} \mathring{p}_i\right)\right),\\
&= \theta((P_2 P_1, Q_2 Q_1, r_1 r_2), \mathring{p}_i),\\
&= \theta((P_2, Q_2, r_2)(P_1, Q_1, r_1), \mathring{p}_i).
\end{aligned}$$

As required. $\square$

Proof of Lemma 5. Let $(P, Q, r) \in \mathbf{G}$, $(H, \eta, \lambda) \in \mathcal{M}$ and $(\Omega, v) \in \mathbb{V}$ be arbitrary. Note that ϕ is linear in H, η and λ, so $\mathrm{D}\phi_X f(\xi, u)$ acts on $f(\xi, u)$ the same way that ϕ_X acts on ξ. Therefore, one has

$$
\begin{aligned}
&\mathrm{D}\phi_X f(\xi, u) \\
&= \mathrm{D}\phi_X \left(H \left(\Omega^\times + \frac{v\eta^\top}{\lambda} - \frac{\eta^\top v}{3\lambda} I_3 \right), -\Omega^\times \eta, -\eta^\top v \right), \\
&= \left(P^{-1} H \left(\Omega^\times + \frac{v\eta^\top}{\lambda} - \frac{\eta^\top v}{3\lambda} I_3 \right) Q, -Q^\top \Omega^\times \eta, \frac{-\eta^\top v}{r} \right), \\
&= f\left(\left(P^{-1}HQ, Q^\top \eta, \frac{\lambda}{r} \right), \left(Q^\top \Omega, \frac{Q^\top v}{r} \right) \right), \\
&= f(\phi(X, \xi), \psi(X, u)).
\end{aligned}
$$

As required. $\square$

Proof of lemma 6. To see the equivariance of h, let $(P, Q, r) \in \mathbf{G}$, $(H, \eta, \lambda) \in \mathcal{M}$ and $\mathring{p}_i \in \mathcal{P}_i$ be arbitrary. Then,

$$
\begin{aligned}
&\rho((P, Q, r), h((H, \eta, \lambda), \mathring{p}_i), \\
&= \rho((P, Q, r), H^{-1}\mathring{p}_i), \\
&= (Q^\top H^{-1}\mathring{p}_i), \\
&= (Q^\top H^{-1} P P^{-1}\mathring{p}_i), \\
&= ((P^{-1}HQ)^{-1} P^{-1}\mathring{p}_i), \\
&= h\left(\left(P^{-1}HQ, Q^\top \eta, \frac{\lambda}{r} \right), P^{-1}\mathring{p}_i \right), \\
&= h\left(\phi\left((P, Q, r), (H, \eta, \lambda) \right), \theta((P, Q, r), \mathring{p}_i) \right).
\end{aligned}
$$

As required. Note that $Q^\top \eta$, $\frac{\lambda}{r}$ don't enter in the equation, so they can be added without loss of generality. $\square$

Proof of Lemma 7. Recall that $\phi((P, Q, r), (H, \eta, \lambda)) = \left(P^{-1}HQ, Q^\top \eta, \frac{\lambda}{r} \right)$. To find $\mathrm{D}_{(P,Q,r)|(I_3,I_3,1)}\phi_\xi$ first choose $a \in \mathfrak{sl}(3)$, $q \in \mathfrak{so}(3)$ and $b \in \mathfrak{mr}(1)$, and then evaluate ϕ applied to $P = e^{ta}$, $Q = e^{tq}$ and $r = e^{tb}$:

$$
\begin{aligned}
&\mathrm{D}_{(P,Q,r)|(I_3,I_3,1)}\phi_\xi(a, q, b) \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \phi((e^{ta}, e^{tq}, e^{tb}), (H, \eta, \lambda)) \mid_{t=0}, \\
&= \frac{\mathrm{d}}{\mathrm{d}t} (e^{-ta} H e^{tq}, e^{-tq}\eta, d e^{-tb}) \mid_{t=0}, \\
&= (-a e^{-ta} H e^{tq} + e^{-ta} H q e^{tq}, -q e^{-tq}\eta, -\lambda b e^{-tb}) \mid_{t=0}, \\
&= (-aIHI + IHqI, -qI\eta, -\lambda b 1), \\
&= (-aH + Hq, -q\eta, -\lambda b),
\end{aligned}
$$

and so

$$
\begin{aligned}
&\mathrm{D}_{(P,Q,r)|(I_3,I_3,1)}\phi_\xi[\Lambda(\xi, u)] \\
&= \left(\mathrm{Ad}_H \left(\frac{v\eta^\top}{\lambda} - \frac{\eta^\top v}{3\lambda} I_3 \right) H + H\Omega^\times, -\Omega^\times \eta, -\lambda \frac{\eta^\top v}{\lambda} \right), \\
&= \left(H \left(\Omega^\times + \frac{v\eta^\top}{\lambda} - \frac{\eta^\top v}{3\lambda} I_3 \right), -\Omega^\times \eta, -\eta^\top v \right), \\
&= f(\xi, u),
\end{aligned}
$$

as required. Now to show that Λ is equivariant

$$
\begin{aligned}
&\mathrm{Ad}_{X^{-1}}(\Lambda(\xi, u)) \\
&= \mathrm{Ad}_{X^{-1}} \left(-\mathrm{Ad}_H \left(\frac{v\eta^\top}{\lambda} - \frac{\eta^\top v}{3\lambda} I_3 \right), \Omega^\times, \frac{\eta^\top v}{\lambda} \right), \\
&= \left(-\mathrm{Ad}_{P^{-1}HQ} \left(\frac{Q^\top v\eta^\top Q}{\lambda} - \frac{\eta^\top v}{3\lambda} I_3 \right), (Q^\top \Omega)^\times, \right. \\
&\qquad \left. \frac{\eta^\top Q Q^\top v}{r\lambda / r} \right), \\
&= \Lambda\left(\left(P^{-1}HQ, Q^\top \eta, \frac{\lambda}{r} \right), \left(Q^\top \Omega, \frac{Q^\top v}{r} \right) \right), \\
&= \Lambda(\phi_X(\xi), \psi_X(u)).
\end{aligned}
$$

As required. $\square$

B. Equivariant observer derivations

This section provides a detailed derivation of the necessary terms and derivatives for the equivariant observer design.

$$
\begin{aligned}
&\mathrm{d}\vartheta^{-1} = \mathrm{D}_{\varepsilon|0}\vartheta^{-1}(\varepsilon) \\
&= \mathrm{D}_{\varepsilon|0} \left(\exp_{\mathbf{SL}(3)}\left((\varepsilon_H)^\wedge_{\mathfrak{sl}(3)} \right), \exp_{\mathbf{SO}(3)}\left(\begin{bmatrix} \varepsilon_\eta \\ 0 \end{bmatrix}^\times \right) e_3, \right. \\
&\qquad \left. \exp(\varepsilon_\lambda) \right), \\
&= \begin{bmatrix} \begin{bmatrix} I_8 & & & \\ -1 & 0_{1,3} & -1 & 0_{1,3} \end{bmatrix} & 0_{9,2} & 0_{9,1} \\ 0_{3,8} & \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} & 0_{3,1} \\ 0_{1,8} & 0_{1,2} & 1 \end{bmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{D}_{e|\xi^\circ}\vartheta(e) \\
&= \mathrm{D}_{e|\xi^\circ} \left(\log_{\mathbf{SL}(3)}(e_H)^\vee, \frac{e_{3\times} e_\eta}{|e_{3\times} e_\eta|} \sin^{-1}(|e_{3\times} e_\eta|), \log(e_\lambda) \right), \\
&= \begin{bmatrix} I_8 & 0_{8\times1} & 0_{8\times3} & 0_{8\times1} \\ 0_{2,8} & 0_{2,1} & \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. & 0_{2,1} \\ 0_{1,8} & 0 & 0_{1,3} & 1 \end{bmatrix}.
\end{aligned}
$$

$$
\mathrm{D}_{E|\mathrm{id}}\phi_{\xi^\circ}(E) = \begin{bmatrix} (*) & I_3 \otimes I_3 & 0_{9,1} \\ 0_{3,9} & I_3 \otimes e_3^\top & 0_{3,1} \\ 0_{1,9} & 0_{1,9} & -1 \end{bmatrix},
$$

where the elements of $(*)$ are given by

$$
\begin{aligned}
\frac{\partial(P^{-1}Q)_{ij}}{\partial P_{nm}}|_{P=Q=I_3} =& \frac{1}{|P|^2} \sum_k \left(|P| \frac{\partial}{\partial P_{nm}} (|P|P^{-1})_{ik} \right. \\
&\left. -(|P|P^{-1})_{ik} \frac{\partial |P|}{\partial P_{nm}} \right) (Q)_{kj}|_{P=Q=I_3}.
\end{aligned}
$$

To determine $\mathrm{D}_{E|\mathrm{id}}\phi_{\xi^\circ}(E)^\dagger : T_{\xi^\circ}\mathcal{M} \to \mathfrak{g}$, taking the pseudo-inverse of $\mathrm{D}_{E|\mathrm{id}}\phi_{\xi^\circ}(E)$ directly does not work as it does not respect the geometry. Instead, the structure of T_{ξ° is used, which has a skew-symmetric first part and a tangent plane to e_3 with zero z-component. This results in

$$
\mathrm{D}_{E|\mathrm{id}}\phi_{\xi^\circ}(E)^\dagger = \begin{bmatrix} -I_9 & (*) & 0_{9,1} \\ 0_{9,9} & (*) & 0_{9,1} \\ 0_{1,9} & 0_{1,3} & -1 \end{bmatrix},
$$

where

$$(*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The origin velocity $\mathring{u}(t) = (\mathring{\Omega}, \mathring{v})$ is expressed as

$$\mathring{u} := \psi_{\hat{X}^{-1}}(u) = \left(\hat{Q}\Omega, \hat{Q}\hat{r}v\right).$$

$$\mathrm{D}_{e|\xi^\circ}\Lambda(e, \mathring{u}) = \begin{bmatrix} (*_1) & (*_2) & (*_3) \\ 0_{9,9} & 0_{9,3} & 0_{9,1} \\ 0_{1,9} & \mathring{v}^\top & -e_3^\top \mathring{v} \end{bmatrix},$$

where the elements are given by

$$(*_1) : \frac{\partial(-\operatorname{Ad}_H U')_{ij}}{\partial H_{nm}} = -\sum_l \left(\frac{\partial}{\partial H_{nm}} H_{im}\right) U'_{ml}(H^{-1})_{lj} - \sum_{l,k} H_{ik}U'_{kl}\frac{\partial}{\partial H_{nm}}(H^{-1})_{lj},$$

$$(*_2) : \frac{\partial\left(-\operatorname{Ad}_H(\mathring{v}\eta^\top - \frac{\eta^\top \mathring{v}}{3} I_3)\right)_{ij}}{\partial \eta_n} = \sum_k -H_{ik}\mathring{v}_k(H^{-1})_{nj} + H_{ik}\frac{\mathring{v}_n}{3}(H^{-1})_{kj},$$

$$(*_3) = \left(\mathring{v}e_3^\top - \frac{e_3^\top \mathring{v}}{3} I_3\right)^\vee.$$

$$\begin{aligned} \mathrm{D}_{e|\xi^\circ} h(e; \theta_{\hat{X}^{-1}}(\mathring{p}_i)) &= \mathrm{D}_{e|\xi^\circ}\pi_{\mathcal{S}^2}\left(e_H^{-1}\hat{P}\mathring{p}_i\right), \\ &= \left[\Pi_{\pi_{\mathcal{S}^2}(\hat{P}\mathring{p}_i)}\hat{P}\mathring{p}_i \otimes I_3 \quad 0_{3,4}\right]. \end{aligned}$$

$$D_{u|0}\Lambda(\xi^\circ, u) = \begin{bmatrix} 0_{9,3} & (*_4) \\ (*_5) & 0_{9,3} \\ 0_{1,3} & e_3^\top \end{bmatrix},$$

where

$$(*_4) = -I_3 \otimes e_3 + \frac{1}{3}\begin{bmatrix} 0_{9,1} & 0_{9,1} & I_3^\vee \end{bmatrix},$$

$$(*_5) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\mathrm{D}_{u|0}\psi_{\hat{X}^{-1}}(u) = \begin{bmatrix} \hat{Q} & 0_{3,3} \\ 0_{3,3} & \hat{Q}\hat{r} \end{bmatrix}.$$

$$\mathrm{D}_{e|\xi^\circ}\phi_{\hat{X}}(e) = \begin{bmatrix} (*_6) & 0_{9,3} & 0_{9,1} \\ 0_{3,9} & \hat{Q}^\top & 0_{3,1} \\ 0_{1,9} & 0_{1,3} & \frac{1}{\hat{r}} \end{bmatrix},$$

$$(*_6) = \frac{\partial(\hat{P}^{-1}e_H\hat{Q})_{ij}}{\partial e_{H_{nm}}}\Big|_{e_H = I_3} = (\hat{P}^{-1})_{in}\hat{Q}_{mj}.$$

$$\mathrm{D}_z|_{\mathring{p}_i}\rho_{\hat{X}^{-1}}(z) = \hat{Q}.$$

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