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Trackability-Based Tracking Control for Stochastic Learning Systems: A Two-Dimensional System Method

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Abstract— **This paper focuses on exploring a novel trackability-based framework for the iterative learning control (ILC) systems subject to stochastic disturbances by using a two-dimensional (2-D) system method. By examining the fundamental trackability property of the stochastic ILC systems, the trackability-based stochastic ILC design and analysis framework is developed, eliminating the need for the common realizability assumption. Under this framework, thanks to the 2-D system method with the Roesser systems, the convergence results for both the output and input errors can be established under a unified condition, regardless of the full column or row rank of the input-output coupling matrix. A simulation example is included to demonstrate the validity of our proposed stochastic ILC design framework for ILC systems.**

I. Introduction

Iterative learning control (ILC), one of the widely-employed intelligent control methods, has been implemented to numerous fields to achieve the trajectory tracking control tasks, such as high-speed trains, batch processes, and robot manipulators (see, e.g., [1]–[3]). The working principle of ILC involves designing an appropriate learning gain matrix to learn and correct the control input in repetitive tasks, which aims to achieve convergence of the control input to the desired input while simultaneously achieving perfect tracking of the prescribed trajectory. Further, ILC possesses a natural advantage in overcoming the influences of iteration-invariant (or repetitive) uncertainties on the prescribed tracking objectives (see, e.g., [4]). Nevertheless, when the ILC systems are subjected to the stochastic disturbances, the repetitive uncertainties are effectively attenuated, leading to primary errors arising from the stochastic disturbances after a limited number of iterations. This constraint significantly narrows the usability of ILC.

To overcome the stochastic disturbances for the ILC systems, many effective stochastic ILC methods have been developed to achieve the tracking objectives such that the stochastic ILC problems can be addressed in the sense of the probability. There generally exist three categories of stochastic ILC methods, i.e., the stochastic approximation based ILC method (see, e.g., [5]–[7]), the stochastic optimization ILC method (see, e.g., [8]–[10]), and the statistics based method (see, e.g., [11]–[13]).

In most of the existing stochastic ILC literature, the realizability assumption is commonly imposed on the stochastic ILC systems (see, e.g., [5]–[15]), which necessitates the input-output coupling matrix to be of full column rank. The validity of this assumption in stochastic ILC systems is open to debate. By utilizing the realizability assumption, it is practical to adopt the indirect analysis methods for ILC systems subject to stochastic disturbances. The control input convergence results can be obtained, enabling the achievement of the tracking objectives that the output can converge to the prescribed trajectory in the probabilistic sense when using the indirect methods to address the stochastic ILC convergence problems (see, e.g., [5]–[15]). However, many stochastic ILC problems may not satisfy the realizability assumption, rendering the indirect convergence analysis method ineffective in dealing with these problems.

Motivated by the above discussions, we intend to propose a novel trackability-based stochastic ILC framework by using two-dimensional (2-D) method. For the stochastic ILC system, we first explore the fundamental trackability property and elucidate the difference between trackability and realizability. Specifically, the results obtained from realizability-based stochastic ILC can be viewed as a special case of trackability-based results. Further, thanks to the 2-D system method with the Roesser systems, we develop the trackability-based stochastic ILC design and analysis framework, eliminating the requirement for the commonly assumed realizability condition in the existing stochastic ILC literature (see, e.g., [5]–[15]). In addition, we resort to a unified condition to develop the convergence results for both the tracking errors and the inputs of stochastic ILC systems, regardless of the full column or row rank of the input-output coupling matrix, which can not be attained in e.g., [5]–[15].

The rest of this paper is organized as follows. In Section II, we introduce our concerned stochastic ILC problems. We establish a trackability-based stochastic ILC design framework in Section III. Section IV includes a simulation example demonstrating the validity of our proposed stochastic ILC framework. Finally, we draw conclusions in Section V.

Notations: Let $\mathbb{Z}_+ \triangleq \{0,1,2,\dots\}$ and $\mathbb{Z}_N \triangleq \{0,1,\dots,N\}$ for $N \in \mathbb{Z}_+\backslash\{0\}$. Let 0 and I be zero matrix and identity matrix with appropriate dimensions, respectively. For any vector $x \in \mathbb{R}^n$, $\|x\|_2$ denotes its 2-norm. For any vector sequence $\{x_k(t) : k \in \mathbb{Z}_+, t \in \mathbb{Z}_N\}$, let $\Delta x_k(t) = x_{k+1}(t) - x_k(t)$. For any matrix $A \in \mathbb{R}^{n\times n}$, $\rho(A)$ denotes its spectral radius.

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For any random variable X, $\mathbb{E}\{X\}$ denotes its expectation.

II. Problem Statement

We consider a stochastic ILC system evolving over both the time axis $t \in \mathbb{Z}_N$ and the iteration axis $k \in \mathbb{Z}_+$:

$$\begin{cases} x_k(t+1) = A(t)x_k(t) + B(t)u_k(t) + w_k(t) \\ \quad y_k(t) = C(t)x_k(t) + v_k(t) \end{cases} \tag{1}$$

where $x_k(t) \in \mathbb{R}^n$, $u_k(t) \in \mathbb{R}^p$, and $y_k(t) \in \mathbb{R}^m$ are the state, input, and output, respectively; $w_k(t) \in \mathbb{R}^n$ and $v_k(t) \in \mathbb{R}^m$ denote the stochastic disturbances; and $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times p}$, and $C(t) \in \mathbb{R}^{m\times n}$ denote the time-varying system matrices. We consider the initial state fixed at $x_k(0) = x_0$, $\forall k \in \mathbb{Z}_+$. Further, we denote a σ-algebra as

$$\mathscr{F}_k = \sigma\{w_i(t), v_i(t), 0 \le i \le k, 0 \le t \le N\}, \ \forall k \in \mathbb{Z}_+.$$

Hence, $\{\mathscr{F}_k : k \in \mathbb{Z}_+\}$ forms a filtration, and then we present a typical assumption regarding the stochastic disturbances:

A1) Let $\{w_k(t) : t \in \mathbb{Z}_N, k \in \mathbb{Z}_+\}$ and $\{v_k(t) : t \in \mathbb{Z}_N, k \in \mathbb{Z}_+\}$ satisfy $\mathbb{E}\{w_{k+1}(t)|\mathscr{F}_k\} = 0$ and $\mathbb{E}\{v_{k+1}(t)|\mathscr{F}_k\} = 0$, $\forall k \in \mathbb{Z}_+, t \in \mathbb{Z}_N$, respectively.

The Assumption A1) encompasses the commonly used white noise assumption concerning the stochastic disturbances, and thus can characterize the stochastic disturbances in many practical applications (see, e.g., [16]). Thanks to the tower property of conditional expectation, the properties of the stochastic disturbances $w_k(t)$ and $v_k(t)$ can be given as

$$\mathbb{E}\{w_k(t)\} = 0, \ \mathbb{E}\{v_k(t)\} = 0, \ \forall k \in \mathbb{Z}_+, t \in \mathbb{Z}_N. \tag{2}$$

Problem statement: Our objective is to determine some control input sequence $\{u_k(t) : t \in \mathbb{Z}_{N-1}, k \in \mathbb{Z}_+\}$ such that for the prescribed trajectory $y_d(t) \in \mathbb{R}^m$, $\forall t \in \mathbb{Z}_N$, the stochastic ILC system (1) can achieve the asymptotic tracking objective:

$$\lim_{k\to\infty} \mathbb{E}\{e_k(t)\} = 0, \ \forall t \in \mathbb{Z}_N \backslash \{0\} \tag{3}$$

where $e_k(t) = y_d(t) - y_k(t)$ denotes the tracking error.

To fulfill the asymptotic tracking objective (3), we introduce two essential definitions for stochastic ILC system (1).

Definition 1. *The prescribed trajectory $y_d(t)$, $t \in \mathbb{Z}_N$ is said to be trackable for stochastic ILC system (1) if some desired inputs $u_d(t) \in \mathbb{R}^m$, $t \in \mathbb{Z}_{N-1}$ exist, together with the initial state condition $x_d(0) = x_0$, such that*

$$\begin{cases} x_d(t+1) = A(t)x_d(t) + B(t)u_d(t) \\ y_d(t+1) = C(t+1)x_d(t+1) \end{cases}, \ t \in \mathbb{Z}_{N-1}. \tag{4}$$

Definition 2. *The prescribed trajectory $y_d(t)$, $t \in \mathbb{Z}_N$ is said to be realizable for stochastic ILC system (1) if a unique desired input $u_d(t) \in \mathbb{R}^m$, $t \in \mathbb{Z}_{N-1}$ exists, together with the initial state condition $x_d(0) = x_0$, such that (4) holds.*

Note that the realizability assumption is generally imposed on classic stochastic ILC systems (see, e.g., [5]–[15]). Based on Definitions 1 and 2, it is clearly that the realizability of the prescribed trajectory $y_d(t)$, $t \in \mathbb{Z}_N$ can be viewed as a special case of the trackability. Therefore, solving trackability-based stochastic ILC problems can enhance the design and analysis methods for existing stochastic ILC results of, e.g., [5]–[15], which heavily rely on the realizability assumption.

With the above discussions, we propose a helpful lemma to develop our trackability-based stochastic ILC results.

Lemma 1. *For the ILC system (1), any prescribed trajectory $y_d(t)$, $\forall t \in \mathbb{Z}_N$*

1) *is realizable if and only if it is trackable when $C(t+1)B(t)$, $\forall t \in \mathbb{Z}_{N-1}$ is of full column rank;*
2) *is trackable, but not realizable when $C(t+1)B(t)$, $\forall t \in \mathbb{Z}_{N-1}$ is of full row rank.*

Proof. See [17, Theorem 1]. ∎

With Lemma 1, we know that trackability is a more essential property for the stochastic ILC system (1) compared to realizability. Thus, the subsequent sections of this paper will focus on developing the trackability-based results.

III. Main Results

In this section, to achieve the asymptotic tracking objective (3) and develop the trackability-based results, we consider two cases separately for the stochastic ILC system (1): $\operatorname{rank}(C(t+1)B(t)) = m$, $\forall t \in \mathbb{Z}_{N-1}$ and $\operatorname{rank}(C(t+1)B(t)) = p$, $\forall t \in \mathbb{Z}_{N-1}$.

A. Case: $\operatorname{rank}(C(t+1)B(t)) = m$, $\forall t \in \mathbb{Z}_{N-1}$

We adopt the following stochastic ILC algorithm for the system (1):

$$u_{k+1}(t) = u_k(t) + \Xi(t)e_k(t+1), \ \forall k \in \mathbb{Z}_+, t \in \mathbb{Z}_{N-1} \tag{5}$$

where $\Xi(t) \in \mathbb{R}^{p\times m}$ denotes the learning gain matrix. We then leverage (1) and (5) to derive

$$\begin{aligned} \Delta x_k(t+1) &= x_{k+1}(t+1) - x_k(t+1) \\ &= A(t)\Delta x_k(t) + B(t)\Xi(t)e_k(t+1) + \Delta w_k(t). \end{aligned} \tag{6}$$

Further, we can derive

$$\begin{aligned} &e_{k+1}(t+1) \\ &= y_d(t+1) - y_k(t+1) + y_k(t+1) - y_{k+1}(t+1) \\ &= e_k(t+1) - C(t+1)A(t)\Delta x_k(t) - C(t+1)\Delta w_k(t) \\ &\quad - C(t+1)B(t)\Xi(t)e_k(t+1) - \Delta v_k(t+1). \end{aligned} \tag{7}$$

Combine (6) and (7) in a 2-D Roesser system as

$$\begin{aligned} &\begin{bmatrix} \Delta x_k(t+1) \\ e_{k+1}(t+1) \end{bmatrix} \\ &= \begin{bmatrix} A(t) & B(t)\Xi(t) \\ -C(t+1)A(t) & I - C(t+1)B(t)\Xi(t) \end{bmatrix} \begin{bmatrix} \Delta x_k(t) \\ e_k(t+1) \end{bmatrix} \\ &\quad + \begin{bmatrix} I & 0 \\ -C(t+1) & -I \end{bmatrix} \begin{bmatrix} \Delta w_k(t) \\ \Delta v_k(t+1) \end{bmatrix}. \end{aligned} \tag{8}$$

Then, we develop the trackability-based results for the stochastic ILC system (1) under the case $\operatorname{rank}(C(t+1)B(t)) = m$, $\forall t \in \mathbb{Z}_{N-1}$ in the following theorem.

Theorem 1. *For any prescribed trajectory $y_d(t)$, $\forall t \in \mathbb{Z}_N$, let Assumption A1) and $\operatorname{rank}(C(t+1)B(t)) = m$, $\forall t \in \mathbb{Z}_{N-1}$ hold, and the stochastic ILC algorithm (5) be applied to* *the stochastic ILC system (1). Then, the asymptotic tracking objective (3) and the convergence of the input $u_k(t)$, i.e.,*

$$\lim_{k\to\infty} \mathbb{E}\{u_\infty(t)-u_k(t)\}=0,\ \forall t\in\mathbb{Z}_{N-1} \tag{9}$$

can be achieved if and only if

$$\rho\left(I-C(t+1)B(t)\Xi(t)\right)<1,\ \forall t\in\mathbb{Z}_{N-1} \tag{10}$$

where $u_\infty(t)$ is dependent on $u_0(t)$, $t\in\mathbb{Z}_{N-1}$.

Proof. We implement two steps to develop the convergence results. *Step i):* The asymptotic tracking objective (3). With (2) and (8), we can derive

$$\begin{aligned}&\begin{bmatrix}\mathbb{E}\{\Delta x_k(t+1)\}\\ \mathbb{E}\{e_{k+1}(t+1)\}\end{bmatrix}\\ &=\begin{bmatrix}A(t) & B(t)\Xi(t)\\ -C(t+1)A(t) & I-C(t+1)B(t)\Xi(t)\end{bmatrix}\\ &\quad\times\begin{bmatrix}\mathbb{E}\{\Delta x_k(t)\}\\ \mathbb{E}\{e_k(t+1)\}\end{bmatrix},\ \forall k\in\mathbb{Z}_+,t\in\mathbb{Z}_{N-1}.\end{aligned} \tag{11}$$

By applying the 2-D system theory from [18, Lemma 3] to (11), we establish that $\lim_{k\to\infty}\mathbb{E}\{e_k(t)\}=0$, $\forall t\in\mathbb{Z}_N\backslash\{0\}$ if and only if $\rho\left(I-C(t+1)B(t)\Xi(t)\right)<1$, $\forall t\in\mathbb{Z}_{N-1}$, where $\mathbb{E}\{\Delta x_k(0)\}=0$, $\forall k\in\mathbb{Z}_+$ is inserted.

Step ii): The convergence of the input (9). Thanks to $\operatorname{rank}(C(t+1)B(t))=m$, $\forall t\in\mathbb{Z}_{N-1}$, there exists a full row rank matrix $\widehat{M}(t)\in\mathbb{R}^{(p-m)\times p}$, $\forall t\in\mathbb{Z}_{N-1}$ such that we can obtain a nonsingular matrix $D(t)\in\mathbb{R}^{p\times p}$ with

$$D(t)=\begin{bmatrix}(C(t+1)B(t))^{\mathrm{T}} & \widehat{M}(t)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}.$$

Then, we can denote its inverse matrix in a specific form as

$$S(t)=D(t)^{-1}=\begin{bmatrix}\Xi(t)\left(C(t+1)B(t)\Xi(t)\right)^{-1} & \widehat{N}(t)\end{bmatrix}$$

where $\widehat{N}(t)\in\mathbb{R}^{p\times(p-m)}$, $\forall t\in\mathbb{Z}_{N-1}$ is a full column rank matrix. Clearly, we have

$$\begin{aligned}C(t+1)B(t)S(t)&=\begin{bmatrix}I_m & 0\end{bmatrix}\\ D(t)\Xi(t)&=\begin{bmatrix}C(t+1)B(t)\Xi(t)\\ 0\end{bmatrix}.\end{aligned}$$

Let $\widehat{u}_k(t)=D(t)u_k(t)=\begin{bmatrix}\widehat{u}_{1,k}(t)^{\mathrm{T}} & \widehat{u}_{2,k}(t)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}$, $\forall k\in\mathbb{Z}_+,t\in\mathbb{Z}_{N-1}$ with $\widehat{u}_{1,k}(t)\in\mathbb{R}^m$ and $\widehat{u}_{2,k}(t)\in\mathbb{R}^{p-m}$. Then, with (5), we can derive

$$\begin{cases}\widehat{u}_{1,k+1}(t)=\widehat{u}_{1,k}(t)+C(t+1)B(t)\Xi(t)e_k(t+1)\\ \widehat{u}_{2,k+1}(t)=\widehat{u}_{2,k}(t)=\widehat{u}_{2,0}(t),\ \forall k\in\mathbb{Z}_+,t\in\mathbb{Z}_{N-1}.\end{cases} \tag{12}$$

Let us revisit (1), and then we have

$$\begin{aligned}y_k(t+1)&=C(t+1)B(t)S(t)\widehat{u}_k(t)+C(t+1)A(t)x_k(t)\\ &\quad+C(t+1)w_k(t)+v_k(t+1)\\ &=\widehat{u}_{1,k}(t)+C(t+1)A(t)x_k(t)\\ &\quad+C(t+1)w_k(t)+v_k(t+1).\end{aligned} \tag{13}$$

Substituting (13) into (12), we have

$$\begin{aligned}&\widehat{u}_{1,k+1}(t)\\ &=\widehat{u}_{1,k}(t)+C(t+1)B(t)\Xi(t)\left(y_d(t+1)-y_k(t+1)\right)\\ &=\left(I-C(t+1)B(t)\Xi(t)\right)\widehat{u}_{1,k}(t)\\ &\quad+C(t+1)B(t)\Xi(t)\left(y_d(t+1)-C(t+1)A(t)x_k(t)\right)\\ &\quad-C(t+1)B(t)\Xi(t)\left(C(t+1)w_k(t)+v_k(t+1)\right).\end{aligned} \tag{14}$$

Subtracting $y_d(t+1)-C(t+1)A(t)x_{k+1}(t)$ from both sides of (14), and then taking the mathematical expectation, yields

$$\begin{aligned}&\mathbb{E}\left\{\widehat{u}_{1,k+1}(t)-y_d(t+1)+C(t+1)A(t)x_{k+1}(t)\right\}\\ &=\left(I-C(t+1)B(t)\Xi(t)\right)\\ &\quad\times\mathbb{E}\left\{\widehat{u}_{1,k}(t)-y_d(t+1)+C(t+1)A(t)x_k(t)\right\}\\ &\quad+\mathbb{E}\left\{C(t+1)A(t)\Delta x_k(t)\right\}.\end{aligned} \tag{15}$$

We then develop the convergence of $\widehat{u}_{1,k}(t)$ in (15) by an enumerative method. For $t=0$, we have

$$\begin{aligned}&\mathbb{E}\left\{\widehat{u}_{1,k+1}(0)-y_d(1)+C(1)A(0)x_0\right\}\\ &=\left(I-C(1)B(0)K(0)\right)\mathbb{E}\left\{\widehat{u}_{1,k}(0)-y_d(1)+C(1)A(0)x_0\right\}.\end{aligned} \tag{16}$$

With the condition (10), we can derive

$$\lim_{k\to\infty}\mathbb{E}\left\{\widehat{u}_{1,k}(0)\right\}=\widehat{u}_{1,\infty}(0)=y_d(1)-C(1)A(0)x_0. \tag{17}$$

Then, we have $u_\infty(0)=S(0)\begin{bmatrix}\widehat{u}_{1,\infty}(0)^{\mathrm{T}} & \widehat{u}_{2,0}(0)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}$, and thus $x_\infty(1)=A(0)x_0+B(0)u_\infty(0)$ can be obtained. Further, we derive $\mathbb{E}\{\Delta x_\infty(1)\}=0$. For $t=1$, we have

$$\begin{aligned}&\mathbb{E}\left\{\widehat{u}_{1,k+1}(1)-y_d(2)+C(2)A(1)x_\infty(1)\right\}\\ &=\left(I-C(2)B(1)K(1)\right)\\ &\quad\times\mathbb{E}\left\{\widehat{u}_{1,k}(1)-y_d(2)+C(2)A(1)x_\infty(1)\right\}.\end{aligned} \tag{18}$$

With the condition (10), we can obtain

$$\begin{aligned}&\lim_{k\to\infty}\mathbb{E}\left\{\widehat{u}_{1,k}(1)\right\}\\ &=\widehat{u}_{1,\infty}(1)=y_d(2)-C(2)A(1)\left(A(0)x_0+B(0)u_\infty(0)\right).\end{aligned} \tag{19}$$

Similarly, repeating the above steps, we can obtain $\lim_{k\to\infty}\mathbb{E}\left\{\widehat{u}_{1,k}(2)\right\}=\widehat{u}_{1,\infty}(2)$. Continuing the enumerative method, we directly present the result for the sake of space efficiency as:

$$\lim_{k\to\infty}\mathbb{E}\left\{\widehat{u}_{1,k}(t)-\widehat{u}_{1,\infty}(t)\right\}=0,\ \forall t\in\mathbb{Z}_{N-1} \tag{20}$$

where $\widehat{u}_{1,\infty}(t)=y_d(t+1)-C(t+1)A(t)x_\infty(t)$. Let us denote

$$\widehat{u}_\infty(t)=\begin{bmatrix}\widehat{u}_{1,\infty}(t)^{\mathrm{T}} & \left(\widehat{M}(t)u_0(t)\right)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}$$

then we leverage (12) and (20) to obtain

$$\lim_{k\to\infty}\mathbb{E}\{\widehat{u}_k(t)-\widehat{u}_\infty(t)\}=0,\ \forall t\in\mathbb{Z}_{N-1}. \tag{21}$$

Then we denote

$$\begin{aligned}u_\infty(t)&=S(t)\widehat{u}_\infty(t)\\ &=\Xi(t)\left(C(t+1)B(t)\Xi(t)\right)^{-1}\widehat{u}_{1,\infty}(t)\\ &\quad+\widehat{N}(t)\widehat{M}(t)u_0(t)\end{aligned}$$

and we utilize $u_k(t)=S(t)\widehat{u}_k(t)$ and (21) to derive

$$\lim_{k\to\infty}\mathbb{E}\{u_\infty(t)-u_k(t)\}=0,\ \forall t\in\mathbb{Z}_{N-1}$$

where $u_\infty(t)$ is given by

$$\begin{aligned}u_\infty(t)&=\widehat{N}(t)\widehat{M}(t)u_0(t)+\Xi(t)\left(C(t+1)B(t)\Xi(t)\right)^{-1}\\ &\quad\times\left(y_d(t+1)-C(t+1)A(t)x_\infty(t)\right)\end{aligned} \tag{22}$$

with $\widehat{N}(t)\widehat{M}(t) = I - \Xi(t)\left(C(t+1)B(t)\Xi(t)\right)^{-1} C(t+1)B(t)$. Clearly, we have

$$u_\infty(0) = \widehat{N}(0)\widehat{M}(0)u_0(0) + K(0)\left(C(1)B(0)K(0)\right)^{-1} \times \left(y_d(1) - C(1)A(0)x_0\right)$$

Then, with (1), we denote

$$x_\infty(1) = A(0)x_0 + B(0)u_\infty(0).$$

By extension, we can obtain $u_\infty(t)$, $\forall t \in \mathbb{Z}_{N-1}$, and $u_\infty(t)$, $\forall t \in \mathbb{Z}_{N-1}$ depends on $u_0(t)$, $\forall t \in \mathbb{Z}_{N-1}$. ■

Remark 1. *Thanks to the 2-D system theory [19], the trackability-based results are developed for the stochastic ILC system (1) in Theorem 1. Under the condition that $C(t+1)B(t)$, $\forall t \in \mathbb{Z}_{N-1}$ is of full row rank, the convergence of both the tracking errors and the input can be guaranteed under a unified condition. Besides, this class of analysis methods has not been explored in the existing stochastic ILC literature.*

B. Case: rank$(C(t+1)B(t)) = p$, $\forall t \in \mathbb{Z}_{N-1}$

We then consider the case where the input-output coupling matrix $C(t+1)B(t)$, $\forall t \in \mathbb{Z}_{N-1}$ is of full column rank, namely, rank$(C(t+1)B(t)) = p$, $\forall t \in \mathbb{Z}_{N-1}$. The trackability-based results for the stochastic ILC system (1) are developed in the following theorem.

Theorem 2. *Given the trackable trajectory $y_d(t)$, $\forall t \in \mathbb{Z}_N$, let Assumption A1) and rank$(C(t+1)B(t)) = p$, $\forall t \in \mathbb{Z}_{N-1}$ hold, and the stochastic ILC algorithm (5) be applied to the stochastic ILC system (1). Then, the asymptotic tracking objective (3) and the convergence of the input $u_k(t)$, i.e.,*

$$\lim_{k\to\infty} \mathbb{E}\{u_d(t) - u_k(t)\} = 0,\ \forall t \in \mathbb{Z}_{N-1} \tag{23}$$

can be achieved if and only if

$$\rho\left(I - \Xi(t)C(t+1)B(t)\right) < 1,\ \forall t \in \mathbb{Z}_{N-1}. \tag{24}$$

Proof. With (1) and (4), we can obtain

$$\delta x_k(t+1) = A(t)\delta x_k(t) + B(t)\delta u_k(t) - w_k(t) \tag{25}$$

where we denote $\delta u_k(t) = u_d(t) - u_k(t)$ and $\delta x_k(t) = x_d(t) - x_k(t)$ for compactness. Using (1), (4), and (5), we can derive

$$\begin{aligned}\delta u_{k+1}(t) &= \delta u_k(t) - \Xi(t)e_k(t+1)\\ &= \delta u_k(t) - \Xi(t)\left(y_d(t+1) - y_k(t+1)\right)\\ &= \delta u_k(t) - \Xi(t)\left(C(t+1)\delta x_k(t+1) - v_k(t+1)\right)\end{aligned} \tag{26}$$

which, together with (25), yields

$$\begin{aligned}\delta u_{k+1}(t) &= \left(I - \Xi(t)C(t+1)B(t)\right)\delta u_k(t)\\ &\quad - \Xi(t)C(t+1)A(t)\delta x_k(t)\\ &\quad + \Xi(t)\left(C(t+1)w_k(t) + v_k(t+1)\right).\end{aligned} \tag{27}$$

Combining (25) and (27) in a 2-D Roesser system as

$$\begin{aligned}&\begin{bmatrix}\delta x_k(t+1)\\ \delta u_{k+1}(t)\end{bmatrix}\\ &= \begin{bmatrix}A(t) & B(t)\\ -\Xi(t)C(t+1)A(t) & I - \Xi(t)C(t+1)B(t)\end{bmatrix}\begin{bmatrix}\delta x_k(t)\\ \delta u_k(t)\end{bmatrix}\\ &\quad + \begin{bmatrix}-I & 0\\ \Xi(t)C(t+1) & \Xi(t)\end{bmatrix}\begin{bmatrix}w_k(t)\\ v_k(t+1)\end{bmatrix}.\end{aligned} \tag{28}$$

Then, thanks to Assumption A1) and the tower property of conditional expectation, we have

$$\begin{aligned}&\begin{bmatrix}\mathbb{E}\{\delta x_k(t+1)\}\\ \mathbb{E}\{\delta u_{k+1}(t)\}\end{bmatrix}\\ &= \begin{bmatrix}A(t) & B(t)\\ -\Xi(t)C(t+1)A(t) & I - \Xi(t)C(t+1)B(t)\end{bmatrix}\\ &\quad \times \begin{bmatrix}\mathbb{E}\{\delta x_k(t)\}\\ \mathbb{E}\{\delta u_k(t)\}\end{bmatrix}.\end{aligned} \tag{29}$$

By applying the 2-D system theory from [18, Lemma 2] to (29), we establish that $\lim_{k\to\infty}\mathbb{E}\{\delta u_k(t)\} = 0$, $\forall t \in \mathbb{Z}_{N-1}$ and $\lim_{k\to\infty}\mathbb{E}\{\delta x_k(t)\} = 0$, $\forall t \in \mathbb{Z}_N$ if and only if $\rho\left(I - \Xi(t)C(t+1)B(t)\right) < 1$, $\forall t \in \mathbb{Z}_{N-1}$, where $\mathbb{E}\{\delta x_k(0)\} = 0$, $\forall k \in \mathbb{Z}_+$ is inserted. Further, by noting that $\mathbb{E}\{e_k(t)\} = C(t)\mathbb{E}\{\delta x_k(t)\}$, the asymptotic tracking objective (3) can be guaranteed. ■

Remark 2. *For the stochastic ILC algorithm (5), we can always choose some learning gain matrix that satisfies the convergence condition (10) (or (24)), which is due to rank$(C(t+1)B(t)) = m$, $\forall t \in \mathbb{Z}_{N-1}$ (or rank$(C(t+1)B(t)) = p$, $\forall t \in \mathbb{Z}_{N-1}$). To be specific, the candidate selection of the gain matrix $\Xi(t)$, $\forall t \in \mathbb{Z}_{N-1}$ can be chosen as*

$$\Xi(t) = \begin{cases}\xi\left(C(t+1)B(t)\right)^{\mathrm{T}}\left(C(t+1)B(t)\left(C(t+1)B(t)\right)^{\mathrm{T}}\right)^{-1}, \text{or}\\ \xi\left(\left(C(t+1)B(t)\right)^{\mathrm{T}}C(t+1)B(t)\right)^{-1}\left(C(t+1)B(t)\right)^{\mathrm{T}}\end{cases}$$

where $0 < \xi < 2$. Hence, we have $I - C(t+1)B(t)\Xi(t) = (1-\xi)I$, $\forall t \in \mathbb{Z}_{N-1}$ (or $I - \Xi(t)C(t+1)B(t) = (1-\xi)I$, $\forall t \in \mathbb{Z}_{N-1}$), namely, the condition (10) (or (24)) holds.

IV. Simulation Example

For illustration, the permanent magnet synchronous motor (PMSM) example (see also [20]) is considered in this section. To achieve the tracking objective (3) by applying our discrete-time stochastic ILC algorithm (5), the sampling time $h = 0.1s$ is adopted, and then the PMSM system can be discretized into the system (1) with

$$A(t) = \begin{bmatrix}\dfrac{L_s - hR(t)}{L_s} & -\dfrac{hL_r\omega}{L_s}\\ \dfrac{hL_s\omega}{L_r} & \dfrac{L_r - hR(t)}{L_r}\end{bmatrix},$$

$$B(t) = \begin{bmatrix}\dfrac{h}{L_s} & 0\\ 0 & \dfrac{h}{L_r}\end{bmatrix},\ \text{and } C(t) = I$$

where $L_r, L_s, R(t)$, and ω are given in Table I.

The stochastic disturbances $w_\Xi(t)$ and $v_\Xi(t)$ follow normally distributed white stochastic processes with $N(0, 0.1^2 I)$. Hence, Assumptions A1) can be ensured. Clearly, $C(t+1)B(t), \forall t \in \mathbb{Z}_{N-1}$ is of both full column and row rank. In this example, we adapt the input voltages to track the prescribed current

$$y_d(t) = \begin{bmatrix}0.03(t-10)^2 & 0.01(t-10)^2\end{bmatrix}^{\mathrm{T}},\ t \in \mathbb{Z}_{20}.$$

TABLE I
PHYSICAL PARAMETERS

	Parameter	Value
Inductance	L_r	1.1 H
Inductance	L_s	1.1 H
Resistance	$R(t)$	$0.5(1.4-0.4e^{-t})\Omega$
Angular velocity	ω	0.1 rad/s

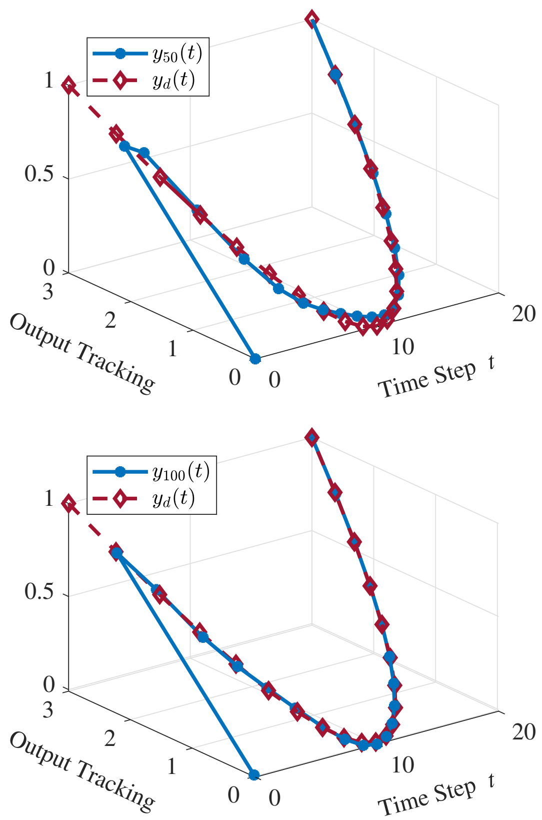

Fig. 1. The tracking performance with the prescribed trajectory after $k=50$ iterations (top) and $k=100$ iterations (bottom).

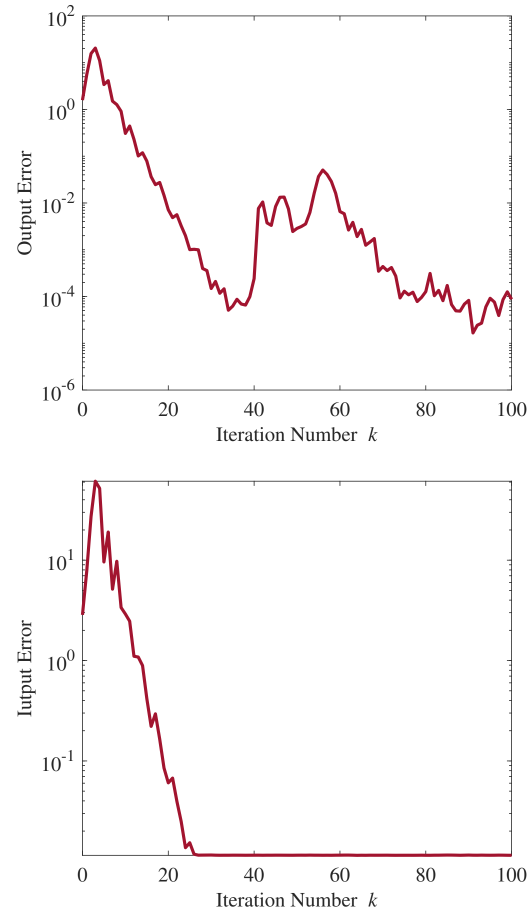

Fig. 2. The error evolutions with respect to the iteration axis. Top: the output error. Bottom: the input error.

We perform the simulation for the stochastic ILC system (1) using ILC algorithm (5) with the initial conditions chosen as $u_0(t)=0$, $t\in\mathbb{Z}_{19}$ and $x_k(0)=0$, $k\in\mathbb{Z}_+$. The learning gain matrix $\Xi(t)$ is given by $\Xi(t)=0.7\left(C(t+1)B(t)\right)^{-1}$, $t\in\mathbb{Z}_{19}$. Thus, both the convergence conditions (10) and (24) can be ensured. The tracking performance of the stochastic ILC system (1) for the 50th and 100th is plotted in Fig. 1. It is clear to see that the output learned from ILC algorithm (5) after $k=100$ iterations closely tracks the prescribed trajectory for $t\in\mathbb{Z}_{20}\backslash\{0\}$. Besides, the evolution of the output error $\max_{t\in\mathbb{Z}_{20}\backslash\{0\}}\|y_d(t)-y_k(t)\|_2$ and the input error $\max_{t\in\mathbb{Z}_{19}}\|u_\infty(t)-u_k(t)\|_2$ for the stochastic ILC system (1) is provided in Fig. 2. Thus, the effectiveness of the stochastic ILC results developed in Theorems 1 and 2 is validated.

V. Conclusions

In this paper, we have established a trackability-based stochastic ILC framework for ILC systems subject to stochastic disturbances, which can eliminate the need for the common realizability assumption. Thanks to the exploration of the trackability property, we have developed the trackability-based convergence results for the stochastic ILC systems by using 2-D system method, which can furnish new insights into the design and analysis of the stochastic ILC. With the trackability-based stochastic ILC framework, we have achieved the convergence for both the input and output errors under a unified condition, avoiding the need for the common realizability assumption. An example has been included to illustrate the validity of our proposed framework for stochastic ILC systems.

In our future work, we will explore the trackability-based stochastic ILC framework for ILC systems subject to initial state disturbances. Additionally, we will focus on establishing trackability-based convergence results in the mean-square sense for stochastic ILC systems.

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