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Reachable Set Estimation for Discrete-time Periodic Piecewise Systems

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Abstract

This paper investigates the reachable set estimation problem for discrete-time periodic piecewise systems subject to bounded-peak disturbances for the first time. Based on the periodic linear-interpolative formulation, the discrete time-scheduling Lyapunov functions with non-jumping or jumping modes are constructed to develop criteria of reachable set estimation that can ensure the asymptotic stability and reachability of the investigated system. Moreover, an index optimizing the bounding region of the desirable reachable set is given via resorting to the ellipsoid technique, and their results are compared. Finally, numerical examples are given to validate the effectiveness of the proposed results.

1. INTRODUCTION

Periodic systems, which can be used to characterize engineering dynamics with cyclic behavior, are intermediate systems connecting time-varying and time-invariant systems, whose applications widely exist in various fields. In recent years, periodic piecewise systems (PPSs), an effective tool for modeling and controlling periodic time-varying dynamics, have attracted increasing research attention since they can get rid of the difficulties caused by traditional methods like the Floquet theory [1]. Apart from periodic properties, PPS is also regarded as a special switched system, in which the subsystem has inherently predetermined switching rules and dwell time over one period. Concerning practical examples of periodic piecewise models are easy to find, such as vibrating conveyor systems [2], mass-spring-damper systems with periodic dynamics [3], and PWM voltage-controlled DC-DC converters [4]. On the basis of switching theory and Lyapunov techniques, many research results on the control and estimation of periodic piecewise systems in the continuous-time domain have been reported. For time-invariant subsystem dynamics, with different methods, the stability, L_2-gain analysis, and the corresponding controller were studied in [5, 6], and moreover, the fault detection observer and the guaranteed cost controller were designed in [7, 8]. To more closely approximate the original periodic system, a periodic piecewise model with time-varying subsystems is developed in [9], followed by the research of this model's L_1 performance and positivity analysis [10], non-fragile controller design [11, 12], H_∞ tracking control scheme [13], Bumpless H_∞ controller design [14], and so on.

The set of all terminal states that a dynamic system can reach with a prescribed initial state and disturbance is called the reachable set, which is an important concept in control engineering and has received much attention in the past decades. In practical engineering, it is usually difficult to accurately capture the characteristics of the system reachable set, which gives rise to the problem of reachable set estimation. A common research means for this problem is to resort to ellipsoid techniques constraining the reachable set to a region as compact as possible. Reachable set estimation can determine whether a state is moving in a prescribed region, which can therefore be used used in practical applications like robot obstacle avoidance [15] and safety monitoring [16]. Specifically, it can be conducted by checking whether the intersection of an insecure region and an estimated reachable set. Up to now, many results concerning reachable set estimation have been represented [15, 17–23]. For instance, In [15], the reachable set estimation and synthesis criteria of discrete-time switching systems were derived, and three optimization frameworks were proposed to shrink bounding

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ellipsoids, where the optimal design parameters can be solved by genetic algorithm. [17] proposed a sufficient condition for the ellipsoidal boundary of the reachable set estimation of singular systems and extends the result to the scenarios of time-varying delay. In [18], the reachable set estimation problem is reconstructed as a chance-constrained optimization problem, and the accuracy of the reachable set estimation was studied by using scenario optimization. Recently, for continuous-time periodic piecewise systems, in [19, 20], a low conservative reachable set estimation condition is given by the subinterval segmentation method and the Bernstein polynomial method, respectively. Moreover, the problem of observer-based output reachable set synthesis for periodic piecewise systems was reported in [21]. However, it is worth noting that discrete-time plants are favored in the engineering field due to their advantages in numerical computations, whereas for the discrete-time periodic piecewise system, corresponding works have not received adequate attention they deserve[24, 25], especially when it comes to reachable set estimation problems.

Motivated by the above, in this work, we focus on the reachable set estimation problem for discrete-time periodic piecewise systems with bounded-peak disturbances. With the discrete time-scheduling Lyapunov functions and ellipsoid techniques, the tractable conditions of estimating reachable sets and optimizing bounding regions for discrete-time periodic piecewise systems are obtained. The main contributions of this work are twofold:

- The criterion of reachable set estimation for discrete-time PPSs is proposed for the first time, and the desirable conditions for optimizing bounding ellipsoids are given. Compared with the continuous plant, the obtained conditions are easy to deal with and beneficial to engineering applications.

- Compared with the results of the estimation of the reachable sets of discrete periodic systems [15], the conditions proposed in this work have a larger solution space because the time-varying Lyapunov function is provided instead of the time-invariant one.

Notations:
$\mathbb{R}^n$ stands for the n-dimensional Euclidean space. The superscript T represents the transpose of a matrix. $M \leq 0$ (resp., $M < 0$) indicates that the matrix is negative semidefinite (resp., negative definite). 0 and I stand for the zero matrix and the identity matrix of appropriate dimensions, respectively. "$*$" in symmetric block matrices is used to denote an ellipsis of the symmetric terms.

2. Problem Formulation and Preliminaries

Consider a discrete-time PPS as

$$x(k+1) = A(k)x(k) + B_w(k)\omega(k), \tag{1}$$

where $x(k) \in \mathbb{R}^{n_x}$ and $y(k) \in \mathbb{R}^{n_y}$ are the system state and the measurement output, respectively. $\omega(k) \in \mathbb{R}^{n_\omega}$ is the bounded-peak disturbance vector and assumed to be measurable, satisfying

$$\omega^T(k)\omega(k) \leq \bar{\omega}^2, \forall k \geq 0, \tag{2}$$

where $\bar{\omega}$ is a known scalar describing the bound of the disturbance. $A(k) = A(k+T_p), B_w(k) = B_w(k+T_p)$ are T_p-periodic matrix functions, where each fundamental period is split into S subintervals $k = \{\ell T_p + k_{i-1}, \dots, \ell T_p + k_i\}$, $\ell = 0, 1, 2, \dots$, $i \in \mathscr{S} = \{1, 2, \dots, S\}$. k_{i-1} is the switching instant in the period from the $(i-1)$th subsystem to ith subsystem, where $k_0 = 0$ and $k_S = T_p$. In ith subsystem, the dwell time is defined as $T_i = k_i - k_{i-1}$ with $\sum_{i=1}^{S} T_i = T_p$, and the system parameter sets $\{A(k), B_w(k)\}$ can be characterized by the time-invariant matrix sets $\{A_i, B_{wi}\}$, $i \in \mathscr{S}$. Then, the system (1) becomes:

$$x(k+1) = A_i x(k) + B_{wi}\omega(k), \tag{3}$$

where $A_{S+1} = A_1, B_{w,S+1} = B_{w1}$ are known constant matrices with appropriate dimensions.

Let us concern the reachability of the system state $x(k)$, the reachable set of the PPS (3) is defined as

$$\mathscr{R}_x = \big\{x \in \mathbb{R}^{n_x} | x(0) = 0, x(k), \omega(k) \text{ satisfy (1) and (2)}, k \geq 0\big\} \tag{4}$$

The reachable set estimation problem of the PPS (3) devotes to finding a region $\bar{\mathscr{E}}_s$ as compact as possible to constrain the reachable set under bounded-peak disturbances assumed in (2), in which $\bar{\mathscr{E}}_s$ is described by $\bar{\mathscr{E}}_s \triangleq \bigcup_{0 \leq k \leq T_p} \mathscr{E}(P(k))$ with

$$\mathscr{E}(P(k)) \triangleq \big\{x \in \mathbb{R}^{n_x} | x^T P(k) x \leq 1, P(k) > 0\big\}, \tag{5}$$

and $P(k)$ is the discrete time-varying matrix function whose specific form will be given in the sequel.

Remark 1. *Notice that, it is not difficult to find that a common method in previous results on reachable set estimation is that the time-invariant Lyapunov matrix is used to determine a bounding ellipsoid to estimate the reachable set of the dynamic system. The bounding ellipsoid is defined as* $E(P) \triangleq \big\{x \in \mathbb{R}^{n_x} | x^T P x \leq 1, P > 0\big\}$. *To ensure the advantage in conservatism and feasible sets, the bounding region described in (5) will be considered in the later development of this paper.*

The following lemma is a basic tool for the reachable set estimation for discetre-time PPSs.

Lemma 1. *Consider the PPS (3) with the bounded-peak disturbance satisfying (2). Given a Lyapunov function $V(k) = V_i(k)$ satisfying $V(0) = 0$ and $V(k) > 0$, for $k \in \{\ell T_p + k_{i-1}, \ldots, \ell T_p + k_i - 1\}, \ell = 0, 1, 2, \ldots, i \in \mathscr{S}$, if there exist scalars $0 < \delta_i < 1$, $i \in \mathscr{S}$, such that*

$$V_i(k+1) - \delta_i V_i(k) - \frac{1-\delta_i}{\bar{\omega}^2}\omega^T(k)\omega(k) \leq 0, \qquad (6)$$

then the system is asymptotically stable and under zero initial conditions, $V(k) \leq 1$, which implies that $\mathscr{R}_x \subseteq \bar{\mathscr{E}}_s \triangleq \bigcup_{0 \leq k \leq T_p} \mathscr{E}(P(k))$, where $\mathscr{E}(P(k))$ defined in (5).

Proof. *As we know, from the previous assumption, the bounded-peak disturbance satisfies (2), and the scalars $0 < \delta_i < 1$, $i \in \mathscr{S}$, intuitively, for $k \in \{\ell T_p + k_{i-1}, \ldots, \ell T_p + k_i - 1\}, \ell = 0, 1, 2, \ldots, i \in \mathscr{S}$, it can be obtained that*

$$\begin{aligned} V_i(k+1) - \delta_i V_i(k) &\leq \frac{1-\delta_i}{\bar{\omega}^2}\omega^T(k)\omega(k), \\ &\leq 1 - \delta_i, \end{aligned} \qquad (7)$$

which implies

$$V_i(k+1) - 1 \leq \delta_i\big(V_i(k) - 1\big), \qquad (8)$$

one can recursively calculate (8) as, over a period T_p,

$$\begin{aligned} V(\ell T_p) - 1 &= V_1(\ell T_p) - 1, \\ &\leq \delta_S^{T_S}\big[V_S\big((\ell-1)T_p + k_{S-1}\big) - 1\big], \\ &\leq \prod_{i=1}^{S}\delta_i^{T_i}\big[V_1\big((\ell-1)T_p\big) - 1\big], \\ &\ \ \vdots \\ &\leq \left(\prod_{i=1}^{S}\delta_i^{T_i}\right)^{\ell}\big[V_1(0) - 1\big]. \end{aligned} \qquad (9)$$

Moreover, for $k \in \{\ell T_p + k_{i-1}, \ldots, \ell T_p + k_i\}, \ell = 0, 1, 2, \ldots, i \in \mathscr{S}$, one has

$$\begin{aligned} V(k) - 1 \leq& \delta_i^{(k - \ell T_p - k_{i-1})}\big[V_i(\ell T_p + k_{i-1}) - 1\big], \\ \leq& \prod_{i=1}^{S}\delta_i^{T_i}\big[V_1(\ell T_p) - 1\big]. \end{aligned} \qquad (10)$$

Thus, combining (9) and (10), it follows that

$$V(k) \leq 1 + \prod_{i=1}^{S}\delta_i^{((\ell+1)T_i)}\big[V_1(0) - 1\big].$$

Therefore, one has $V(k) \leq 1$, as $\ell \to \infty$. The proof is completed.

3. Main Results

3.1. Stability and Reachability Analysis

In this section, the stability and reachability of the discrete-time PPS (3) will be discussed. The discrete time-scheduling Lyapunov function with periodic parameters is used. A sufficient condition is proposed to ensure the desirable bounding region for the estimation of the PPS (3).

Before proceeding, for $k \in \{\ell T_p + k_{i-1}, \ldots, \ell T_p + k_i - 1\}, \ell = 0, 1, 2, \ldots, i \in \mathscr{S}$, one constructs a discrete time-scheduling Lyapunov function with periodic parameters:

$$\begin{aligned} V(k) &= V_i(k) = x^T(k)p_i(k)x(k), \\ V(k+1) &= V_i(k+1) = x^T(k+1)p_i(k+1)x(k+1), \end{aligned}$$

where

$$\begin{aligned} p_i(k) &= \sigma_i(k)p_i + \big(1 - \sigma_i(k)\big)p_{i-1}, \\ p_i(k+1) &= \sigma_i(k+1)p_i + \big(1 - \sigma_i(k+1)\big)p_{i-1}, \end{aligned} \qquad (11)$$

with $\sigma_i(k) = \frac{k + \ell T_p - k_{i-1}}{T_i}$, $\sigma_i(k+1) = \frac{k+1+\ell T_p - k_{i-1}}{T_i}$. $p_i(k + T_p) = p_i(k), p_i(k+1+T_p) = p_i(k+1)$, $i \in \mathscr{S}$, where $p_i > 0, p_{i-1} > 0$ are constant matrices, and $p_{S+1} = p_1, p_S = p_0$.

Theorem 1. *Consider the discrete-time PPS (3) with bounded-peak disturbance in (2), under zero initial conditions, if there exist positive symmetric matrices p_{i-1}, p_i, $p_{i+1}, i \in \mathscr{S}$, with $p_{S+1} = p_1$, $p_S = p_0$, and scalars $0 < \delta_i < 1$, such that*

$$\begin{bmatrix} \Theta_{i,11} & \Theta_{i,12} \\ * & \Theta_{i,22} \end{bmatrix} \leq 0, \qquad (12)$$

$$\begin{bmatrix} \Sigma_{i,11} & \Sigma_{i,12} \\ * & \Sigma_{i,22} \end{bmatrix} \leq 0, \qquad (13)$$

where

$$\begin{aligned} \Theta_{i,11} =& A_i^T\big(p_{i-1} + \frac{1}{T_i}(p_i - p_{i-1})\big)A_i - \delta_i p_{i-1}, \\ \Theta_{i,12} =& A_i^T\big(p_{i-1} + \frac{1}{T_i}(p_i - p_{i-1})\big)B_{wi}, \\ \Theta_{i,22} =& B_{wi}^T\big(p_{i-1} + \frac{1}{T_i}(p_i - p_{i-1})\big)B_{wi} - \frac{1-\delta_i}{\bar{\omega}^2}I, \\ \Sigma_{i,11} =& A_i^T p_i A_i - \delta_i p_i + \frac{\delta_i}{T_i}(p_i - p_{i-1}), \\ \Sigma_{i,12} =& A_i^T p_i B_{wi}, \\ \Sigma_{i,22} =& B_{wi}^T p_i B_{wi} - \frac{1-\delta_i}{\bar{\omega}^2}I. \end{aligned}$$

Then the investigated system is asymptotically stable and its state converges to the ellipsoids $\mathscr{E}(p_i(k))$.

Proof. First, to analyze the asymptotical stability and reachability of the system (3), one can choose the discrete time-scheduling Lyapunov function candidate (11) with Lyapunov matrix (11). It follows that, for $k \in \{\ell T_p + k_{i-1}, \dots, \ell T_p + k_i - 1\}, \ell = 0,1,2,\dots,\ i \in \mathscr{S}$,

$$\begin{aligned}
&V_i(k+1) - \delta_i V_i(k) - \frac{1-\delta_i}{\bar{\omega}^2}\omega^T(k)\omega(k) \\
=&x^T(k+1)p_i(k+1)x(k+1) - \delta_i x^T(k)p_i(k)x(k) \\
&- \frac{1-\delta_i}{\bar{\omega}^2}\omega^T(k)\omega(k) \\
=&\big(A_i x(k) + B_{wi}\omega(k)\big)^T p_i(k+1)\big(A_i x(k) + B_{wi}\omega(k)\big) \\
&- \delta_i x^T(k)p_i(k)x(k) - \frac{1-\delta_i}{\bar{\omega}^2}\omega^T(k)\omega(k) \\
=&\eta^T(k)\Pi_i(k)\eta(k),
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
\eta(k) &= \begin{bmatrix} x^T(k) & \omega^T(k) \end{bmatrix}^T, \\
\Pi_i(k) &= \begin{bmatrix} \Lambda_i(k) & A_i^T p_i(k+1)B_{wi} \\ * & B_{wi}^T p_i(k+1)B_{wi} - \frac{1-\delta_i}{\bar{\omega}^2}I \end{bmatrix}, \\
\Lambda_i(k) &= \begin{bmatrix} A_i^T p_i(k+1)A_i - \delta_i p_i(k) \end{bmatrix},
\end{aligned}$$

In the light of the convex property of $P_i(k)$, one can consider the switching instant $\ell T_p + k_{i-1}$, which can obtained that

$$p_i(k) = p_{i-1}, \;\; p_i(k+1) = p_{i-1} + \frac{1}{T_i}(p_i - p_{i-1}). \tag{15}$$

In what follows, let us now consider the switching instant $\ell T_p + k_i - 1$, and then it is easily to derive that

$$p_i(k) = p_i + \frac{1}{T_i}(p_i - p_{i-1}), \;\; p_i(k+1) = p_i. \tag{16}$$

With (12)-(13), combining (15)-(16), one has $\Pi_i(k) \le 0$. According to (14), one can derive that $V_i(k+1) - \delta_i V_i(k) - \frac{1-\delta_i}{\bar{\omega}^2}\omega^T(k)\omega(k) \le 0$ holds. Hence, it can be known from Lemma 1 that $V_i(k) \le 1$, for $k \in \{\ell T_p + k_{i-1}, \dots, \ell T_p + k_i\}, \ell = 0,1,2,\dots,\ i \in \mathscr{S}$, which implies that $\bar{\mathscr{E}}_s \triangleq \bigcup_{0 \le k \le T_p} \mathscr{E}(p_i(k))$, where $\mathscr{E}(p_i(k))$ defined in (5). The proof of Theorem 1 is completed.

Note that, the above results are based on the fact that adjacent subsystems can share information about adjacent time instant (so-called non-jump mode). Specifically, for the $(i+1)$th subsystem, that is, $k \in \{\ell T_p + k_i, \dots, \ell T_p + k_{i+1}\}$, one has that $P_{i+1}(k) = P_i$ when k is at the initial time instant $\ell T_p + k_i$, which is obviously consistent with information from the last instant of the previous subsystem. For the sake of practical engineering, one considers the more general case where the Lyapunov matrices have bounded mode-dependent jumps at the switching instant of the adjacent subsystem, that is, $V_{m,i+1}(\ell T_p + k_i) \le \mu_i V_{m,i}(\ell T_p + k_i)$, in which the $P_{m,i}(k)$ is given as

$$p_{m,i}(k) = \sigma_i(k)p_{i,i+1} + \big(1 - \sigma_i(k)\big)p_{i,i-1}, \tag{17}$$

and $p_{m,i}(k + T_p) = p_{m,i}(k)$, $i \in \mathscr{S}$. Moreover, for $k \in \{\ell T_p + k_{i-1}, \ell T_p + k_i - 1\}, \ell = 0,1,2,\dots,\ i \in \mathscr{S}$, it follows that

$$\begin{aligned}
&p_{m,i}(\ell T_p + k_{i-1}) = p_{i,i-1}, \\
&p_{m,i}(\ell T_p + k_{i-1} + 1) = p_{i,i-1} + \frac{1}{T_i}(p_{i,i+1} - p_{i,i-1}),
\end{aligned} \tag{18}$$

$$\begin{aligned}
&p_{m,i}(\ell T_p + k_i - 1) = p_{i+1,i} - \frac{1}{T_i}(p_{i+1,i} - p_{i,i-1}), \\
&p_{m,i}(\ell T_p + k_i) = p_{i+1,i}.
\end{aligned} \tag{19}$$

Thus, the following corollary can be obtained as follows.

Corollary 1. *Consider the discrete-time PPS (3) with bounded-peak disturbance in (2), under zero initial conditions, if there exist positive symmetric matrices* $p_{i,i-1}$, $p_{i,i+1}, i \in \mathscr{S}$, *and scalars* $\mu_i > 1$, $0 < \delta_i < 1$, *with* $p_{1,0} \le \mu_S p_{S,S+1}$, $p_{i+1,i} \le \mu_i p_{i,i+1}$, $i = 1,2\dots,S-1$, *such that*

$$\begin{bmatrix} \Omega_{i,11} & \Omega_{i,12} \\ * & \Omega_{i,22} \end{bmatrix} \le 0, \tag{20}$$

$$\begin{bmatrix} \Delta_{i,11} & \Delta_{i,12} \\ * & \Delta_{i,22} \end{bmatrix} \le 0, \tag{21}$$

where

$$\begin{aligned}
\Omega_{i,11} =& A_i^T\big(p_{i,i-1} + \frac{1}{T_i}(p_{i,i+1} - p_{i,i-1})\big)A_i - \delta_i p_{i,i-1}, \\
\Omega_{i,12} =& A_i^T\big(p_{i,i-1} + \frac{1}{T_i}(p_{i,i+1} - p_{i,i-1})\big)B_{wi}, \\
\Omega_{i,22} =& B_{wi}^T\big(p_{i,i-1} + \frac{1}{T_i}(p_{i,i+1} - p_{i,i-1})\big)B_{wi} - \frac{1-\delta_i}{\bar{\omega}^2}I, \\
\Sigma_{i,11} =& A_i^T p_{i,i+1}A_i - \delta_i p_{i,i+1} + \delta_i \frac{1}{T_i}\big(p_{i,i+1} - p_{i,i-1})\big), \\
\Sigma_{i,12} =& A_i^T p_{i,i+1}B_{wi}, \\
\Sigma_{i,22} =& B_{wi}^T p_{i,i+1}B_{wi} - \frac{1-\delta_i}{\bar{\omega}^2}I.
\end{aligned}$$

Then the investigated system is asymptotically stable and its state converges to the ellipsoids $\mathscr{E}(p_{m,i}(k))$.

Proof. First, the Lyapunov function candidate $V_{m,i}(k)$ with Lyapunov matrix (17) is constructed and given as $V(k) = V_{m,i}(k) = x^T(k)P_{m,i}(k)x(k)$. With (20)-(21), combining (18)-(19), one can know that, for $k \in \{\ell T_p + k_{i-1}, \dots, \ell T_p + k_i - 1\}, \ell = 0,1,2,\dots,\ i \in \mathscr{S}$, $V_{m,i}(k+$

$1) - \delta_i V_{m,i}(k) - \frac{1-\delta_i}{\bar{\omega}^2}\omega^T(k)\omega(k)$. According to the proof line similar to lemma 1, it follows that $V(\ell T_p) - 1 = V_{m,1}(\ell T_p) - 1$, and

$$
\begin{aligned}
&V_{m,1}(\ell T_p) - 1,\\
\leq&\mu_S[V_{m,S}(\ell T_p) - 1 + 1] - 1,\\
\leq&\mu_S \delta_S^{T_S}\big[V_{m,S}\big((\ell-1)T_p - k_{S-1}\big) - 1\big] + (\mu_S - 1),\\
\leq&\mu_S \delta_S^{T_S}\big[\mu_{S-1}V_{m,S-1}\big((\ell-1)T_p - k_{S-1}\big) - 1\big] + (\mu_S - 1),\\
\leq&\prod_{i=1}^{S}\delta_i^{T_i}\mu_i\big[V_{m,1}\big((\ell-1)T_p\big) - 1\big] + \phi_i,\\
&\vdots\\
\leq&\left(\prod_{i=1}^{S}\delta_i^{T_i}\mu_i\right)^{\ell}\big[V_{m,1}(0) - 1\big] + \ell\phi_i,
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_i =& \prod_{i=1}^{S}\delta_i^{T_i}\mu_i(\mu_2 - 1) + \prod_{i=2}^{S}\delta_i^{T_i}\mu_i(\mu_3 - 1)\\
&+ \delta_S^{T_S}\mu_S(\mu_{S-1} - 1) + \cdots + (\mu_S - 1).
\end{aligned}
$$

For $k \in \{\ell T_p + k_{i-1}, \ldots, \ell T_p + k_i\}, \ell = 0, 1, 2, \ldots, i \in \mathscr{S}$, one has

$$
\begin{aligned}
V(k) - 1 \leq& \delta_i^{(k-\ell T_p - k_{i-1})}\big[V_{m,i}(\ell T_p + k_{i-1}) - 1\big],\\
\leq& \delta_i^{T_i}\big[V_{m,i}(\ell T_p + k_{i-1}) - 1\big],\\
&\vdots\\
\leq& \delta_S^{T_S}\prod_{i=1}^{S-1}\delta_i^{T_i}\mu_i\big[V_{m,1}(\ell T_p) - 1\big] + \vartheta_i.
\end{aligned}
\tag{22}
$$

where

$$
\begin{aligned}
\vartheta_i =& \delta_S^{T_S}\prod_{i=1}^{S-1}\delta_i^{T_i}\mu_i(\mu_2 - 1) + \delta_S^{T_S}\prod_{i=2}^{S-1}\delta_i^{T_i}\mu_i(\mu_3 - 1)\\
&+ \delta_S^{T_S}\prod_{i=S-1}^{S-1}\delta_i^{T_i}\mu_i(\mu_{S-2} - 1) + \cdots + \delta_S^{T_S}(\mu_{S-1} - 1).
\end{aligned}
$$

Thus, combining (9) and (10), it follows that

$$
\begin{aligned}
V(k) \leq& 1 + \delta_S^{(\ell+1)T_S}\mu_S^{\ell}\prod_{i=1}^{S-1}\delta_i^{(\ell+1)T_i}\mu_i^{\ell+1}\big[V_1(0) - 1\big]\\
&+ \ell\phi_i\delta_S^{T_S}\prod_{i=1}^{S-1}\delta_i^{T_i}\mu_i + \vartheta_i.
\end{aligned}
$$

Since $\mu_i > 1$, $0 < \delta_i < 1$, one has $V(k) \leq 1$, as $\ell \to \infty$. The proof is completed.

Remark 2. *Compared with the case of non-jumping $P_i(k)$, the results based on the bounded mode-dependent jumping $P_{m,i}(k)$ will be less conservative and more conducive to engineering applications. Note that if we desire to further reduce the conservatism of the result, one can partition the Lyapunov matrix into smaller intervals [19]. Whereas for computing resources, the proposed conditions in this paper are more economical.*

3.2. Optimization of the bounding ellipsoids

Notice that, with Theorem 1 and Corollary 1, the bounding ellipsoids $\mathscr{E}(p_i(k))$ and $\mathscr{E}(p_{m,i}(k))$ are obtained. In what follows, our goal is to optimize these ellipsoids to make them as compact as possible. In most existing results on reachable set estimation, a common method is to introduce a positive definite scalar ε, which can not only be used to optimize the bounding region [15] but also can be used to compare the conservatism of estimation results [19]. Specifically one can achieve it by providing conditions $\varepsilon I \leq p_i(k)$ or $\varepsilon I \leq p_{m,i}(k)$ while maximizing ε. It can be ensured that, for all k, the following inequalities hold.

$$
\varepsilon x^T(k)x(k) \leq \varepsilon x^T(k)p_i(k)x(k) \leq 1, \tag{23}
$$

$$
\text{or } \varepsilon x^T(k)x(k) \leq \varepsilon x^T(k)p_{m,i}(k)x(k) \leq 1. \tag{24}
$$

By applying the Schur complement equivalence to inequalities (23)-(24), according to the properties of the matrix functions $p_i(k)$ and $p_{m,i}(k)$ at time instant $\ell T_p + k_{i-1}$ and $\ell T_p + k_i - 1$, one enables from (23)-(24) that the following inequalities

$$
\begin{bmatrix} -p_{i-1} & I \\ * & -\theta I \end{bmatrix} \leq 0, \quad \begin{bmatrix} \varphi_i & I \\ * & -\theta I \end{bmatrix} \leq 0, \tag{25}
$$

$$
\text{or} \quad \begin{bmatrix} -p_{i,i-1} & I \\ * & -\theta I \end{bmatrix} \leq 0, \quad \begin{bmatrix} \nu_i & I \\ * & -\theta I \end{bmatrix} \leq 0, \tag{26}
$$

where $\varphi_i = -p_i + \frac{1}{T_i}(P_i - P_{i-1})$, $\nu_i = -p_{i,i+1} + \frac{1}{T_i}(P_{i,i+1} - P_{i,i-1})$, $\theta = \varepsilon^{-1} > 0$. Combined with the criteria in Theorem 1 and Corollary 1, the optimization problem of reachable set estimation for discrete-time PPSs can be characterized as follows:

Minimize θ subject to

$$
\begin{cases} (12)-(13),(25), & \text{for Theorem 1}, \\ (20)-(21),(26), & \text{for Corollary 1}. \end{cases}
$$

Therefore, the bounding region can be minimized by ellipsoids $\bigcup_{i\in\mathscr{S}} E(p_{i-1}), E(p_i), E(p_{i,i-1}), E(p_{i+1,i})$ which are all contained in the ball $\mathscr{B}(\varepsilon) = \{x \in \mathbb{R}^n | x^T\varepsilon x \leq 1, \varepsilon > 0\}$.

4. Numerical Example

Consider a PPS containing three subsystems, whose fundamental period is $T_p = 20$. The dwell time is specified as $T_1 = 4, T_2 = 6, T_3 = 10$. With (2)-(3), the relevant parameters are given as

$$k \in [\ell T_p, \ell T_p + 4),$$
$$A_1 = \begin{bmatrix} 0.65 & 0.2 \\ 0.128 & -0.6 \end{bmatrix}, \quad B_{w1} = \begin{bmatrix} -0.4200 \\ 0.3717 \end{bmatrix},$$

$$k \in [\ell T_p + 4, \ell T_p + 10),$$
$$A_2 = \begin{bmatrix} 0.60 & -0.15 \\ 0.2 & 0.19 \end{bmatrix}, \quad B_{w2} = \begin{bmatrix} -0.3633 \\ 0.6857 \end{bmatrix},$$

$$k \in [\ell T_p + 10, (\ell + 1) T_p),$$
$$A_3 = \begin{bmatrix} 0.24 & -0.4 \\ 0.2 & 0.12 \end{bmatrix}, \quad B_{w3} = \begin{bmatrix} 0.2897 \\ -0.5238 \end{bmatrix}.$$

Given $\delta_1 = 0.696$, $\delta_2 = 0.592$, $\delta_3 = 0.652$, and consider the bounded-peak disturbance signal $\omega(k) = 1.21 * sin(k)$ with $\bar{\omega} = 1.4641$. In order to compare the advantages of estimation criteria developed by $p_i(k)$ and $p_{m,i}(k)$, we use the SeDuMi solver in MATLAB based on the YALMIP language, and the results are given as follows:

Table 1: Comparisons of θ

	The criterion of Theorem 1	The criterion of Corollary 1
θ	2.5742	2.5739

One can observe from Table I that the estimation result developed with $p_{m,i}(k)$ (Corollary 1) is less conservative compared with non-jumping ones. Moreover, with zero initial state $x = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, the reachable set $\mathscr{R}_x$ and its bounding region are shown in the following figures.

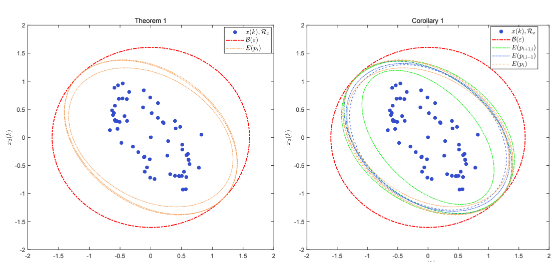

Figure 1: Bounding regions of reachable set (Theorem 1 and Corollary 1)

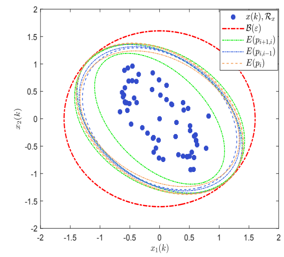

Figure 2: Comparison of bounding regions of reachable set

The bounding ball constructed by ε, the bounding ellipsoids $E(p_i)$ (Theorem 1), and the bounding ellipsoids $E(p_{i,i-1})$, $E(p_{i+1,i})$ (Corollary 1) are shown in Figure 1, respectively. It can observer that the bounding region obtained by subinterval switching information (Theorem 1 and Corollary 1) is smaller than the ball $\mathscr{B}(\varepsilon)$ developed by a positive scalar ε. Then, the comparative result of $E(p_i)$ and $E(p_{i,i-1})$,$E(p_{i+1,i})$ is given in Figure 2. It shows that the bounding region $E(p_{i+1,i})$ is more compact, which is consistent with the results of Table 1. Therefore, this means that the proposed criteria is desirable, where Corollary 1 is more conducive to engineering applications.

5. Conclusions

In this paper, the reachable set estimation problem of discrete-time PPSs is studied for the first time. By constructing discrete time-scheduling Lyapunov functions with non-jumping or jumping modes at switching instant, the estimation criteria of reachable sets for discrete-time PPSs are established, and sufficient conditions for optimizing the bounding region of reachable sets are given by using ellipsoid techniques. The simulation results show that the proposed condition based on bounded mode-dependent jumping Lyapunov matrices is less conservative and can shrink the reachable set to a smaller region, which is more advantageous in engineering applications.

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