2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Attitude Dynamics Modelling: Fractional Consensus Approach

Jerzy Baranowski[1], Waldemar Bauer[1], Karolina Dukała[2], Dorota Mozyrska[3], Małgorzata Wyrwas[3]

***Abstract*— In this paper we propose a consensus model using fractional calculus, which is an emerging topic in multi-agent modeling. Fractional models have infinite memory and can be understood as a relatively simple extension of traditional calculus. We propose a model structure motivating it by psychological research. For such model we also provide a stability analysis allowing results on possibilities of consensus arising in the modelled group of agents. To achieve this, we use fractional difference equations, which illustrate our considerations for agent groups of increasing complexity.**

I. INTRODUCTION

The shaping of attitudes and opinions in societies and groups is a topic of constant interest, practically from the dawn of time. How people vote, what they support, who do they like, who do they hate - these questions are investigated both by researchers and by users of such knowledge. What is missing, are sufficient mathematical tools that would allow us to quantise such changes and possibly predict the dynamically changing attitudes. The focus of this paper is the discussion of possible modeling tools joining psychology, consensus modeling and fractional calculus that would allow better investigation on opinion–shaping phenomena. Consensus modeling in psychology typically refers to the process of reaching agreement or consensus among experts or participants in a study regarding certain psychological phenomena, theories, or assessments. It can involve various techniques and methodologies, such as expert panels, or group discussions, to arrive at a shared understanding or judgment. Using the idea of models with fractional order we introduce inside the group pf agents a kind of memory in decisions. The most important goal in opinion dynamics is to find a model with which recognise and potentially predict the tendency of a group of individuals into the direction of common opinion. When a group of a team, a committee or consumers, called generally agents, takes with time the same or very close similar opinion or way of behaviours we can say that consensus is reached by the group. We assume that each agent/expert in the group has his/her own opinion. However, each agent is willing to cooperate and possibly change his/her opinion. They can have meetings, discussions, some interactions between them. Such kinds of interactions between agents are described by the interconnection topology of the system (system of agents). The consensus reaching in multi–agent systems is under investigation for around two decades. There are classical models, introduced by Krause in [1] or sometimes referred to as the Hegselmann–Krause model in [2]. The various kind of topological type consensus models were widely studied for leaderless or leader-following models for integer-order multiagent systems in continuous-time or discrete-time cases. Important theoretic results include models on isolated time scales [3], flocking phenomena [4], [5] and control methods [6]. Examples of consensus modelling in multi–agent systems include vehicle formations [7], [8] and fixed and switching networks [9].

Recently, fractional calculus in both continuous-time and discrete-time cases gained considerable development as an interesting extension multi-agent modeling methodology. Works among the others include, earlier introductory works of the authors [10], and multiple applications by other authors, see for example [11], [12]. The main difference of fractional models from integer order ones is the infinite memory horizon of the models. Fractional differential and difference equation solutions are not generated by semigroups and because of that cannot ignore the long memory effects. Such effects are however obviously present in modelling of human opinions, as these opinions depend of humans' experiences.

The main contribution of this paper is the development of new interdisciplinary model for attitude dynamics and study of its certain properties in continuous and discrete time domains. We use the model with commensurate order, it means the same for each agent. As the order is a parameter that can be interpreted as a memory inside a process, we assume that each agent has the same parameter memory. What value of the order we should use, in fact it is not decided. We straggle with more theoretical investigations, without using any exact data-set. To analyse what can happen inside the model when we use different orders. We use the knowledge from psychology to determine the parametric structure of the model. Such model is formulated as a system of nonlinear fractional difference equations which can be used to study the behaviour of individual agents. We also formulate the model of opinion differences, which along with the use of stability theory allows determination of whether consensus among the agents in the system is possible. Our results are illustrated with numerical examples of agent networks of increasing complexity.

The paper is organised as follows. Firstly, we introduce

*(1) Work partially realised in the scope of project titled "Process Fault Prediction and Detection". Project was financed by The National Centre for Research and Development on the base of decision no. UMO-2021/41/B/ST7/03851. (3) The work was supported by Bialystok University of Technology grant No. WZ/WI-IIT/2/2023 and funded by the resources for research by the Ministry of Education and Science.

[1]J. Baranowski and W. Bauer are with Department of Automatic Control and Robotics, AGH University of Science & Technology, Kraków, Poland jb@agh.edu.pl, wby@agh.edu.pl

[2]K. Dukała is SWPS University of Social Sciences and Humanities, Katowice, Poland dukalakarolina@gmail.com

[3]D. Mozyrska and M. Wyrwas are with Faculty of Computer Science, Bialystok University of Technology, Wiejska 45A, 15-351 Białystok, Poland d.mozyrska@pb.edu.pl, m.wyrwas@pb.edu.pl

fractional calculus preliminaries, that are used for model construction and subsequent analysis which are followed by the concepts of modeling attitude and opinion changes in psychology. Then we present our model, along with the justification of its structure and parameters. Then we illustrate how the model represents attitude dynamics with the use of simulations and providing psychological context. Paper ends with some plans for future research, especially addressing experimental validation and parameter identifiability.

Fractional calculus is a potent methodology for modelling of advanced and complex processes. It has been applied to many scientific and engineering fields and verified to be a powerful tool in modeling most physical processes with memory effect, which cannot be well described by integer–order equations. For a comprehensive review of theory and applications of fractional calculus, we refer the reader to monographs by Hilfer [13], Kaczorek [14], Ostalczyk [15] or Podlubny [16].

Let $(h\mathbb{N})_c := \{c, c+h, c+2h, ...\}$ and $\mathbb{N}_c := \{c, c+1, c+2, ...\}$, where $c \in \mathbb{R}$ and $h > 0$. Let us recall the generalised binomial $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)...(\alpha-k+1)}{k!}$, where $\alpha \in \mathbb{R}$. Let $a^{(\alpha)} : \mathbb{N}_0 \to \mathbb{R}$ be the sequence of coefficients defined as follows: $a^{(\alpha)}(k) := (-1)^k \binom{\alpha}{k}$. In numerical calculations more efficient method of computing the values of defined sequence is the following recurrence $a^{(\alpha)}(0) := 1\,,\; a^{(\alpha)}(k+1) := \left(1 - \frac{\alpha+1}{k+1}\right) a^{(\alpha)}(k)\,, \quad k \in \mathbb{N}_0$. Using this sequence the following difference operator is defined.

Definition 1: [see [15], [17]] Let $\alpha \in \mathbb{R}$. The *Grünwald–Letnikov difference operator* Δ_h^α *of order* α *for a function* $x : (h\mathbb{N})_0 \to \mathbb{R}$ is defined by

$$\left(\Delta_h^\alpha x\right)(t) := h^{-\alpha} \sum_{i=0}^{k} a^{(\alpha)}(i) x(t - ih)\,, \tag{1}$$

where $i \in \mathbb{N}_0$, $t = kh \in (h\mathbb{N})_0$, $\alpha \in \mathbb{R}$ and $h > 0$.
Observe that the Grünwald–Letnikov difference operator can be extended to vector functions in the componentwise manner, then one can study the fractional order difference systems.

The following difference linear system is considered in direction of stability notions:

$$\left(\Delta_h^\alpha x\right)(t+h) = Ax(t)\,, \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, $\alpha \in (0,1]$, $h > 0$ and $x : (h\mathbb{N})_0 \to \mathbb{R}^n$ is the state. The only equilibrium point of linear systems with the Grünwald-Letnikov difference is the trivial solution $x \equiv 0$.

Let $\overline{x}(k) := \left(x_1(kh), x_2(kh), \ldots, x_n(kh)\right)^{\mathrm{T}} \in \mathbb{R}^n$. The fractional order difference system (2) is called *stable* if, for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\|\overline{x}(0)\| < \delta$ implies $\|\overline{x}(k)\| < \epsilon$, for all $k \in \mathbb{N}_0$ and it is *asymptotically stable* if it is stable and $\lim_{k\to+\infty} \overline{x}(k) = 0$.

Let $\mathrm{Spec}(A)$ be the set of eigenvalues of matrix A.

Proposition 1 (see [18]): If the following conditions are satisfied

(i) for all $i = 1, \ldots, n$

$$\arg \lambda_i \in \left[\alpha\frac{\pi}{2}, 2\pi - \alpha\frac{\pi}{2}\right]\,, \tag{3}$$

(ii) for all $i = 1, \ldots, n$

$$|\lambda_i| < |w_i|\,, \tag{4}$$

where $\arg \lambda_i$ and $|\lambda_i|$ are respectively the main argument and modulus of $\lambda_i \in \mathrm{Spec}(A)$ and

$$|w_i| = \left(\frac{2}{h}\left|\sin\frac{\arg\lambda_i - \alpha\frac{\pi}{2}}{2-\alpha}\right|\right)^\alpha\,, \tag{5}$$

then system (2), with $\alpha \in (0,1]$, is asymptotically stable.

Corollary 2: For the scalar system (2) with $A = \lambda < 0$ we have that it is asymptotically stable if $|\lambda| < \left(\frac{2}{h}\right)^\alpha$.

Remark 1: There is a possibility to have asymptotical stability in case if $\lambda = 0 \in \mathrm{Spec}(A)$, when zero is the single eigenvalue of A, however exact conditions for such case are an open problem.

II. Attitude and Opinion Change

Research on attitude and opinion change has a long history in psychology. Usually two frameworks of dynamics in opinion change are describe [20]. First, most common, is based on cause - effect assumptions. Usually researchers, during well-controlled experiments, are manipulating one or more variables and examine its impact on the dependent variable (e.g. opinion) in time. Within this methodology the role of a number of variables, which may affects people's opinions and attitudes were discovered and described.

It is well known effect that process of attitude change sometimes may appear as a result of discussion and logic consideration of opinions expressed during discussion (e.g. [21], [22], [23], [24]), but in most cases the mechanism of attitude change lays in persuasion. In this terms, opinion change is not a result of logical reasoning, weighing arguments etc., but stems from unconscious, intuitive and irrational heuristics and cognitive biases. E.g. a well-known literature on cognitive effort shows that not the content of the message is important in attitude change, but variables like being in good mood [25], fast speech [26]; inducing fear in message [27], clear intentions [28], the difference between two opinions [29], order of presentation [30] and much more (for a review: [31]).

Two of the most important variables in opinion change are persuasiveness of sender (source) and susceptibility to persuasion of a recipient of the message (audience). It is well known effect that the possibility to change one's opinion depends on attributes of a source like physical attractiveness (e.g. [32], [33], [34]), his/her expertise [35], gender (e.g. [36]), aggressiveness [37], trustworthiness [34], power [38], majority status [39] and more. On the other hand the susceptibility to persuasion of recipient (called persuadability or suggestibility) plays an important role too. People who are less intelligent [40], have lower self-esteem [41], lower need for cognition [42], lower dogmatism [43], younger [44] are more likely to change their opinions and attitudes.

Due to the newest psychological theories (called dual process models, like elaboration likelihood model; [45], [46]) attitudes change could be caused not by one, but at least by two different processes at the same time. Within this framework any variable (like previous mentioned persuasiveness or suggestibility) could operate differently in different situations. Hence, it is possible that in some cases the high attractiveness of a sender could encourage thinking and leads to opinion change or activating peripheral routes of cognitive processing and did not leads to opinion change, but even to opinion polarization. So the effect of persuasiveness and susceptibility to persuasion on opinion change is not clear and depends on different circumstances and variables.

One of the most important is strength of the opinion or personal engagement in opinion. A lot of experiments proved that the more important the opinion is for a recipient of message (namely, for his/her identity), the less likely he/she will change it [31], [47]. And the stronger the opinion the more resistant it is to change [48]. But the pattern is not as clear. For example, Petty, Cacioppo, and Goldman in [29] showed that, when the issue was important, the expressiveness of a source of a message plays almost no role in opinion change. But when the issue was unimportant, the opinion change was stronger when the persuasiveness of a source was higher. On the other hand, Kameda et. al. [49] showed that opinions in important topic were influenced most strongly by "cognitively central" group members, whose initial beliefs about the discussion topic overlapped the most with other members. Also there are some evidence suggests that "minors can change the opinion of many", only when the issue is important for them, the group is coherent and consequent despite the persuasiveness of units (for review see: [50]). To sum up it may be said that at least three important variables plays and important role in opinion change: widely understood persuasiveness, susceptibility to suggestion and importance of the opinion for self, but the relationship between them is not clear.

In the next section we will present the proposed model representing this relationship. We consider the discrete time domain. We also explain the model construction and discuss its behaviour.

III. Fractional model of opinion dynamics

In our research we investigate consensus model structures, that could be used in possibly general environment of agents. The desired model had the need of agent individualisation, possibility of structure changing, and easy addition of additional group members.

Moreover, our main goal was to include in the model psychologically justified parameters such as opinion weight, susceptibility, persuasibility and persons internal consistency. This model was created as all the consensus models are i.e. in a sense that each agents opinion is defined by their relation to those of other agents. In our model each agent is described by his own fractional difference equation such as:

$$\left(\Delta_h^\alpha x_i\right)(t+h) = s_i \sum_{j\neq i} p_j (x_j(t) - x_i(t)) \mathrm{e}^{-w_i(x_i(t)-x_j(t))^2} \quad i \in \{1,\ldots,N\}\,, \tag{6}$$

where

- $x_j(t)$ is the value/level of the opinion of j-th agent at time/step t;
- N is a number of agents in the system;
- s_i is the agent's susceptibility. Here we interpret it as a coefficient determining how interactions can influence the agent. If s_i is equal to 0, agent is completely resistant to others opinions, this coefficient is potentially unbounded;
- w_i is the agent's personal opinion weight, this is the source of systems main nonlinearity. The interpretation is as follows, if w_i is 0, then agents does not have much confidence in their opinion, that is why interaction with anybody can sway them towards their point of view. If w_i is large, then only opinions close to agent's own can influence them;
- p_i is the persuasibility, this coefficient represents how agent is good at convincing others. Agents with high persuasibility will convince others faster to their point of view;
- α is the model order, which corresponds to the internal consistency of the model, in general it can be different for each agent. If $\alpha = 1$, then we have the traditional differential equation. However, if order α is reduced below one, then fractional dynamics introduces influence of agent's history. In particular fractional model can represent agent's individual opinion evolution, even if at a moment removed from other interactions.

Note that a discrete form of the system (6) allows for both simulating solutions and applying the results presented earlier to investigate model behavior.

Most important aspect of system modelling is the investigation of stability. The investigation of behaviour of zero equilibrium of original system is not very interesting, as it corresponds to people with general neutral opinion. This also limits investigation to the case that everyone becomes neutral to the issue. More interesting is the investigation of differences between opinions. That is why we focus on consensus in the model in the following section.

IV. Consensus in the attitude dynamics model

In this section we assume for simplicity, that all orders are equal to α. In order to investigate consensus in the model we need to define new variables. Let us define $\eta_{i,j}(t) := x_i(t) - x_j(t)$, $1 \le j < i \le N$. Then we get a new state vector

$$\eta = (\eta_{2,1},\ \eta_{3,1},\ \eta_{3,2},\ \eta_{4,1},\ \eta_{4,2},\ \eta_{4,3},\ \ldots,\ \eta_{N,1},\ \eta_{N,2},\ \ldots,\ \eta_{N,N-1})^{\mathsf{T}}\,, \tag{7}$$

where $\eta \in \mathbb{R}^{0.5N(N-1)}$. Then, we get the following equation:

$$\left(\Delta_h^\alpha \eta\right)(t+h) = f(\eta(t))\,, \tag{8}$$

where the vector f is a function of $\eta_{i,j}$.

Remark 2: Construction of f can be easily explained using the following example. Let us consider $\eta_{2,1} = x_2 - x_1$, it can be easily seen that

$$\begin{aligned}\left(\Delta_h^\alpha x_1\right)(t+h) &= s_1 \sum_{j\neq 1} p_j (x_j(t) - x_1(t)) \mathrm{e}^{-w_1(x_1(t)-x_j(t))^2} \\ &= s_1 \Big(\sum_{j>1} p_j \eta_{j,1}(t) \mathrm{e}^{-w_1(\eta_{j,1}(t))^2} \Big)\end{aligned} \tag{9}$$

and similarly

$$\begin{aligned}\left(\Delta_h^\alpha x_2\right)(t+h) = s_2 \Big(& - p_1 \eta_{2,1}(t) \mathrm{e}^{-w_2(\eta_{2,1}(t))^2} + \\ & + \sum_{j>2} p_j \eta_{j,2}(t) \mathrm{e}^{-w_2(\eta_{j,2}(t))^2} \Big)\end{aligned} \tag{10}$$

in consequence

$$\begin{aligned}&(\Delta_h^\alpha \eta_{2,1})(t+h) = \\ &= \left(\Delta_h^\alpha x_2\right)(t+h) - \left(\Delta_h^\alpha x_1\right)(t+h) = \\ &= -\left(s_2 p_1 \mathrm{e}^{-w_2(\eta_{2,1}(t))^2} + s_1 p_2 \mathrm{e}^{-w_1(\eta_{2,1}(t))^2}\right)\eta_{2,1}(t) + \\ &\quad + s_2 \sum_{j>2} p_j \eta_{j,2}(t) \mathrm{e}^{-w_2(\eta_{j,2}(t))^2} + \\ &\quad - s_1 \sum_{j>2} p_j \eta_{j,1}(t) \mathrm{e}^{-w_1(\eta_{j,1}(t))^2} = f_1(\eta(t)) .\end{aligned} \tag{11}$$

In general one gets

$$\begin{aligned}\left(\Delta_h^\alpha x_k\right)(t+h) = s_k \Big(& - \sum_{j<k} p_j \eta_{k,j}(t) \mathrm{e}^{-w_k(\eta_{k,j}(t))^2} + \\ & + \sum_{j>k} p_j \eta_{j,k}(t) \mathrm{e}^{-w_k(\eta_{j,k}(t))^2} \Big) .\end{aligned} \tag{12}$$

Then for $k \geq 2$ and $\ell = 1, 2, \ldots, k-1$ one gets

$$\begin{aligned}&(\Delta_h^\alpha \eta_{k,\ell})(t+h) = \\ &= \left(\Delta_h^\alpha x_k\right)(t+h) - \left(\Delta_h^\alpha x_\ell\right)(t+h) = \\ &= -\left(s_k p_\ell \mathrm{e}^{-w_k(\eta_{k,\ell}(t))^2} + s_\ell p_k \mathrm{e}^{-w_\ell(\eta_{k,\ell}(t))^2}\right)\eta_{k,\ell}(t) + \\ &\quad + s_k \sum_{j>k} p_j \eta_{j,k}(t) \mathrm{e}^{-w_k(\eta_{j,k}(t))2} + \\ &\quad - s_k \sum_{j\neq \ell, j<k} p_j \eta_{k,j}(t) \mathrm{e}^{-w_k(\eta_{k,j}(t))^2} + \\ &\quad + s_\ell \sum_{j<\ell} p_j \eta_{\ell,j}(t) \mathrm{e}^{-w_\ell(\eta_{\ell,j}(t))^2} \\ &\quad - s_\ell \sum_{j\neq k, j>\ell} p_j \eta_{j,\ell}(t) \mathrm{e}^{-w_\ell(\eta_{j,\ell}(t))^2} = f_\iota(\eta(t)) ,\end{aligned} \tag{13}$$

where $\iota := \frac{k(k-1)}{2} - k + \ell + 1$.

As it can be seen, analysis of those functions might be complicated, it can be however easily seen that zero is the equilibrium point of the system. This equilibrium is important at it corresponds to the consensus between the agents - i.e. common opinions among them.

Let us linearize system (8) at $\eta^* = 0 \in \mathbb{R}^{0.5N(N-1)}$. Then we get the following system:

$$\left(\Delta_h^\alpha \eta\right)(t+h) = M\eta(t) , \tag{14}$$

where $M := f'(0)$.

Let $m \in \mathbb{N}$. In order to see how the matrix M looks like we introduce the following matrices:

$$\mathbb{P}_m = \begin{pmatrix} p_1 & p_2 & \ldots & p_m \\ p_1 & p_2 & \ldots & p_m \\ \vdots & \vdots & \ddots & \vdots \\ p_1 & p_2 & \ldots & p_m \end{pmatrix} \in \mathbb{R}^{m\times m},$$

$$\mathbf{1}_m := \begin{pmatrix} 1 & \ldots & 1 \end{pmatrix}^T \in \mathbb{R}^{m\times 1},$$

$$\mathbf{0}_m := \begin{pmatrix} 0 & \ldots & 0 \end{pmatrix}^T \in \mathbb{R}^{m\times 1},$$

$$\mathbf{p}_m := \begin{pmatrix} p_1 & \ldots & p_m \end{pmatrix} \in \mathbb{R}^{1\times m},$$

$\mathcal{S}_m := \operatorname{diag}(s_1, s_2, \ldots, s_m)$ is the diagonal matrix of dimension $m \times m$, and

$$\begin{aligned}D_m &:= - s_m \mathbb{P}_{m-1} - p_m \mathcal{S}_{m-1} \\ &= \begin{pmatrix} -s_m p_1 - s_1 p_m & \ldots & -s_m p_{m-1} \\ -s_m p_1 & \ldots & -s_m p_{m-1} \\ \vdots & \ddots & \vdots \\ -s_m p_1 & \ldots & -s_m p_{m-1} \\ -s_m p_1 & \ldots & -s_m p_{m-1} - s_{m-1} p_m \end{pmatrix} .\end{aligned}$$

Note that $D_m \in \mathbb{R}^{(m-1)\times(m-1)}$. Let $\mathbb{S}_n$, A_n and M_n be matrices defined respectively in recursive way as follows: $\mathbb{S}_2 := \begin{pmatrix} -s_1 & s_2 \end{pmatrix} \in \mathbb{R}^{1\times 2}$,

$$\mathbb{S}_n := \begin{pmatrix} \mathbb{S}_{n-1} & \mathbf{0}_{0.5(n-2)(n-1)} \\ -\mathcal{S}_{n-1} & s_n \mathbf{1}_{n-1} \end{pmatrix} \in \mathbb{R}^{0.5n(n-1)\times n} ,$$

for $n \geq 3$, $A_2 := \begin{pmatrix} -s_1 p_2 & s_2 p_1 \end{pmatrix}^T \in \mathbb{R}^{2\times 1}$,

$$A_n := \begin{pmatrix} A_{n-1} & -p_n \mathcal{S}_{n-1} \\ \mathbf{0}^T_{0.5(n-1)(n-2)} & s_n \mathbf{p}_{n-1} \end{pmatrix} \in \mathbb{R}^{n\times 0.5n(n-1)} , \tag{15}$$

for $n \geq 3$ and

$$M_2 := -s_1 p_2 - s_2 p_1 \tag{16a}$$

$$M_n := \begin{pmatrix} M_{n-1} & p_n \mathbb{S}_{n-1} \\ A_{n-1} & D_n \end{pmatrix} , n \geq 3 , \tag{16b}$$

where $M_n \in \mathbb{R}^{0.5n(n-1)\times 0.5n(n-1)}$.

Observe that for N agents we get $M = M_N$, where M_N is defined in the recursive way by (16).

Particularly, for $N = 2$ agents we have only one difference $\eta = x_2 - x_1$ and $M = M_2 = -s_1 p_2 - s_2 p_1$ Then linearized systems are only one-dimensional: $\left(\Delta_h^\alpha \eta\right)(t+h) = -(s_2 p_1 + s_1 p_2)\eta(t)$.

Moreover, for $N = 3$ one gets

$$M = M_3 = \begin{pmatrix} M_2 & -s_1 p_3 & s_2 p_3 \\ -s_1 p_2 & -s_1 p_3 - s_3 p_1 & -s_3 p_2 \\ s_2 p_1 & -s_3 p_1 & -s_3 p_2 - s_2 p_3 \end{pmatrix} ,$$

while for $N = 4$ we get

$$M = M_4 = \left(\begin{array}{c|c} M_3 & p_4 \mathbb{S}_3 \\ \hline A_3 & D_4 \end{array}\right) ,$$

where

$$p_4 \mathbb{S}_3 = \left(\begin{array}{cc|c} -s_1 p_4 & s_2 p_4 & 0 \\ \hline -s_1 p_4 & 0 & s_3 p_4 \\ 0 & -s_2 p_4 & s_3 p_4 \end{array}\right) ,$$

$$A_3 = \left(\begin{array}{c|cc} -s_1p_2 & -s_1p_3 & 0 \\ s_2p_1 & 0 & -s_2p_3 \\ \hline 0 & s_3p_1 & s_3p_2 \end{array}\right)$$

and

$$D_4 = \begin{pmatrix} -s_1p_4 - s_4p_1 & -s_4p_2 & -s_4p_3 \\ -s_4p_1 & -s_4p_2 - s_2p_4 & -s_4p_3 \\ -s_4p_1 & -s_4p_2 & -s_4p_3 - s_3p_4 \end{pmatrix}.$$

Moreover, for $N = 5$:

$$M = M_5 = \left(\begin{array}{c|c} M_4 & p_5\mathbb{S}_4 \\ \hline A_4 & D_5 \end{array}\right),$$

where

$$p_5\mathbb{S}_5 = \left(\begin{array}{ccc|c} -s_1p_5 & s_2p_5 & 0 & 0 \\ -s_1p_5 & 0 & s_3p_5 & 0 \\ 0 & -s_2p_5 & s_3p_5 & 0 \\ \hline -s_1p_5 & 0 & 0 & s_4p_5 \\ 0 & -s_2p_5 & 0 & s_4p_5 \\ 0 & 0 & -s_3p_5 & s_4p_5 \end{array}\right),$$

$$A_4 = \left(\begin{array}{ccc|ccc} -s_1p_2 & -s_1p_3 & 0 & -s_1p_4 & 0 & 0 \\ s_2p_1 & 0 & -s_2p_3 & 0 & -s_2p_4 & 0 \\ 0 & s_3p_1 & s_3p_2 & 0 & 0 & -s_3p_4 \\ \hline 0 & 0 & 0 & s_4p_1 & s_4p_2 & s_4p_3 \end{array}\right)$$

and

$$D_5 = \begin{pmatrix} d_1 & -s_5p_2 & -s_5p_3 & -s_5p_4 \\ -s_5p_1 & d_2 & -s_5p_3 & -s_5p_4 \\ -s_5p_1 & -s_5p_2 & d_3 & -s_5p_4 \\ -s_5p_1 & -s_5p_2 & -s_5p_3 & d_4 \end{pmatrix}$$

for $d_i := -s_5p_i - s_ip_5$, $i = 1, 2, 3, 4$.

Note that in [19] the relations between the asymptotic stability of nonlinear fractional order difference systems and their linearizations are presented. In [19] we consider systems where one of the Caputo-type, Riemann-Liouville-type or Grünwald-Letnikov-type difference operators (denoted by Υ_h^α in [19]) is used on the left hand side of the system. Then $(\Upsilon_h^\alpha \eta)(t) = (\Delta_h^\alpha \eta)(t + h)$. Additionally, the Mittag-Leffler function $E_{\alpha,\beta}(M, k)$ used in the paper [25] (now [19]) can be expressed as follows $E_{\alpha,\beta}(M, k) = \sum_{i=0}^{\infty}(-1)^{k-i}\binom{-i\alpha-\beta}{k-i}M^i$. Therefore the results presented in [19] can be used for systems with the Grünwald-Letnikov-type operators and the following lemma gives the relations between asymptotic stability of systems (8) and (14):

Lemma 3: Assume that $f(\eta) - M\eta = o(\|\eta\|)$ uniformly as $\|\eta\| \to 0$. If the linear system (14) is asymptotically stable and there exist $k_0 \in \mathbb{N}_0$, $m_\beta \geq 1$ and $q \in (0, 1)$ such that $\| \sum_{i=0}^{\infty}(-1)^{k-i}\binom{-i\alpha-\beta}{k-i}M^i\| \leq m_\beta q^k$ for $k \geq k_0$, then the zero solutions of the nonlinear system (8) is asymptotically stable.

Let us now present the behaviour of the considered models.

Proposition 4: Let x_i for $i \in \{1, \ldots, N\}$ evaluate according to system (6) for $\alpha \in (0, 1]$. If for each $\lambda \in \mathrm{Spec}(M)$ conditions (i) and (ii) from Proposition 1 hold, then system (8) is asymptotically stable, i.e. $|x_i(kh) - x_j(kh)| \to 0$ with $k \to \infty$, for $\eta_{i,j}(0)$ small enough.

Proof: The fact that for each $\lambda \in \mathrm{Spec}(M)$ conditions (i) and (ii) from Proposition 1 hold for $\alpha \in (0, 1]$ is equivalent to asymptotic stability of system $(\Delta_h^\alpha \eta)(t+qh) = M\eta(t)$ by Proposition 1. Then for $\eta_{i,j}(0)$ small enough $\eta(t) \to 0$ with $t \to \infty$ and consequently the thesis holds. ■

Corollary 5: For the scalar system $(\Delta_h^\alpha \eta)(t + h) = -(s_2p_1 + s_1p_2)\eta(t)$ we have that it is asymptotically stable if $s_2p_1 + s_1p_2 < \left(\frac{2}{h}\right)^\alpha$. And then, initially nonlinear discrete-time is locally asymptotically stable.

Corollary 6: For $N = 3$ agents we receive the following system of differences between opinions: $\begin{pmatrix} (\Delta_h^\alpha \eta_{2,1})(t + h) \\ (\Delta_h^\alpha \eta_{3,1})(t + h) \\ (\Delta_h^\alpha \eta_{3,2})(t + h) \end{pmatrix} = M_3\eta(t)$. The matrix of the system has only real eigenvalues, one of them is equal zero. Hence, the system is stable if $s_1(p_2 + p_3) + s_2(p_1 + p_3) + s_3(p_1 + p_3) < \left(\frac{2}{h}\right)^\alpha$. And then, initially nonlinear discrete–time is locally stable.

V. Numerical simulations

We consider now particular examples with different situations between agents in systems, consisting of progressively more complex group of agents and discuss their behaviour using psychological observations.There are presented and described examples, where act two, three, five or fifty agents. All simulations were conducted in Maple using discrete fractional differences presented in the preliminaries.

Example 1: Let us consider firstly the group of two agents with parameters presented inside the equations

$$\begin{aligned} (\Delta_h^\alpha x_1)(t) &= 20 \cdot 0.3(x_2(t) - x_1(t))\mathrm{e}^{-0.3(x_2(t)-x_1(t))^2} \\ (\Delta_h^\alpha x_2)(t) &= 0.7(x_1(t) - x_2(t))\mathrm{e}^{-0.7(x_1(t)-x_2(t))^2} \end{aligned}$$

with initial values $x_1(0) = 1$, $x_2(0) = 5$. Observe that we have $s_1p_2 + s_2p_1 = 6.7 < (200)^\alpha$, for $\alpha > \frac{\ln 6.7}{\ln 200} \approx 0.359$. Hence, for orders big enough system reaches consensus based on Corollary 5. Description of the situation presented by by plots in Figure 1 is the following

- Agent 2 is more susceptible. Agent 1 is more persuasible.
- First reaction of Agent 1 is to withdraw from his position to more neutral stance. It is caused by confrontation with Agent 2, whose opinion is radically different and at the same time much more important.
- It can be interpreted as a defensive reaction to usually more emotionally invested disputant. Typically less invested person initially becomes disinterested in the subject.
- However after the initial turmoil agents start to discuss the issue that interests them. That allows getting a consensus.

Example 2: Now we consider the group of $N = 3$ agents and behaviour represented by system (6) with vector parameters $p := [20, 1, 20]$, $w = [0.8, 0.1, 0.1]$, $s = [0.8, 0.1, 0.1]$ and initial values of opinions: $x_1(0) = 1$, $x_2(0) = 5$, $x_3(0) = 8$. Observe that we have $s_1p_2 + s_1p_3 + s_2p_1 + s_2p_3 + s_3p_1 + s_3p_2 = 22.9 < (200)^\alpha$, for $\alpha > \frac{\ln 22.9}{\ln 200} \approx 0.591$.

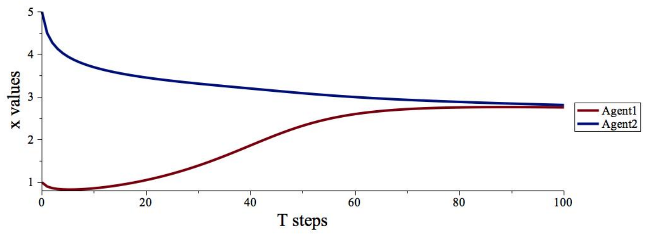

Fig. 1. System with two agents presented in Example 1. Differing susceptibilities cause differences in ways of approaching consensus. $x_1(0) = 1$, $x_2(0) = 5$, $h = 0.01$, $\alpha = 0.9$, $T = 100$ steps.

Hence, for orders big enough system reaches consensus based on Corollary 6.

Description of the situation presented by by plots in Figure 2 is the following

- Agent 1 has a relatively neutral opinion on the discussed issue. However, his opinion is very important to him. He is however most susceptible of the group.
- Agents 2 and 3 have much more radical opinion on the issue. However both of them are not very invested in their opinions (weight of 0.1). Agent 3 is much more persuasive person than Agent 2.
- Agent 1 meeting two radicals decides to withdraw to almost fully neutral position.
- Agents 2 and 3 observing his behaviour start to alter their stance towards more neutral position.
- When Agents 2 and 3 come position that is more acceptable to Agent 1 he starts moving towards consensus.
- Difference in relative changes of opinion is explainable by the fact, that opinion of Agent 1 was much more important for him.

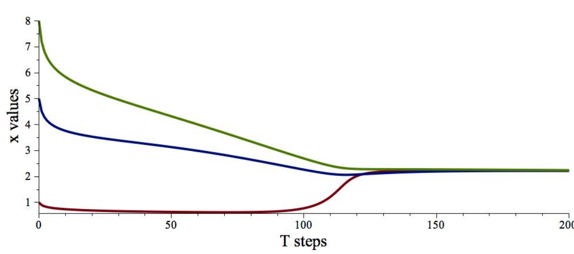

Fig. 2. System with three agents presented in Example 2. We observe behaviour consistent with Corollary 6. $x_1(0) = 1$, $x_2(0) = 5$, $x_3(0) = 8$, $h = 0.01$, $\alpha = 0.9$, $p := [20, 1, 20]$, $w = [0.8, 0.1, 0.1]$, $s = [0.8, 0.1, 0.1]$, $T = 200$ steps

Example 3: Population of agents from Example 2 was supplemented by two more. Behaviour of agents is represented by system (6) with vector parameters $p := [20, 1, 20, 5, 60]$, $w = [0.8, 0.1, 0.1, 0.9, 0.9]$, $s = [0.8, 0.1, 0.1, 0.8, 0.9]$ and initial values of opinions: $x(0) = [1, 5, 8, 10, 15]$. It is interesting that we can observe reaching the consensus in time period $T = 1000$ steps only for narrow interval of order. In numerical simulations, done in Maple, we found that then $\alpha \in (0.84; 0.91)$. For greater order, but less than 1, the time of reaching the consensus for all agents need more steps. However, for $\alpha = 1$ there is no consensus in the systems, see Figure 4, similarly, but with different kind of divergency for order smaller than 0.84.

Description of the situation presented by by plots in Figure 3 is the following

- Agent 4 is more radical, however he is not persuasible and is rather susceptible. His opinion is very important to him.
- Agent 5 is most radical of the group, and is at least 3 times as persuasible as the rest of the group. He is also most susceptible. His opinion is very important to him.
- As before we can see, that presence of a radical causes others to move towards more neutral stances.
- High persuasibility of Agent 5 allows him to obtain common opinion with agents 3 and 4 rather quickly and soon after that with agent 2.
- Agent 1 keeps his opinion close to full neutrality.
- Over time however, high susceptibility of this new group leader (Agent 5) causes him to become more neutral. It is an influence of rather persuasible Agent 1.
- Effect of Agent 1 can be understood as influence of his consistency. It is well known in psychology (see [51]) that consistent minority can influence majority.

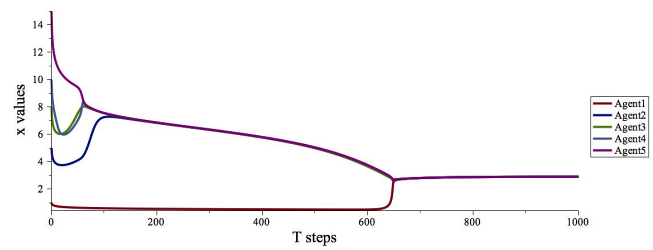

Fig. 3. System with five agents from Example 3. Only certain orders of $\alpha \in (0.84; 0.91)$ allow reaching consensus. $x(0) = [1, 5, 8, 10, 15]$, $h = 0.01$, $\alpha = 0.9$, $p := [20, 1, 20, 5, 60]$, $w = [0.8, 0.1, 0.1, 0.9, 0.9]$, $s = [0.8, 0.1, 0.1, 0.8, 0.9]$, $T = 1000$ steps.

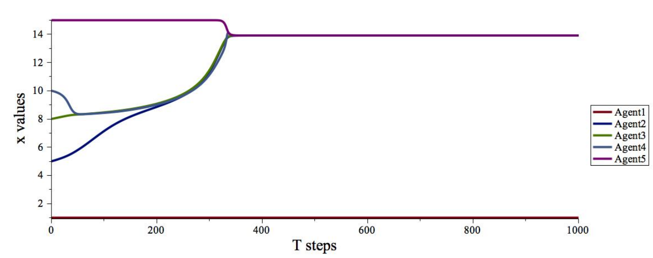

Fig. 4. System with five agents from Example 3. Integer order model does not reach consensus. $x(0) = [1, 5, 8, 10, 15]$, $h = 0.01$, $\alpha = 1$, $p := [20, 1, 20, 5, 60]$, $w = [0.8, 0.1, 0.1, 0.9, 0.9]$, $s = [0.8, 0.1, 0.1, 0.8, 0.9]$, $T = 1000$ steps.

Example 4: We introduce now to the system the 6th agent with opposite value of opinion with minus at the beginning and different values of parameters: p_6, w_6, s_6. We can observe that with the same values for the previous five agents and new vector parameters $p := [20, 1, 20, 5, 60, 20]$, $w =$

$[0.8, 0.1, 0.1, 0.9, 0.9, 0.1]$, $s = [0.8, 0.1, 0.1, 0.8, 0.9, 0.1]$ and initial values of opinions: $x(0) = [1, 5, 8, 10, 15, -5]$ for order $\alpha = 0.9$ consensus is still reached, see Figure 5. Moreover, the condition for local stability is not fulfilled in this case.

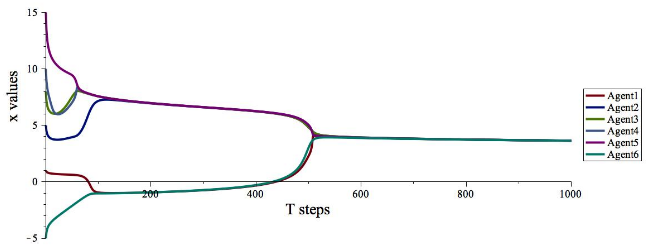

Fig. 5. Six agent system from Example 4. Despite not fulfilling linearized condition for stability the consensus is still reached. $x(0) = [1, 5, 8, 10, 15, -5]$, $h = 0.01$, $\alpha = 0.9$, $p := [20, 1, 20, 5, 60, 20]$, $w = [0.8, 0.1, 0.1, 0.9, 0.9, 0.1]$, $s = [0.8, 0.1, 0.1, 0.8, 0.9, 0.1]$, $T = 1000$ steps.

However, for the order $\alpha = 1$ we do not have consensus for both with five or with six agents (Fig. 6). There are two clusters of opinions, similarly like in Figure 4.

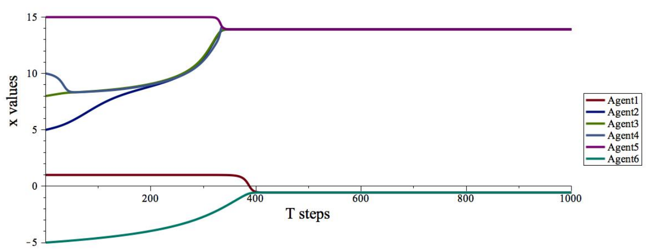

Fig. 6. Six agent system from Example 4. Integer order model exhibits clustering. $x(0) = [1, 5, 8, 10, 15, -5]$, $h = 0.01$, $\alpha = 1$, $p := [20, 1, 20, 5, 60, 20]$, $w = [0.8, 0.1, 0.1, 0.9, 0.9, 0.1]$, $s = [0.8, 0.1, 0.1, 0.8, 0.9, 0.1]$, $T = 1000$ steps.

As it can be observed in all above examples behaviour of agents is actually consistent with people's behaviour that can be observed in psychological context. We can also observe that the fractional order seems justified as integer order cases in Examples 3 and 4 show hard to explain behaviour.

There is some evidence suggesting, that our model is empirically valid. For example research on interrogative suggestibility [52], social influence [53] and false confessions [54] show, that people who are more prone to suggestions, changes their opinion according to opinion of more pursuable person, and magnitude of this change is depended on susceptibility. This supports our Example 1. On the other hand [55] in two well-designed experiments showed that in group of three persons the influence of confident agents might drastically change one's opinion in repeated interactions. Participants who took part in discussion changed their opinion to achieve consensus with more confident persons (namely, people who has very high personal opinion weight). Similar expert's effects were shown by Moussaid et. al [56]: persons with high confidence (personal opinion weight) changed opinion of participants with low confidence. It seems that participants tried to adjust their opinions to opinion of more confident interlocutor, despite individual characteristics of interlocutor (eg. susceptibility). This experiments partially supports our Example 2.

VI. Conclusions

The future step is to provide an experimental validation of the proposed model. At the moment we intend to perform two type of experiments including actual subjects. We will investigate binary interactions between people in large groups. Because of such limitation, the identification of parameters should be easier, and on the other hand realisation of multiple pairs will result in proper statistical distribution. Second type of experiments will include large group of people interacting with each other in a closed room using monitoring to map the interactions along with their durations. Both before and after the experiment the participants will be surveyed in order to determine their initial and post interaction attitudes. Second significant area of further investigation are the mathematical properties of the model. Analysis of nonlinear systems of fractional differential equations is an emerging topic, which requires careful analysis. One of the methods that can be used is the fractional analog of Lyapunov theory.

References

[1] U. Krause, A discrete nonlinear and non-autonomous model of consensus formation, Proc. Commun. Difference Equations (2000) 227–236.

[2] V. D. Blondel, J. M. Hendrickx, F. N. Tsitsiklis, On Krause's multi-agent consensus model with state-dependent connectivity, IEEE Transactions on Automatic Control 5 (11) (2009) 2586–2597.

[3] E. Girejko, L. Machado, A. B. Malinowska, N. Martins, Krause model of opinion dynamics on isolated time scales, Mathematical Methods in the Applied Sciences (2016) 13p.

[4] D. Lee, M. W. Spong, Stable flocking of multiple inertial agents on balanced graphs, IEEE Trans. Autom. Control 52 (8) (2007) 1469–1475.

[5] R. Olfati-Saber, Flocking for multi-agent dynamic systems: Algorithms and theory, IEEE Trans. Autom. Control 51 (3) (2006) 401–420.

[6] G. Xie, L. Wang, Consensus control for a class of networks of dynamic agents, Int. J. Robust Nonlinear Control 17 (10-11) (2007) 941–959.

[7] G. Lafferriere, A. Williams, J. Caughman, J. J. P. Veerman, Decentralized control of vehicle formations, Mathematical Problems in Engineering 54 (9) (2005) 899–910.

[8] W. Ren, E. M. Atkins, Distributed multi-vehicle coordinated control via local information exchange, Int. J. Robust Nonlinear Control 17 (10–11) (2007) 1002–1033.

[9] H. G. Tanner, A. Jadbabaie, G. J. Pappas, Flocking in fixed and switching networks, IEEE Trans. Autom. Control 52 (5) (2007) 863–868.

[10] E. Girejko, D. Mozyrska, Opinion dynamics and fractional operators, in: D. Baleanu, J. T. Machado (Eds.), International Conference on Fractional Differentiation and its Applications : FDA2014, University of Catania, Catania, 2014, p. 4 pages.

[11] G. Ren, Y. Yu, C. Xu, X. Hai, Consensus of fractional multi-agent systems by distributed event-triggered strategy, Nonlinear Dynamics 96 (2019) 541–555. doi:10.1007/s11071-018-4580-8.

[12] Y. Ye, H. Su, Leader-following consensus of nonlinear fractional-order multi-agent systems over directed networks, Nonlinear Dynamics 96 (2019) 1391–1403. doi:10.1007/s11071-019-04861-6.

[13] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific Publishing Company, Singapore, 2000.

[14] T. Kaczorek, Fractional positive linear systems, Kybernetes 38 (7/8) (2009) 1059–1078.

[15] P. Ostalczyk, Discrete fractional calculus. Applications in control and image processing, World Scientific Publishing Co Pte Ltd, vol. Series in Computer Vision - Vol. 4, 2016.
[16] I. Podlubny, Fractional Differential Equations. Mathematics in Sciences and Engineering, 198, Academic Press, San Diego, 1999.
[17] T. Kaczorek, Selected problems of fractional systems theory, Vol. 411, Springer, 2011.
[18] D. Mozyrska, M. Wyrwas, Stability of discrete fractional linear systems with positive orders, IFAC-PapersOnLine 11 (1) (2017) 8115–8120. doi:10.1016/j.ifacol.2017.08.1250.
[19] D. Mozyrska, M. Wyrwas, Stability by linear approximation and the relation between the stability of difference and differential fractional systems, Mathematical Methods in the Applied Sciences 40 (11) (2017) 4080–4091. doi:10.1002/mma.4287.
[20] A. Nowak, M. Lewenstein, Dynamical systems: A tool for social psychology?, Academic Press, 1994.
[21] J. Habermas, The theory of communicative action, volume i, Boston: Beacon (1984).
[22] J. S. Fishkin, The voice of the people: Public opinion and democracy (1997).
[23] M. Neblo, Thinking through democracy: Between the theory and practice of deliberative politics, Acta politica 40 (2) (2005) 169–181.
[24] J. T. Machado, Mathematical and computational modeling of political systems, Nonlinear Dynamics 96 (2019) 1471–1490. doi:10.1007/s11071-019-04865-2.
[25] R. Ziegler, Mood and processing of proattitudinal and counterattitudinal messages, Personality and Social Psychology Bulletin 39 (4) (2013) 482–495. arXiv:http://dx.doi.org/10.1177/0146167213478020, doi:10.1177/0146167213478020.
[26] J. MacLachlan, M. H. Siegel, Reducing the costs of tv commercials by use of time compressions, Journal of Marketing Research 17 (1) (1980) 52–57.
[27] K. Witte, M. Allen, A meta-analysis of fear appeals: implications for effective public health campaigns., Health education & behavior : the official publication of the Society for Public Health Education 27 5 (2000) 591–615.
[28] E. Walster, L. Festinger, The effectiveness of "overheard" persuasive communications, Journal of Abnormal and Social Psychology 65 (6) (1962) 395–402, cited By 73. doi:10.1037/h0041172.
[29] R. E. Petty, J. T. Cacioppo, R. Goldman, Personal involvement as a determinant of argument-based persuasion., Journal of personality and social psychology 41 (5) (1981) 847.
[30] C. P. Haugtvedt, D. T. Wegener, Message order effects in persuasion: An attitude strength perspective, Journal of Consumer Research 21 (1) (1994) 205. arXiv:/oup/backfile/Content_public/Journal/jcr/21/1/10.1086/209393/2/21-1-205.pdf, doi:10.1086/209393.
[31] R. E. Petty, D. T. Wegener, The handbook of social psychology, volume 1 and 2 (1998).
[32] J. Horai, N. Naccari, E. Fatoullah, The effects of expertise and physical attractiveness upon opinion agreement and liking, Sociometry 37 (4) (1974) 601–606.
[33] M. Snyder, M. Rothbart, The effects of communicator attractiveness on opinion change, Canadian Journal of Behavioral Sciences 3 (1971) 377–387.
[34] J. R. Priester, R. E. Petty, The influence of spokesperson trustworthiness on message elaboration, attitude strength, and advertising effectiveness, Journal of Consumer Psychology 13 (4) (2003) 408–421.
[35] C. L. Hafer, K. L. Reynolds, M. A. Obertynski, Message comprehensibility and persuasion: Effects of complex language in counterattitudinal appeals to laypeople, Social Cognition 14 (4) (1996) 317–337.
[36] A. H. Eagly, L. L. Carli, Sex of researchers and sex-typed communications as determinants of sex differences in influenceability: a meta-analysis of social influence studies., Psychological Bulletin 90 (1) (1981) 1.
[37] C. J. Wigley III, J. McCroskey, J. Daly, M. Martin, M. Beatty, Verbal aggressiveness, Communication and personality: Trait perspectives (1998) 191–214.
[38] S. T. Fiske, B. Morling, L. E. Stevens, Controlling self and others: A theory of anxiety, mental control, and social control, Personality and Social Psychology Bulletin 22 (2) (1996) 115–123.
[39] W. Wood, S. Lundgren, J. A. Ouellette, S. Busceme, T. Blackstone, Minority influence: a meta-analytic review of social influence processes. (1994).
[40] N. Rhodes, W. Wood, Self-esteem and intelligence affect influenceability: The mediating role of message reception., Psychological Bulletin 111 (1) (1992) 156.
[41] I. L. Janis, Personality correlates of susceptibility to persuasion1, Journal of Personality 22 (4) (1954) 504–518. doi:10.1111/j.1467-6494.1954.tb01870.x.
[42] J. T. Cacioppo, R. E. Petty, The need for cognition., Journal of personality and social psychology 42 (1) (1982) 116.
[43] M. Rokeach, The open and closed mind, The Personnel and Guidance Journal 39 (8) (1961) 674–680. doi:10.1002/j.2164-4918.1961.tb02039.x.
[44] S. J. Ceci, M. Bruck, Suggestibility of the child witness: A historical review and synthesis., Psychological bulletin 113 (3) (1993) 403.
[45] R. E. Petty, J. T. Cacioppo, The elaboration likelihood model of persuasion, in: Advances in experimental social psychology, Vol. 19, Academic Press, 1986, pp. 123–205.
[46] R. E. Petty, P. Brinol, D. Kenrick, N. Goldstein, S. Braver, A multi-process approach to social influence, Six degrees of social influence: Science, application, and the psychology of Robert Cialdini (2012) 49–58.
[47] J. R. Zuwerink, P. G. Devine, Attitude importance and resistance to persuasion: It's not just the thought that counts., Journal of Personality and Social Psychology 70 (5) (1996) 931.
[48] R. E. Petty, J. A. Krosnick, Attitude Strength: Antecedents and Consequences, Taylor & Francis, 1995.
[49] T. Kameda, Y. Ohtsubo, M. Takezawa, Centrality in sociocognitive networks and social influence: An illustration in a group decision-making context., Journal of personality and social psychology 73 (2) (1997) 296.
[50] W. Wood, Attitude change: Persuasion and social influence, Annual review of psychology 51 (1) (2000) 539–570.
[51] S. Moscovici, E. Lage, M. Naffrechoux, Influence of a consistent minority on the responses of a majority in a color perception task, Sociometry 32 (4) (1969) 365–380.
[52] G. Gudjonsson, The psychology of interrogations and confessions: a handbook, Wiley series in psychology of crime, policing, and law, Wiley, 2003.
[53] S. E. Asch, Effects of group pressure upon the modification and distortion of judgments., in: Groups, leadership and men; research in human relations., Carnegie Press SN -, Oxford, England, 1951, pp. 177–190.
[54] S. M. Kassin, K. L. Kiechel, The social psychology of false confessions: Compliance, internalization, and confabulation, Psychological Science 7 (3) (1996) 125–128. doi:10.1111/j.1467-9280.1996.tb00344.x.
[55] D. Mallinson, P. Hatemi, The effects of information and social conformity on opinion change, PLoS One 13 (5) (5 2018). doi:10.1371/journal.pone.0196600.
[56] M. Moussaïd, J. E. Kämmer, P. P. Analytis, H. Neth, Social influence and the collective dynamics of opinion formation, PLOS ONE 8 (11) (2013) 1–8. doi:10.1371/journal.pone.0078433.