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Stochastic Relaxation of the Maximum Allowable Delay for a Class of Networked Control Systems

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Abstract— **Stability in networked control systems has been typically addressed in the deterministic setting using the concepts of Maximum Allowable Transmit Interval (MATI) and Maximum Allowable Delay (MAD). This work looks to extend the analysis to the stochastic setting by giving conditions for uniform stability in probability when the communication delay follows a probability distribution that includes values over the deterministic MAD. Analytical conditions are given and validated through simulations using a concrete example.**

I. INTRODUCTION

The widespread use of feedback control loops that share the communication medium, particularly in the industrial setting, has stimulated the development of a branch of control theory known as control over networks, or networked control systems (NCSs) [1], [2]. NCSs modernize the notion of feedback loop by explicitly integrating in the analysis a communication channel between sensors and the controller and/or between the controller and actuators [3]. In an NCS, the communication network plays a key role in the performance of the system; hence, its presence must be taken into account from the earliest design stage [4].

Focusing on maintaining closed-loop stability achieved by a nominal non-networked control law, several efforts have been made by the control community to establish requirements on the network "quality of service" [5] to ensure stability of the system after the inclusion of the communication network. The introduction of the Maximum Allowable Transmit Interval (MATI) and Maximum Allowable Delay (MAD) [6] of an NCS represents a concrete step in this regard, which enables linking communications and control. The MATI and MAD refer to timeliness of information. The MATI can be viewed as an upper bound on the sampling period [2], while the MAD is an upper bound on the information delay the control loop tolerates [7].

Based on these concepts, a large body of recent literature has focused on obtaining values for MATI and MAD that are not conservative and get close to a "true value" [5], [7], [8], [9]. However, MATI and MAD are deterministic bounds and stability claims hold only in the deterministic setting. This is certainly a limitation given that in a realistic NCS the network is inherently stochastic, which makes it impossible to guarantee the fulfillment of a bound at every transmission event, particularly for the MAD.

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Inspired by the findings in [10], which show that stability is preserved when MATI and MAD are violated with a low probability, and supported by recent works where conditions for stability in probability have been formulated for systems with stochastic communication protocols [9], [11], [12] and event-triggered control laws regulated by stochastic processes [13], in this work we contribute by deriving conditions for uniform stability in probability for a class of NCSs that face a stochastic communication delay taking values over the deterministic MAD with positive probability.

The rest of this manuscript is organized as follows. Section II introduces preliminaries. In Section III the problem of interest is formulated. The main result of this work is given in Section IV. Numerical experiments are presented in Section V. Finally, conclusions are stated in Section VI.

II. PRELIMINARIES

A. Notation and Basic Definitions

In this work, $\mathbb{N}$ denotes the set of positive integers, $\mathbb{Z}_{\geq 0}$ the set of non-negative integers, $\mathbb{R}$ the set of real numbers, $\mathbb{R}_{\geq 0}$ the set of non-negative real numbers, and $\mathbb{R}^N$ the Euclidean space of dimension N. $\mathbb{B}$ denotes the closed unit ball and $\mathbb{B}^{\circ}$ the corresponding open ball. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$; it is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. Given a compact set $\mathcal{A} \subset \mathbb{R}^N$, $\rho : \mathbb{R}^N \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{PD}(\mathcal{A})$ if it is continuous and $\rho(x) = 0 \iff x \in \mathcal{A}$.

B. Stochastic Hybrid Systems

To conduct the analysis, we adopt the stochastic hybrid systems framework introduced in [14], [15]. Consider the following stochastic hybrid system

$$\begin{aligned} \dot{x} &= F(x),\ x \in \mathcal{C} \\ x^+ &\in G(x, v^+),\ x \in \mathcal{D},\ v \sim \mu(\cdot), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $\mathcal{C} \subset \mathbb{R}^n$ and $\mathcal{D} \subset \mathbb{R}^n$ are the flow set and jump set, respectively; $F : \mathbb{R}^n \to \mathbb{R}^n$ is the flow map and $G : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is the jump map. The notation v^+ is a placeholder for a sequence of independent identically distributed (i.i.d.) input random variables $v_i : \Omega \to \mathbb{R}^m$, $i \in \mathbb{N}$, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. μ denotes the distribution function derived from the i.i.d. random variables.

Throughout this work, the following standing assumption is made use of.

Assumption 1: Consider system (1):

1) $\mathcal{C}$, $\mathcal{D} \subset \mathbb{R}^n$ are closed.

2) $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

3) a) $G : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is locally bounded.

b) $v \mapsto \mathrm{graph}(G(\cdot, v)) := \{(x, y) \in \mathbb{R}^{2n} : y \in G(x, v)\}$ is measurable with closed values.

The solution concept for system (1) is built as follows. A *relaxed hybrid arc* is a mapping $\phi : H \to \mathbb{R}^n$ such that H is a hybrid time domain [14] and, for each $j \in \mathbb{Z}_{\geq 0}, t \mapsto \phi(t, j)$ is locally absolutely continuous. Given the measurable space $(\Omega, \mathcal{F})$, a *stochastic hybrid arc* is a mapping x defined on Ω such that $\mathrm{x}(\omega)$ (also written x_ω) is a *relaxed hybrid arc* for each $\omega \in \Omega$ and the set-valued mapping from Ω to $\mathbb{R}^{n+2}$ defined by

$$\omega \mapsto \mathrm{graph}(\mathrm{x}_\omega) := \{(t, j, z) \in \mathbb{R}^{n+2} : (t, j) \in \mathrm{dom}\, \mathrm{x}_\omega, \\ z = \mathrm{x}_\omega(t, j)\},$$

is $\mathcal{F}$-measurable [15] with closed values.

Given the sequence of i.i.d. random variables $v := \{v_i\}_{i=1}^\infty$, let $\{\mathcal{F}_i\}_{i=1}^\infty$ denote the natural filtration of $\mathcal{F}$ with respect to v and define $\mathrm{graph}(\mathrm{x}_\omega)_{\leq j} := \mathrm{graph}(\mathrm{x}_\omega) \cap (\mathbb{R}_{\geq 0} \times \{1, \ldots, j\} \times \mathbb{R}^n)$. An $\{\mathcal{F}_i\}_{i=1}^\infty$ *adapted stochastic hybrid arc* is a *stochastic hybrid arc*, x, such that the mapping $\omega \mapsto \mathrm{graph}(\mathrm{x}_\omega)_{\leq j}$ is $\mathcal{F}_j$-measurable for each $j \in \mathbb{Z}_{\geq 0}$. A solution to (1), starting at $x \in \mathbb{R}^n$ with input $v(\omega)$, is an *adapted stochastic hybrid arc*, x, such that for almost every $\omega \in \Omega$:

1) $\mathrm{x}_\omega(0, 0) = x$
2) for each $j \in \mathbb{Z}_{\geq 0}$, if $I^j := \{t : (t, j) \in \mathrm{dom}\, \mathrm{x}_\omega\}$ has nonempty interior, then, for every $t \in I^j(\omega)$:
   a) $\mathrm{x}_\omega(t, j) \in \mathcal{C}$
   b) $\mathrm{x}_\omega(t, j) - \mathrm{x}_\omega(T_j, j) = \int_{T_j}^t F(\mathrm{x}_\omega(s, j))\mathrm{d}s$
3) if $(t, j), (t, j + 1) \in \mathrm{dom}\, \mathrm{x}_\omega$ then
   a) $\mathrm{x}_\omega(t, j) \in \mathcal{D}$
   b) $\mathrm{x}_\omega(t, j + 1) \in G(x_\omega(t, j), v_{j+1}(\omega))$

For a solution x starting at a point $x \in \mathcal{O}$, we use the notation $\mathrm{x} \in \mathcal{S}_r(\mathcal{O})$.

At this point, the stability notions used in this work are introduced.

Definition 1 ([13]): A compact set $\mathcal{A}$ is said to be

1) *Uniformly Lyapunov stable in probability:* if for each $\epsilon > 0$ and $\rho > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} &\mathrm{x} \in \mathcal{S}_r(\mathcal{A} + \delta\mathbb{B}) \\ &\quad \Rightarrow \mathbb{P}(\mathrm{graph}(\mathrm{x}) \subset \mathbb{R}^2 \times (\mathcal{A} + \epsilon\mathbb{B})) \geq 1 - \rho \end{aligned} \tag{2}$$

2) *Uniformly Lagrange stable in probability:* if for each $\delta > 0$ and $\rho > 0$ there exists $\epsilon > 0$ such that (2) holds.
3) *Uniformly globally stable in probability (UGSp):* if it is both uniformly Lyapunov stable in probability and uniformly Lagrange stable in probability.

In what follows, basic definitions are stated, to conclude with a key result presenting sufficient conditions to prove the uniformly globally stable in probability property.

Let $\mathcal{V} := \cup_{\omega \in \Omega, i \in \mathbb{N}} \mathrm{v}_i(\omega)$. A function $V : \mathrm{dom}\, V \to \mathbb{R}$ is a *certification candidate for* $\mathcal{H} = (\mathcal{C}, F, \mathcal{D}, G, \mu)$, and we write $V \in D(\mathcal{H})$, if the following conditions are satisfied.

C1: $\mathcal{C} \cup \mathcal{D} \cup G(\mathcal{D} \times \mathcal{V}) \subset \mathrm{dom}\, V$

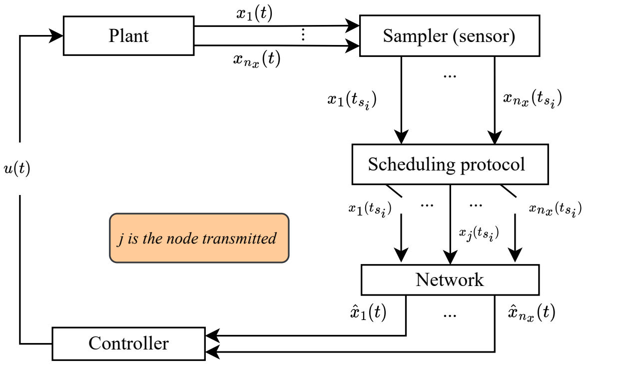


Fig. 1: NCS under study, a communication network from sensor to controller is explicitly included.

C2: $0 \leq V(x)$ for all $x \in \mathcal{C} \cup \mathcal{D} \cup G(\mathcal{D} \times \mathcal{V})$

C3: The quantity $\int_{\mathbb{R}^m} \sup_{g \in G(x,v)} V(g)\mu(\mathrm{d}v)$ is well defined (and finite) for each $x \in \mathcal{D}$, using the convention that $\sup_{g \in G(x,v)} V(g) = 0$ when $G(x, v) = \emptyset$.

Definition 2: Given a closed set $S \subset \mathbb{R}^n$, let $V : \mathrm{dom}\, V \to \mathbb{R}$ be a certification candidate for $\mathcal{H}_{\cap S} := (\mathcal{C} \cap S, F, \mathcal{D} \cap S, G \cap S, \mu)$. For all $x \in \mathcal{C} \cap S$ we define

$$\mathcal{L}_S V(x) := \sup_{f \in F(x)} \langle \nabla V(x), f \rangle, \tag{3}$$

and for all $x \in \mathcal{D} \cap S$ we define

$$\Delta_S V(x) := \int_{\mathbb{R}^m} \sup_{g \in G(x,v) \cap S} V(g)\mu(\mathrm{d}v) - V(x). \tag{4}$$

These quantities are well-defined and finite for each $x \in \mathcal{C} \cap S$, respectively $x \in \mathcal{D} \cap S$, under the stochastic hybrid basic conditions of *Assumption 1*.

Lemma 1 ([15]): Consider the stochastic hybrid system (1) and let *Assumption 1* hold, the compact set $\mathcal{A} \subset \mathbb{R}^n$ is UGSp if there exist $\mathcal{K}_\infty$ functions α_1, α_2 and $V \in D(\mathcal{H})$ such that

$$\alpha_1(|x|_\mathcal{A}) \leq V(x) \quad \forall x \in \mathcal{C} \cup \mathcal{D} \cup G(\mathcal{D} \times \mathcal{V}) \tag{5a}$$

$$V(x) \leq \alpha_2(|x|_\mathcal{A}) \quad \forall x \in \mathcal{C} \cup \mathcal{D} \tag{5b}$$

$$\mathcal{L}_{\mathbb{R}^n} V(x) \leq 0 \quad \forall x \in \mathcal{C} \tag{5c}$$

$$\Delta_{\mathbb{R}^n} V(x) \leq 0 \quad \forall x \in \mathcal{D} \tag{5d}$$

III. Problem Statement

The NCS under study considers a communication network from sensor to controller and is illustrated in Fig. 1. In this setup, we consider a closed loop formed by a continuous-time plant and a stabilizing state-feedback controller,

$$\begin{aligned} \dot{x} &= f(x, u) \\ u &= \kappa(\hat{x}), \end{aligned} \tag{6}$$

where $x \in \mathrm{R}^{n_x}$ denotes the state of the plant, $u \in \mathbb{R}^{n_u}$ denotes the control input and $\hat{x} \in \mathbb{R}^{n_x}$ is the most recent state measurement available at the controller. f is assumed to be continuous and κ is continuously differentiable.

The operation of the NCS is as follows. The plant has a set of output nodes, which sample periodically one or more states of the plant at transmission instants t_{s_i}, satisfying

$t_{s_{i+1}} - t_{s_i} = \lambda \ \forall i \in \mathbb{N}$. The information is sent over the network to the controller, according to a protocol that determines which state node has access to the network. The sampling period satisfies $\lambda \leq \tau_{mati}$, where τ_{mati} denotes the MATI. The output arrives to the controller after a transmission delay v_i. It is assumed that there exists a MAD, denoted as τ_{mad}, that guarantees closed loop stability if $v_i < \tau_{mad} \ \forall i \in \mathbb{N}$.

The problem of interest in this work considers a stochastic transmission delay modeled as a random variable v_i that distributes according to a probability density function μ with bounded support, up to $\bar{d}$, such that

$$\int_0^{\bar{d}} \mu(\mathrm{d}v) = 1. \tag{7}$$

We consider $\tau_{mad} < \bar{d} < \lambda$, i.e., the delay can take values exceeding τ_{mad} with positive probability. Additionally, we consider that the stochastic delay satisfies the small delay condition [7], i.e., the controller updates always occur before the next plant transmission. The update at $t_{s_i} + v_i$ satisfies

$$\hat{x}((t_{s_i} + v_i)^+) = x(t_{s_i}) + h(i, e(t_{s_i})), \tag{8}$$

where $e = \hat{x} - x$ and the function h is related to the protocol that determines which node is granted access to the network [11]. In particular, the components h_j of h satisfy

$$h_j(i, e(t_{s_i})) = \begin{cases} 0, & \text{if } j \text{ is transmitted at instant } t_{s_i} \\ e_j(t_{s_i}), & \text{if } j \text{ is not transmitted at } t_{s_i}, \end{cases} \tag{9}$$

for $j \in \{1, 2, ..., n_x\}$.

To conduct the analysis, we formulate the NCS model into a stochastic hybrid system framework, similar to [7]. Considering the variable $s \in \mathbb{R}^{n_x}$, which stores the sampled state in the value $h - e$ at instant t_{s_i}, a timer $\tau_t \in [0, \lambda]$ to model the transmission interval, a timer $\tau_d \in \mathbb{R}_{\geq 0}$ to model the stochastic transmission delay, a counter $k \in \mathbb{N}$ to track the sampling/transmission events and a binary variable $q \in \{0, 1\}$ that indicates if the next event is a transmission ($q = 0$) or an update ($q = 1$), the stochastic hybrid model describing the NCS is given by

$$\dot{\xi} := (\dot{x}, \dot{e}, \dot{s}, \dot{\tau}_t, \dot{\tau}_d, \dot{k}, \dot{q}) = (F(x, e), G(x, e), 0, -1, -q, 0, 0),$$
$$\xi \in \mathcal{C} := \{\mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times [0, \lambda] \times \mathbb{R}_{\geq 0} \times \mathbb{N} \times \{0, 1\}\} \tag{10}$$

$$\xi^+ := (x^+, e^+, s^+, \tau_t^+, \tau_d^+, k^+, q^+) = \begin{cases} (x, e, h(k, e) - e, \lambda, v, k + 1, 1), & \text{if } \xi \in \mathcal{D}_1 \\ (x, s + e, -s - e, \tau_t, M, k, 0), & \text{if } \xi \in \mathcal{D}_2 \end{cases}, \tag{11}$$

where F and G are functions depending on f and κ, $\mathcal{C}$ denotes the flow set, and $\mathcal{D}_1 = \{\xi \in \mathcal{C} | \tau_t = 0\}$ and $\mathcal{D}_2 = \{\xi \in \mathcal{C} | \tau_d = 0\}$ are the jump sets, $M \in \mathbb{R}_{\geq 0}$ is an arbitrary constant value and $v \sim \mu(\cdot)$ is the random delay value. Note that *Assumption 1* holds.

The problem considered in this work is summarized as follows.

Problem 1: Consider the NCS described by (10)-(11). Suppose that the closed-loop (6) remains stable whenever $\lambda < \tau_{mati}$ and $v_i < \tau_{mad}$, $\forall i \in \mathbb{N}$. Derive conditions that ensure stability in probability when v_i is allowed to surpass τ_{mad} with a positive probability, tabulated in $\mu(\cdot)$.

IV. Main Result

To conduct the analysis, we rely on the following assumptions, in the spirit of [7] and [13].

Assumption 2: There exist a continuously differentiable function $W : \mathbb{N} \times \{0, 1\} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, $\mathcal{K}_\infty$ functions $\underline{\beta}_W$ and $\overline{\beta}_W$, a constant $\epsilon_1 \in (0, 1)$, and a measurable function $\epsilon_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that $\forall k \in \mathbb{N}$, $\forall v \in [0, \bar{d}]$, $\forall q \in \{0, 1\}$, and $\forall s, e \in \mathbb{R}^{n_x}$ the following hold:

$$\underline{\beta}_W(|(e, s)|) \leq W(k, q, e, s) \leq \overline{\beta}_W(|(e, s)|) \tag{12a}$$
$$W(k^+, 1, e^+, s^+) \leq \epsilon_1 W(k, 0, e, s) \tag{12b}$$
$$W(k^+, 0, e^+, s^+) \leq \epsilon_2(v) W(k, 1, e, s) \tag{12c}$$

The next assumption demands a robust stability property on the controlled system with respect to the network.

Assumption 3: There exist a continuously differentiable function $V : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, $\mathcal{K}_\infty$-functions $\underline{\beta}_V$, $\overline{\beta}_V$, continuous functions $H_i : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, positive definite functions ρ_i and σ_i and constants $\gamma_i > 0$, for $i \in \{0, 1\}$, and $\eta > 0$ such that $\forall k \in \mathbb{N}$, $q \in \{0, 1\}$, $s, e \in \mathbb{R}^{n_x}$ and almost all $x \in \mathbb{R}^{n_x}$, the following hold:

$$\underline{\beta}_V(|x|) \leq V(x) \leq \overline{\beta}_V(|x|) \tag{13}$$

$$\begin{aligned} \langle \nabla V(x), F(x, e) \rangle \leq & (1 - q)[-\rho_0(|x, e|) - H_0^2(x) - \\ & \sigma_0(W(k, q, e, s)) + \gamma_0^2 W(k, q, e, s)] \\ & + q[-\rho_1(|x, e|) - H_1^2(x) \\ & - \sigma_1(W(k, q, e, s)) + \gamma_1^2 W(k, q, e, s)] \end{aligned} \tag{14}$$

with $\sigma_0(W(k, 0, e, s)) \geq 2\eta W(k, 0, e, s)$.

Remark 1: In contrast to [7], [13], Assumption 2, referring to the stability of the communication protocol [11], allows ϵ_2 to be a random variable. Assumption 3 entails the stability of closed loop (6) before the addition of the network; stronger than those in [7], [13], a domination condition on the communication protocol term, relying on the existence of η, has been requested to face the stochastic delays.

To state the following condition, let us define strictly increasing positive functions by

$$\begin{aligned} &\phi_0(\tau_t) : [0, \lambda] \to \mathbb{R}_+, \quad \text{with} \quad \dot{\phi}_0(\tau_t) \geq \frac{1}{\delta}, \quad \delta > 0, \\ &\dot{\phi}_1(\tau_d) = L_1 \phi_1(\tau_d) + \gamma_1 \phi_1^2(\tau_d) + \gamma_1. \end{aligned} \tag{15}$$

The following condition, which is similar to the one in [13], will be shown critical for the analysis, as it limits the growth rate of the terms associated to the network in flow.

Condition 1: There exist constants $L_i \geq 0$, $i = 0, 1$, and $K \geq 0$ such that $\forall k \in \mathbb{N}$, $q \in \{0, 1\}$, $s \in \mathbb{R}^{n_x}$, $x \in \mathbb{R}^{n_x}$, and almost all $e \in \mathbb{R}^{n_x}$, the following holds:

$$\begin{aligned} \left\langle \frac{\partial W(k, q, e, s)}{\partial e}, G(x, e) \right\rangle \leq & (1 - q)[L_0 W(k, q, e, s) \\ & + 2K\sqrt{W(k, q, e, s)} H_0(x)] \\ & + q[L_1 W(k, q, e, s) \\ & + 2\sqrt{W(k, q, e, s)} H_1(x)], \end{aligned} \tag{16}$$

with

$$L_0 = \frac{\eta}{\delta\gamma_0^2\phi_0(\lambda)}, \quad K = \frac{\sqrt{\eta}}{\delta\gamma_0^2\phi_0(\lambda)}. \tag{17}$$

The main result of this work is now introduced.

Theorem 1: Consider the NCS described by (10)-(11) and let *Assumptions 2, 3* hold. If *Condition 1* is satisfied and

$$\int_0^{\bar{d}} \phi_1(v)\mu(\mathrm{d}v) < \frac{\delta\gamma_0^2\phi_0(0)}{\gamma_1\epsilon_1}, \tag{18}$$

$$\int_0^{\bar{d}} \epsilon_2(v)\mu(\mathrm{d}v) < \frac{\gamma_1\phi_1(0)}{\delta\gamma_0^2\phi_0(\lambda)}, \tag{19}$$

then the set $\mathcal{W} = \{0\}\times\{0\}\times\{0\}\times[0,\lambda]\times[0,\bar{d}]\times\mathbb{N}\times\{0,1\}$ is UGSp.

Proof: Consider the following certification candidate

$$\begin{aligned} U(\xi) &= V(x) + (1-q)\delta W(k,q,e,s)\gamma_0^2\phi_0(\tau_t) \\ &\quad + qW(k,q,e,s)\gamma_1\phi_1(\tau_d), \end{aligned} \tag{20}$$

where ϕ_0 and ϕ_1 are the solutions of (15). For all $\xi \in \mathcal{C}$ we define

$$F_{SHS}(\xi) = (F(x,e), G(x,e), 0, -1, -q, 0, 0). \tag{21}$$

During flows, and due to Assumption 3, we have that for $q = 0$, for all (τ_t, τ_d, k, q) and almost all (x, e, s),

$$\begin{aligned} &\mathcal{L}U(\xi) = \langle \nabla U(\xi), F_{SHS}(\xi)\rangle \\ &= \frac{\partial V(x)}{\partial x}F(x,e) + \delta\gamma_0^2\phi_0(\tau_t)\frac{\partial W(k,0,e,s)}{\partial e}G(x,e) \\ &- \delta\gamma_0^2 W(k,0,e,s)\dot{\phi}_0(\tau_t) \\ &\le -\rho_0(|x,e|) - H_0^2(x) - \sigma_0(W(k,0,e,s)) \\ &+ \gamma_0^2 W(k,0,e,s) + \delta\gamma_0^2\phi_0(\tau_t)\frac{\eta}{\delta\gamma_0^2\phi_0(\lambda)}W(k,0,e,s) \\ &+ 2\delta\gamma_0^2\phi_0(\tau_t)\frac{\sqrt{\eta}}{\delta\gamma_0^2\phi_0(\lambda)}\sqrt{W(k,0,e,s)}H_0(x) \\ &- \delta\gamma_0^2 W(k,0,e,s)\dot{\phi}_0(\tau_t) \\ &\le -\rho_0(|x,e|) - H_0^2(x) - 2\eta W(k,0,e,s) \\ &+ \eta W(k,0,e,s) + 2\sqrt{\eta}\sqrt{W(k,0,e,s)}H_0(x) \\ &= -\rho_0(|x,e|) - \left(H_0(x) - \sqrt{\eta W(k,0,e,s)}\right)^2 \le 0. \end{aligned} \tag{22}$$

Similarly, for $q = 1$, for all (τ_t, τ_d, k, q) and almost all (x, e, s),

$$\begin{aligned} &\mathcal{L}U(\xi) = \langle \nabla U(\xi), F_{SHS}(\xi)\rangle \\ &= \frac{\partial V(x)}{\partial x}F(x,e) + \gamma_1\phi_1(\tau_d)\frac{\partial W(k,1,e,s)}{\partial e}G(x,e) \\ &- \gamma_1 W(k,1,e,s)\dot{\phi}_1(\tau_d) \\ &\le -\rho_1(|x,e|) - H_1^2(x) - \sigma_1(W(k,1,e,s)) + \gamma_1^2 W(k,1,e,s) \\ &+ \gamma_1\phi_1(\tau_d)\left[L_1 W(k,1,e,s) + 2\sqrt{W(k,1,e,s)}H_1(x)\right] \\ &- \gamma_1 W(k,1,e,s)\left[L_1\phi_1(\tau_d) + \gamma_1\phi_1^2(\tau_d) + \gamma_1\right] \\ &= -\rho_1(|x,e|) - H_1^2(x) - \sigma_1(W(k,1,e,s)) \\ &+ 2\gamma_1\phi_1(\tau_d)\sqrt{W(k,1,e,s)}H_1(x) - \gamma_1^2 W(k,1,e,s)\phi_1^2(\tau_d) \\ &= -\rho_1(|x,e|) - \sigma_1(W(k,1,e,s)) \\ &- \left(H_1(x) - \gamma_1\phi_1(\tau_d)\sqrt{W(k,1,e,s)}\right)^2 \le 0. \end{aligned} \tag{23}$$

During jumps, we have that $\forall\xi \in \mathcal{D}_1$,

$$\begin{aligned} \xi &= (x, e, s, 0, M, k, 0) \\ U(\xi) &= V(x) + \delta\gamma_0^2\phi_0(0)W(k,0,e,s) \\ \xi^+ &= (x, e, h(k,e) - e, \lambda, v, k+1, 1) \\ U(\xi^+) &= V(x) + \gamma_1\phi_1(v)W(k+1,1,e,h(k,e)-e), \end{aligned} \tag{24}$$

and then,

$$\begin{aligned} &\int_0^{\bar{d}} U(\xi^+)\mu(\mathrm{d}v) - U(\xi) = V(x)\int_0^{\bar{d}}\mu(\mathrm{d}v) \\ &+ \gamma_1 W(k+1,1,e,h(k,e)-e)\int_0^{\bar{d}}\phi_1(v)\mu(\mathrm{d}v) - V(x) \\ &- \delta\gamma_0^2\phi_0(0)W(k,0,e,s) \\ &= \gamma_1 W(k+1,1,e,h(k,e)-e)\int_0^{\bar{d}}\phi_1(v)\mu(\mathrm{d}v) \\ &- \delta\gamma_0^2\phi_0(0)W(k,0,e,s) \\ &\le \gamma_1\int_0^{\bar{d}}\phi_1(v)\mu(\mathrm{d}v)\epsilon_1 W(k,0,e,s) \\ &- \delta\gamma_0^2\phi_0(0)W(k,0,e,s) \\ &= W(k,0,e,s)\left(\gamma_1\epsilon_1\int_0^{\bar{d}}\phi_1(v)\mu(\mathrm{d}v) - \delta\gamma_0^2\phi_0(0)\right) \\ &\le 0. \end{aligned} \tag{25}$$

Similarly, $\forall\xi \in \mathcal{D}_2$,

$$\begin{aligned} \xi &= (x, e, s, \tau_t, 0, k, 1) \\ U(\xi) &= V(x) + \gamma_1\phi_1(0)W(k,1,e,s) \\ \xi^+ &= (x, s+e, -s-e, \tau_t, M, k, 0) \\ U(\xi^+) &= V(x) + \delta\gamma_0^2\phi_0(\tau_t)W(k,0,s+e,-s-e), \end{aligned} \tag{26}$$

and then,

$$\begin{aligned} &\int_0^{\bar{d}} U(\xi^+)\mu(dv) - U(\xi) = V(x)\int_0^{\bar{d}}\mu(\mathrm{d}v) \\ &+ \delta\gamma_0^2\phi_0(\tau_t)\int_0^{\bar{d}} W(k,0,s+e,-s-e)\mu(\mathrm{d}v) - V(x) \\ &- \gamma_1\phi_1(0)W(k,1,e,s) \\ &= \delta\gamma_0^2\phi_0(\tau_t)\int_0^{\bar{d}} W(k,0,s+e,-s-e)\mu(\mathrm{d}v) \\ &- \gamma_1\phi_1(0)W(k,1,e,s) \\ &\le \delta\gamma_0^2\phi_0(\tau_t)W(k,1,e,s)\int_0^{\bar{d}}\epsilon_2(v)\mu(\mathrm{d}v) \\ &- \gamma_1\phi_1(0)W(k,1,e,s) \\ &= W(k,1,e,s)\left(\delta\gamma_0^2\phi_0(\tau_t)\int_0^{\bar{d}}\epsilon_2(v)\mu(\mathrm{d}v) - \gamma_1\phi_1(0)\right) \\ &\le 0. \end{aligned} \tag{27}$$

Invoking *Lemma 1* yields that $\mathcal{W}$ is UGSp. ■

V. Numerical Results

To validate the analytical result, we conduct numerical experiments on a closed loop involving a linear system and a state-feedback controller, namely,

$$\dot{x} = Ax + Bu, \quad u = K\hat{x}, \tag{28}$$

with $A = 1$, $B = 5$ and $K = -0.4$. By using the NCS model given in (10) and (11), we have $F(x, e) = A_{11}x + A_{12}e$ and $G(x, e) = A_{21}x + A_{22}e$, with $A_{11} = A + BK$, $A_{12} = BK$, $A_{21} = -(A + BK)$ and $A_{22} = -BK$.

To simulate the NCS, we use the HyEQ Toolbox for Matlab, considering in all cases the initial condition $\xi_0 = (1, -1, 1, 0, 1, 0, 0)$. First, we simulate the NCS without delay with the objective of obtaining the experimental "true" MATI, resulting in $\tau_{mati} = 1.098$s. Then, setting the sampling rate as $\lambda = 1.09s$, we obtain the deterministic MAD, resulting in $\tau_{mad} = 0.184$s.

Considering the MAD, four scenarios are defined for μ based on a slotted principle, yielding the discrete distributions shown in Fig. 2. It can be seen that each scenario includes a positive probability for a value over the MAD.

To verify *Assumption 2*, assume that $W(k, q, e, s) = e^T e$ and, hence, (12a) holds trivially. Since only one node samples the state of the system, is follows that $W(k^+, 1, e^+, s^+) = W(k, 0, e, s)$ in the transmission jump; therefore, (12b) holds with $\epsilon_1 = 1$. To obtain ϵ_2, we simulate the NCS setting the delay as a constant value for each possible delay value; at each update jump, we calculate the ratio between $W(k^+, 0, e^+, s^+)$ and $W(k, 1, e, s)$, with $W(k, 1, e, s) \neq 0$, and take the maximum on (k, e, s) to obtain an approximated value for ϵ_2, so that (12c) –and hence *Assumption 2*– holds. Table I shows the values obtained. Note that since ϵ_2 is a strictly increasing function, it also distributes according to μ.

TABLE I: Values for ϵ_2 obtained in simulations

Delay (μs)	0.22	2.2	220	220000
ϵ_2	1.1×10^{-13}	1.1×10^{-11}	1.1×10^{-7}	2.39

We now proceed to verify *Condition 1*. Firstly, we compute (16) for $q = 0$

$$\begin{aligned} &\left\langle \frac{\partial(e^T e)}{\partial e}, A_{21}x + A_{22}e \right\rangle = 2e^T A_{21}x + 2e^T A_{22}e \\ &\leq 2\sqrt{W(k, 0, e, s)}|A_{21}x| + 2|A_{22}|W(k, 0, e, s) \\ &\leq 2K\sqrt{W(k, 0, e, s)}H_0(x) + L_0 W(k, 0, e, s), \end{aligned} \tag{29}$$

which is satisfied if $H_0(x) = |A_{21}x|$ and

$$L_0 = \frac{\eta}{\delta\gamma_0^2\phi_0(\lambda)} \geq 2|A_{22}|, \quad K = \frac{\sqrt{\eta}}{\delta\gamma_0^2\phi_0(\lambda)} \geq 1 \tag{30}$$

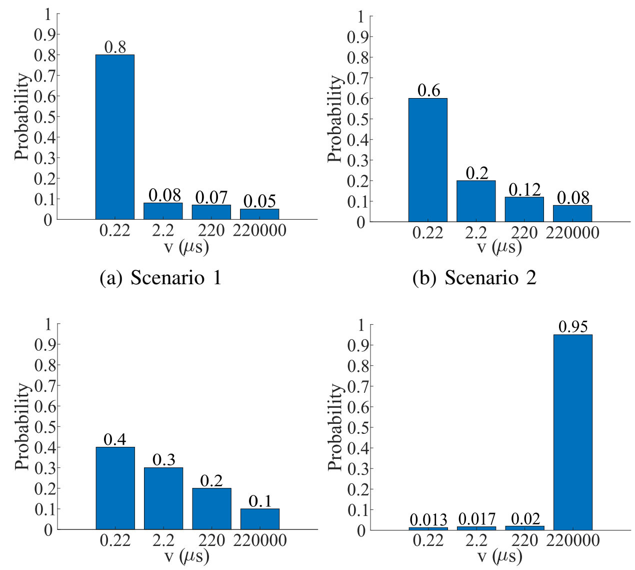
considering η sufficiently large. Then, for $q = 1$

$$\begin{aligned} &\left\langle \frac{\partial(e^T e)}{\partial e}, A_{21}x + A_{22}e \right\rangle = 2e^T A_{21}x + 2e^T A_{22}e \\ &\leq 2\sqrt{W(k, 1, e, s)}|A_{21}x| + 2|A_{22}|W(k, 1, e, s) \\ &\leq 2\sqrt{W(k, 1, e, s)}H_1(x) + L_1 W(k, 1, e, s), \end{aligned} \tag{31}$$

which is satisfied if $H_1(x) = |A_{21}x|$ and $L_1 = 2|A_{22}|$. From this, we define $H(x) = H_0(x) = H_1(x)$. Then, we consider $V(x) = x^T P x$ (note that (13) holds). We can derive from (14) the values for γ_0 and γ_1, such that *Assumption 3* holds. In particular, for this example, we will consider $\rho(|x, e|) = \rho_0(|x, e|) = \rho_1(|x, e|)$ and $\gamma = \gamma_0 = \gamma_1$ so we will proceed indistinctly for $q = 0$ and $q = 1$ as follows

$$\begin{aligned} &\langle \nabla V(x), A_{11}x + A_{12}e \rangle = 2x^T P A_{11}x + 2x^T P A_{12}e \\ &\leq -\rho(|x, e|) - H^2(x) + \gamma^2 W(k, q, e, s) \end{aligned} \tag{32}$$

Defining $\rho(|x, e|) = \nu|x|^2 + \nu|e|^2$, (32) results in the following LMI



(a) Scenario 1 (b) Scenario 2 (c) Scenario 3 (d) Scenario 4

Fig. 2: Discrete probability distributions ruling the transmission delay in each scenario.

$$\begin{pmatrix} A_{11}^T P + P A_{11} + \nu I + A_{21}^T A_{21} & P A_{12} \\ A_{12}^T P & (\nu - \gamma^2) I \end{pmatrix} \leq 0 \tag{33}$$
$$P = P^T > 0$$

Solving (33) with the YALMIP toolbox and setting $\nu = 0.01$, we obtain $\gamma = 2.33$.

Regarding ϕ_0, we can approximate it as a linear function and by setting $\delta = 0.1$, $\phi_0(0) = 2.35$ and $\phi_1(0) = 0.4$, (15) can be solved. Fig. 3 shows the resulting ϕ_0 and ϕ_1.

To validate *Theorem 1* we simulate the scenarios in Fig. 2. Since we are working with probabilities, we ran a total of 500 realizations for each scenario. From this, we can calculate the probability of the state converging to the origin for the different delay distribution used. Fig. 4 shows the state trajectory for one representative realization to illustrate the expected behavior of the system in each scenario.

For the probability distribution in scenario 1, the conditions (18) and (19) in *Theorem 1* hold and the simulations indeed indicate that the system is stable with probability 1. This result is meaningful since it implies that the MAD can be violated in 5% of the transmissions without compromising the stability of the NCS. For scenarios 2, 3 and 4, the conditions in *Theorem 1* do not hold. However, in scenario 2, the amount by which the conditions are violated is lower than in scenario 3, and in scenario 3 are lower than in scenario 4. This fact impacts the simulation results, where scenario 2 yielded a 75.8% probability of being stable, while scenarios 3 and 4, yielded a 18.2% and 0%, respectively.

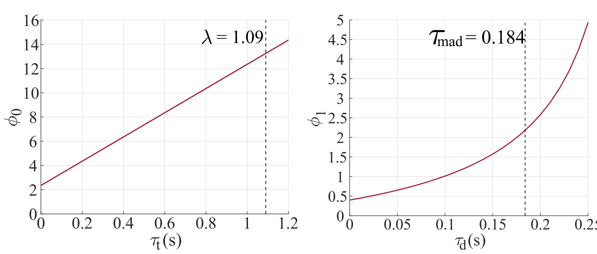
The representative trajectories in Fig. 4 graphically illustrate the different dynamical behavior in each scenario. While in scenario 1 convergence is clean, stability deteriorates progressively as the delay distribution makes violating the MAD more likely. These results suggest that quantifying the amount by which the conditions in *Theorem 1* are violated is relevant to assess the dynamical behavior of the NCS.

VI. Conclusions

In this work, analytical conditions are given that ensure uniform stability in probability for a class of NCSs subject to a stochastic transmission delay that takes values over the MAD. It is shown that the deterministic MAD can be violated with a low probability without compromising the stability of the system. The results are verified in the context of a simulated example.

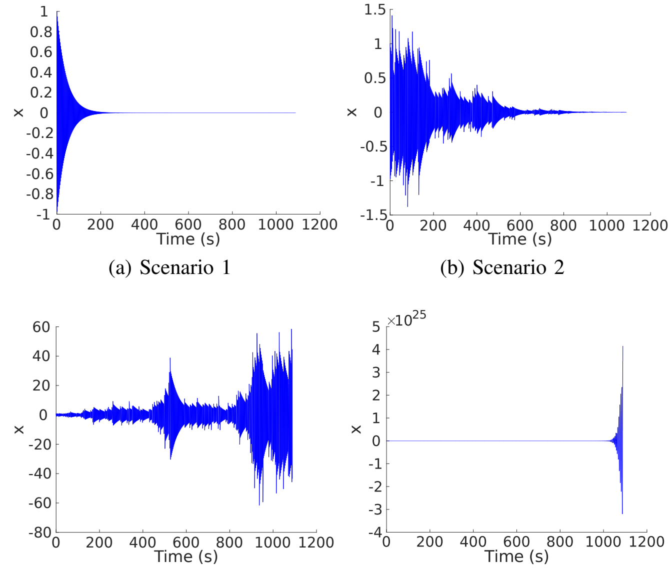
The results of this work are of practical relevance when an NCS, designed in an ideal setting and hence with a known MAD, is operated in a realistic scenario where the channel is no longer deterministic and becomes stochastic. If an estimation of the delay distribution is available, our results allow giving a guarantee for uniform stability in probability.

Future work includes extending the results for a more general NCS setup and verifying the stability conditions in a real implementation.



(a) ϕ_0 (b) ϕ_1

Fig. 3: Functions ϕ_i used in the numerical analysis.



(a) Scenario 1 (b) Scenario 2 (c) Scenario 3 (d) Scenario 4

Fig. 4: Representative state trajectory for each scenario.

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