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Nonlinear Quasi-unknown Input Observer Design using Dissipativity

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Abstract— **This paper addresses the design of nonlinear quasi-unknown input observers for systems that can be interpreted as an interconnection of a linear dynamical subsystem with a static nonlinear feedback subject to additive perturbations. As assumed classically an exosystem is considered describing the dynamics of the perturbation, meaning that the dynamical mechanisms giving rise to the perturbation are sufficiently well known but the underlying initial condition is unknown. Extending classical results based on the Sylvester equation, and combining them with well-established dissipativity concepts and design methods, a new approach for quasi-unknown input observer design is obtained. It simplifies previous work on general unknown input observer design by exploiting the structural knowledge about the exosystem, without extending the state dimension. The approach is illustrated with numerical case examples.**

I. INTRODUCTION

Dissipativity has turned out as a key structural analysis and design tool for linear and nonlinear systems (see, e.g., [1], [2])

Unknown input observer design has attained considerable attention since the early work of Hautus and others (see, e.g., [3], [4]). This holds true also for nonlinear systems, for which different approaches have been presented, including dissipative observer design considering dissipativity properties for the system with respect to the unknown input [5], as well as alternative matrix-inequality based approaches [6], sliding mode observers [7], [8], high-gain observers [9], [10] and continuous-discrete Kalman Filter adaptations [11], [12].

In case that some dynamics of the input are known, i.e., a so-called exosystem model is at hand, more structured approaches are possible [13], [14]. This has been widely used in literature, e.g., for process systems [15], [16] and for the particular case of slowly varying perturbations is directly related to the idea of the proportional-integral observers [17], [18]. As these approaches include the estimation of the perturbation using the exosystem model the order of the observer is augmented in comparison to the dimension of the system state.

On the other hand, it is known that for linear systems reduced order unknown input observers can be designed in case that the perturbation has vector relative degree one and there are as many measurements as unknown inputs [3], because under this condition one can divide the state space into one part that is directly influenced by the unknown input and another one that is not. The observer is then designed for the part which is not directly influenced, requiring certain detectability or observability properties. This idea has also been extended to some classes of nonlinear systems [5], [6] and can in principle be applied also to extended state systems, that include the exosystem model to overcome the dimension augmentation problem mentioned above.

Yet another approach has been exploited, e.g., in [19] for a linear system that corresponds to an early-lumping PDE approximation, where the observer is composed of two parts: one that provides an estimate that is influenced by the unknown input and in consequence does not provide the correct values, and a second part that provides an asymptotic correction mechanism so that a suitably combined estimate converges to the actual state value. The main underlying assumption is that an observer does exist for the unperturbed case and that an associated Sylvester equation has a solution, which depends on the observer and the exosystem.

Having these studies as points of departure, in the present one the problem of quasi unknown input observer design from [19] is extended to nonlinear systems exploiting concepts and results from dissipativity theory for observer design [2], [5], [20]. In difference to [5] the exosystem model is explicitly accounted for in the problem solution, leading to a reduced dimension of the underlying dissipativity matrix inequalities and imposing a particular structure of the solution based on an associated Sylvester equation.

II. PROBLEM FORMULATION

Consider a system with dynamics

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{G}\boldsymbol{\psi}(\boldsymbol{\sigma}) + \boldsymbol{\varphi}(t, \mathbf{y}, \mathbf{u}) + \mathbf{D}\boldsymbol{v} \quad (1a)$$

$$\boldsymbol{\sigma} = \mathbf{H}\mathbf{x} \quad (1b)$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \quad (1c)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ denotes the state vector at time $t \geq 0$, $\mathbf{u}(t) \in \mathbb{R}^m$ represents control inputs, and $\mathbf{y}(t) \in \mathbb{R}^p$ are measured outputs. The initial state is denoted by $\mathbf{x}(0) = \mathbf{x}_0$. The function $\boldsymbol{\psi} : \mathbb{R}^r \to \mathbb{R}^s$ is assumed Lipschitz continuous in $\boldsymbol{\sigma}(t) \in \mathbb{R}^r$ where $\boldsymbol{\sigma}$ is a linear, *not* necessarily measured, function of the state. The variable $\boldsymbol{\varphi} : \mathbb{R}_{\geq 0} \times \mathbb{R}^{p+m} \to \mathbb{R}^n$ is a possibly nonlinear function of the known quantities $t, \mathbf{y}, \mathbf{u}$ and assumed piecewise continuous in t and Lipschitz continuous in $\mathbf{y}$ and $\mathbf{u}$. The variable $\boldsymbol{v}(t) \in \mathbb{R}^q$ is an external disturbance generated by the linear time-invariant exogenous system (exo-system)

$$\dot{\mathbf{w}} = \mathbf{E}\mathbf{w}, \quad \mathbf{w}(0) = \mathbf{w}_0 \quad (2a)$$

$$\boldsymbol{v} = \boldsymbol{\Gamma}\mathbf{w} \quad (2b)$$

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with state $\mathbf{w}(t) \in \mathbb{R}^d$ and *unknown* initial condition $\mathbf{w}_0$. The following assumption is placed on this exo-system.

Assumption 1: The dynamic matrix $\mathbf{E} \in \mathbb{R}^{d\times d}$ of the exo-system is assumed to be real and its spectrum lying on the imaginary axis only, i.e., $\mathrm{Re}\{\lambda_i(\mathbf{E})\} = 0,\ i = 1, \dots, d$ and non-defective. That means,

$$\mathbf{E} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}, \quad \mathbf{\Lambda} := \mathrm{diag}\left([j\Omega_1 \ \dots \ j\Omega_d]\right) \tag{3}$$

with frequencies Ω_i and regular matrix

$$\mathbf{V} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_d\end{bmatrix}, \tag{4}$$

with eigenvectors $\mathbf{v}_i$ associated to $\lambda_i,\ i = 1, \dots, d$.

The subsequent considerations aim at designing a state observer that asymptotically estimates the state $\mathbf{x}$ from the measurements $\mathbf{y}$ only, i.e., without knowledge of $\boldsymbol{v}$. To that end, the following observer structure is proposed

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{G}\boldsymbol{\psi}(\hat{\boldsymbol{\sigma}}) + \boldsymbol{\varphi}(t, \mathbf{y}, \mathbf{u}) - \mathbf{L}(\mathbf{C}\hat{\mathbf{x}} - \mathbf{y}) \tag{5a}$$

$$\hat{\boldsymbol{\sigma}} = \mathbf{H}\hat{\mathbf{x}} - \mathbf{N}_\sigma(\mathbf{C}\hat{\mathbf{x}} - \mathbf{y}) \tag{5b}$$

$$\bar{\mathbf{x}} = \hat{\mathbf{x}} - \mathbf{N}_y(\mathbf{C}\hat{\mathbf{x}} - \mathbf{y}) \tag{5c}$$

where $\hat{\mathbf{x}} \in \mathbb{R}^n$ represents the state of the observer and $\mathbf{L}$, $\mathbf{N}_\sigma$ and $\mathbf{N}_y$ are constant observer gains to be designed in the following. Note that (5a) essentially constitutes a classical Luenberger observer where in the nonlinearity $\boldsymbol{\psi}(\hat{\boldsymbol{\sigma}})$ the estimate of $\boldsymbol{\sigma}$, denoted by $\hat{\boldsymbol{\sigma}}$, is considered. The variable $\bar{\mathbf{x}}$, which is governed by the algebraic correction (5c), represents the estimate of the state. Hence, the goal is to find corrections $\mathbf{L}$, $\mathbf{N}_\sigma$ and $\mathbf{N}_y$ such that the estimation error $\bar{\mathbf{x}} - \mathbf{x}$ converges to zero exponentially. To that end, the errors $\tilde{\mathbf{x}} = \hat{\mathbf{x}} - \mathbf{x}$ and $\tilde{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma}$ are defined, leading to the observation error dynamics in the two-subsystem interconnection form

$$\dot{\tilde{\mathbf{x}}} = \mathbf{A}_\mathrm{L}\tilde{\mathbf{x}} + \mathbf{G}\boldsymbol{\nu} - \mathbf{D}\boldsymbol{v} \tag{6a}$$

$$\tilde{\boldsymbol{\sigma}} = \mathbf{H}_\mathrm{N}\tilde{\mathbf{x}} \tag{6b}$$

$$\boldsymbol{\nu} = \tilde{\boldsymbol{\psi}}(\tilde{\boldsymbol{\sigma}}; \boldsymbol{\sigma}) \tag{6c}$$

with $\mathbf{H}_\mathrm{N} = \mathbf{H} - \mathbf{N}_\sigma\mathbf{C}$, $\mathbf{A}_\mathrm{L} = \mathbf{A} - \mathbf{L}\mathbf{C}$ and

$$\tilde{\boldsymbol{\psi}}(\tilde{\boldsymbol{\sigma}}) := \boldsymbol{\psi}(\boldsymbol{\sigma} + \tilde{\boldsymbol{\sigma}}) - \boldsymbol{\psi}(\boldsymbol{\sigma}), \quad \tilde{\boldsymbol{\psi}}(\mathbf{0}; \boldsymbol{\sigma}) = \mathbf{0} \quad \forall \quad \boldsymbol{\sigma}. \tag{6d}$$

For the observer design procedure the following notions from dissipativity theory are exploited.

III. Notions from Dissipativity Theory

Following the notions and ideas in [1], [2], [21]–[24] a system with state $\mathbf{x}(t) \in \mathbb{R}^n$, input $\mathbf{u}(t) \in \mathbb{R}^m$ and output $\mathbf{y}(t) \in \mathbb{R}^p$ is called dissipative with respect to a given supply rate $\omega(\mathbf{y}, \mathbf{u})$ if there exists a storage function $\mathcal{S}\succeq 0$ for which it holds true that

$$\mathcal{S}(\mathbf{x}(t)) \leq \mathcal{S}(\mathbf{x}(0)) + \int_0^t \omega(\mathbf{y}(\tau), \mathbf{u}(\tau))\mathrm{d}\tau, \tag{7}$$

or if $\mathcal{S}$ is differentiable

$$\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}t} = \frac{\partial\mathcal{S}}{\partial\mathbf{x}}\dot{\mathbf{x}} \leq \omega(\mathbf{y}, \mathbf{u}). \tag{8}$$

In this case, the system is called strictly state dissipative with dissipation rate κ, if

$$\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}t} \leq -\kappa\|\mathbf{x}\|^2 + \omega(\mathbf{y}, \mathbf{u}). \tag{9}$$

For a quadratic supply rate

$$\omega(\mathbf{y}, \mathbf{u}) = \begin{bmatrix}\mathbf{y}\\ \mathbf{u}\end{bmatrix}^\intercal \begin{bmatrix}\mathbf{Q} & \mathbf{S}\\ \mathbf{S}^\mathrm{T} & \mathbf{R}\end{bmatrix} \begin{bmatrix}\mathbf{y}\\ \mathbf{u}\end{bmatrix} \geq 0 \tag{10}$$

the system is called $(\mathbf{Q}, \mathbf{S}, \mathbf{R})$ strictly state dissipative with rate κ if (9) holds true with ω given by (10).

In the following, let $\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C})$ denote a linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{11a}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{11b}$$

with vectors and states of appropriate dimension. Considering the quadratic storage function

$$\mathcal{S}(\mathbf{x}) = \mathbf{x}^\mathrm{T}\mathbf{P}\mathbf{x}, \quad \mathbf{P} = \mathbf{P}^\mathrm{T} \succ 0 \tag{12}$$

it follows that $\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is $(\mathbf{Q}, \mathbf{S}, \mathbf{R})$ strictly state dissipative with rate κ if

$$\begin{bmatrix}\mathbf{P}\mathbf{A} + \mathbf{A}^\mathrm{T}\mathbf{P} + \kappa\mathbf{I} & \mathbf{P}\mathbf{B}\\ \mathbf{B}^\mathrm{T}\mathbf{P} & \mathbf{0}\end{bmatrix} \preceq \begin{bmatrix}\mathbf{C}^\mathrm{T}\mathbf{Q}\mathbf{C} & \mathbf{C}^\mathrm{T}\mathbf{S}\\ \mathbf{S}^\mathrm{T}\mathbf{C} & \mathbf{R}\end{bmatrix}. \tag{13}$$

On the other hand, a static, memoryless map $\boldsymbol{\varphi}(\mathbf{u})$ with $\boldsymbol{\varphi}(\mathbf{0}) = \mathbf{0}$ is called $(\mathbf{Q}, \mathbf{S}, \mathbf{R})$ dissipative if the associated supply rate in (10) is non-negative, i.e.

$$\omega(\boldsymbol{\varphi}, \mathbf{u}) = \begin{bmatrix}\boldsymbol{\varphi}\\ \mathbf{u}\end{bmatrix}^\intercal \begin{bmatrix}\mathbf{Q} & \mathbf{S}\\ \mathbf{S}^\mathrm{T} & \mathbf{R}\end{bmatrix} \begin{bmatrix}\boldsymbol{\varphi}\\ \mathbf{u}\end{bmatrix} \geq 0. \tag{14}$$

Based on these concepts it is straightforward to obtain the following result.

Lemma 1 ([20]): Consider the system interconnection

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{15a}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{15b}$$

$$\mathbf{u} = -\boldsymbol{\varphi}(\mathbf{y}) \tag{15c}$$

and let $\boldsymbol{\varphi}$ be $(\mathbf{Q}, \mathbf{S}, \mathbf{R})$ dissipative, and $\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C})$ be $(-\mathbf{R}, \mathbf{S}^\mathrm{T}, -\mathbf{Q})$ strictly state dissipative with rate $\kappa > 0$. Then $\mathbf{x} = \mathbf{0}$ is exponentially stable.

Note that in the case that the matrices $\mathbf{Q}$ and $\mathbf{R}$ satisfy certain properties, further results can be derived, like the following.

Lemma 2: Let $\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C})$ be $(\mathbf{Q}, \mathbf{S}, \mathbf{R})$ strictly state dissipative with rate $\kappa > 0$ and let $\mathbf{Q}\preceq 0$, and $\mathbf{R}\succ 0$ and κ be such that $\mathbf{W} \prec \mathbf{0}$ with

$$\mathbf{W} = -\kappa\mathbf{I} + \mathbf{C}^\mathrm{T}\mathbf{Q}\mathbf{C} + \left(\mathbf{P}\mathbf{B} - \mathbf{C}^\mathrm{T}\mathbf{S}\right)\mathbf{R}^{-1}\left(\mathbf{B}^\mathrm{T}\mathbf{P} - \mathbf{S}^\mathrm{T}\mathbf{C}\right). \tag{16a}$$

Then $\mathbf{A}$ is Hurwitz.

Proof: By assumption $\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is $(\mathbf{Q}, \mathbf{S}, \mathbf{R})$ strictly state dissipative, so that (13) implies that there exists $\mathbf{P} = \mathbf{P}^\mathrm{T} \succ 0$ for which it holds true that

$$\begin{bmatrix}\mathbf{P}\mathbf{A} + \mathbf{A}^\mathrm{T}\mathbf{P} + \kappa\mathbf{I} - \mathbf{C}^\mathrm{T}\mathbf{Q}\mathbf{C} & \mathbf{P}\mathbf{B} - \mathbf{C}^\mathrm{T}\mathbf{S}\\ \mathbf{B}^\mathrm{T}\mathbf{P} - \mathbf{S}^\mathrm{T}\mathbf{C} & -\mathbf{R}\end{bmatrix} \preceq \mathbf{0}. \tag{17}$$

As assumed $\mathbf{R} \succ \mathbf{0}$ so that (17) holds true if and only if the associated Schur complement satisfies (see, e.g., [2])

$$\mathbf{PA} + \mathbf{A}^{\mathrm{T}}\mathbf{P} + \kappa\mathbf{I} - \mathbf{C}^{\mathrm{T}}\mathbf{QC} - \left(\mathbf{PB} - \mathbf{C}^{\mathrm{T}}\mathbf{S}\right)\mathbf{R}^{-1}\left(\mathbf{B}^{\mathrm{T}}\mathbf{P} - \mathbf{S}^{\mathrm{T}}\mathbf{C}\right) \prec \mathbf{0}$$

or equivalently

$$\mathbf{PA} + \mathbf{A}^{\mathrm{T}}\mathbf{P} \prec \mathbf{W} \tag{18}$$

with $\mathbf{W}$ as given in (16a). By assumption, $\mathbf{W} \prec 0$. In consequence, from $\mathbf{P} \succ 0$ it follows that $\mathbf{A}$ must be Hurwitz. ■

IV. Dissipative Quasi-unknown-Input Observer

For the subsequent analysis of the observation error dynamics (6) assume that in virtue of Lemma 1 the linear subsystem $\Sigma(\mathbf{A}_{\mathrm{L}}, \mathbf{G}, \mathbf{H}_{\mathrm{N}})$ is $(-\mathbf{R}, -\mathbf{S}^{\mathrm{T}}, -\mathbf{Q})$ strictly state dissipative[1] with rate $\kappa > 0$. Thus, for given $\mathbf{Q}, \mathbf{S}, \mathbf{R}$ and $\mathbf{N}_{\sigma}$ the gain $\mathbf{L}$ can be chosen such that the inequality

$$\begin{bmatrix} \mathbf{A}_{\mathrm{L}}^{\mathrm{T}}\mathbf{P} + \mathbf{PA}_{\mathrm{L}} + \kappa\mathbf{I} & \mathbf{PG} \\ \mathbf{G}^{\mathrm{T}}\mathbf{P} & 0 \end{bmatrix} \preceq \begin{bmatrix} -\mathbf{H}_{\mathrm{N}}^{\mathrm{T}}\mathbf{RH}_{\mathrm{N}} & -\mathbf{H}_{\mathrm{N}}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}} \\ -\mathbf{SH}_{\mathrm{N}} & -\mathbf{Q} \end{bmatrix} \tag{19}$$

holds true. In consequence, for $\mathbf{D}\boldsymbol{v} = \mathbf{0}$, the observation error exponentially converges to zero for any nonlinearity $\tilde{\boldsymbol{\psi}}$ which is $(\mathbf{Q}, \mathbf{S}, \mathbf{R})$ dissipative (cp. [20]), and for $\mathbf{Q} \prec \mathbf{0}$ and $\mathbf{R}$ so that Lemma 2 applies (with the appropriate change of notation), the matrix $\mathbf{A}_{\mathrm{L}}$ is Hurwitz.

To design the observer in the presence of the disturbance $\boldsymbol{v}$ and to proof the main result, some preliminary considerations are discussed. Given that the basic idea in the dissipative observer design resides in designing the properties of the linear subsystem of the observation error dynamics in accordance with the ones of the nonlinear subsystem, first the underlying problem for the linear subsystem is addressed in the following.

A. Design of the linear subsystem

Consider the series connection of the linear systems $\Sigma_1(\mathbf{E}, \mathbf{0}, \mathbf{\Gamma})$ (system 1) and $\Sigma_2(\mathbf{A}_{\mathrm{L}}, -\mathbf{D}, \mathbf{C})$ (system 2) with state vector $\mathbf{w}$ and $\tilde{\mathbf{x}}$, respectively in the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \tilde{\mathbf{x}} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{\mathrm{L}} & -\mathbf{D\Gamma} \\ \mathbf{0} & \mathbf{E} \end{bmatrix}\begin{bmatrix} \tilde{\mathbf{x}} \\ \mathbf{w} \end{bmatrix} \tag{20a}$$

$$\tilde{\mathbf{y}} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \tilde{\mathbf{x}} \\ \mathbf{w} \end{bmatrix} \tag{20b}$$

This system resorts to the estimation error dynamics presented in (6) with $\mathbf{w}$ generated by (2) where $\mathbf{G}$ is set to zero. System (20) is referred to hereinafter as composite system. Assume that system Σ_2 is minimal, i.e., completely controllable and observable, that $\mathbf{D}$ has full column rank, and $\mathbf{C}, \mathbf{\Gamma}$ full row rank. Then, according to [25, Lemma 20.5] the following Lemma holds:

Lemma 3: The composite system looses observability if and only if λ is a pole of Σ_1 and a zero of Σ_2 such that there exists an $\mathbf{v} \in$ null space of $(\lambda\mathbf{I} - \mathbf{E})$ and $\mathbf{\Gamma v} \in$ null space of $\mathbf{C}(\lambda\mathbf{I} - \mathbf{A}_{\mathrm{L}})^{-1}\mathbf{D}$.

This is in particular of interest if the goal is to also reconstruct the states of the exosystem.

For the subsequent analysis, let

$$\mathbf{e} \triangleq \tilde{\mathbf{x}} - \mathbf{\Pi w}, \tag{21}$$

with matrix $\mathbf{\Pi} \in \mathbb{R}^{n \times q}$. Then, with (20) one obtains

$$\dot{\mathbf{e}} = \mathbf{A}_{\mathrm{L}}\tilde{\mathbf{x}} - \mathbf{D\Gamma w} - \mathbf{\Pi E w} \tag{22}$$

which, by further using (21), yields

$$\dot{\mathbf{e}} = \mathbf{A}_{\mathrm{L}}\mathbf{e} + [\mathbf{A}_{\mathrm{L}}\mathbf{\Pi} - \mathbf{\Pi E} - \mathbf{D\Gamma}]\mathbf{w}. \tag{23}$$

In consequence, if $\mathbf{\Pi}$ is chosen such that the Sylvester equation

$$\mathbf{A}_{\mathrm{L}}\mathbf{\Pi} - \mathbf{\Pi E} = \mathbf{D\Gamma} \tag{24}$$

holds true, the dynamics of $\mathbf{e}$ are governed by the autonomous system

$$\dot{\mathbf{e}} = \mathbf{A}_{\mathrm{L}}\mathbf{e}. \tag{25}$$

Consequently, if $\mathbf{A}_{\mathrm{L}}$ is a Hurwitz matrix, $\mathbf{e}$ converges to zero asymptotically and, according to (21),

$$\tilde{\mathbf{x}}(t) = \mathbf{\Pi w}(t) \tag{26}$$

is established. The associated solution reads, by once again exploiting (21) evaluated at $t = 0$, as

$$\mathbf{e}(t) = \mathrm{e}^{\mathbf{A}_{\mathrm{L}}t}\mathbf{e}(0) = \mathrm{e}^{\mathbf{A}_{\mathrm{L}}t}\left[\tilde{\mathbf{x}}(0) - \mathbf{\Pi w}(0)\right]. \tag{27}$$

Consequently, inserting the above equation into (21) and reformulating for $\tilde{\mathbf{x}}$ provides the solution for the state $\tilde{\mathbf{x}}$ of the composite system in the form

$$\tilde{\mathbf{x}}(t) = \mathrm{e}^{\mathbf{A}_{\mathrm{L}}t}\left[\tilde{\mathbf{x}}(0) - \mathbf{\Pi w}(0)\right] + \mathbf{\Pi w}(t). \tag{28}$$

Correspondingly, for the output one obtains

$$\tilde{\mathbf{y}}(t) = \mathbf{C}\mathrm{e}^{\mathbf{A}_{\mathrm{L}}t}\left[\tilde{\mathbf{x}}(0) - \mathbf{\Pi w}(0)\right] + \mathbf{C\Pi w}(t). \tag{29}$$

B. Design for the nonlinear system

Based on these preliminary considerations, the main result of the paper is formulated:

Theorem 4: Consider system (1) with disturbance $\boldsymbol{v}$ generated by (2). Let Assumption 1 hold true and $\tilde{\boldsymbol{\psi}}$ be $(\mathbf{Q}, \mathbf{S}, \mathbf{R})$-dissipative, $\Sigma(\mathbf{A}_{\mathrm{L}}, \mathbf{G}, \mathbf{H}_{\mathrm{N}})$ be $(-\mathbf{R}, -\mathbf{S}^{\mathrm{T}}, -\mathbf{Q})$ strictly state dissipative with rate $\kappa > 0$, where $\mathbf{Q} \prec \mathbf{0}$ and $\mathbf{R}$ being such that Lemma 2 applies. Then, if there exist $\mathbf{N}_{\sigma}$ and $\mathbf{N}_y$ such that

$$\mathbf{H\Pi} - \mathbf{N}_{\sigma}\mathbf{C\Pi} = \mathbf{0}, \quad \mathbf{\Pi} - \mathbf{N}_y\mathbf{C\Pi} = \mathbf{0} \tag{30}$$

holds, where $\mathbf{\Pi}$ satisfies the Sylvester equation (24), the estimate $\bar{\mathbf{x}}$ obtained from the observer (5) converges to the state $\mathbf{x}$ exponentially.

Proof: Let $\mathbf{e}$ be defined according to (21) with $\mathbf{\Pi}$ being the unique solution of (24) which exists since the spectra of $\mathbf{A}_{\mathrm{L}}$ and $\mathbf{E}$ are disjoint (see, e.g. [26]) due to Assumption 1,

[1]Note that to account for the sign in (6c) in comparison to (15c), the entries with $\mathbf{S}$ are multiplied by -1.

and $\mathbf{A}_\mathrm{L}$ being Hurwitz in virtue of Lemma 2. Then with (6a) and (2), as well as taking into account (24) it holds that

$$\begin{aligned}\dot{\mathbf{e}} &= \mathbf{A}_\mathrm{L}\tilde{\mathbf{x}} + \mathbf{G}\boldsymbol{\nu} - \mathbf{D}\boldsymbol{v} - \boldsymbol{\Pi}\boldsymbol{\Gamma}\mathbf{E}\mathbf{w} \\ &= \mathbf{A}_\mathrm{L}(\mathbf{e} + \boldsymbol{\Pi}\boldsymbol{v}) + \mathbf{G}\boldsymbol{\nu} - \mathbf{D}\boldsymbol{v} - \boldsymbol{\Pi}\boldsymbol{\Gamma}\mathbf{E}\mathbf{w} \\ &= \mathbf{A}_\mathrm{L}\mathbf{e} + [\mathbf{A}_\mathrm{L}\boldsymbol{\Pi}\boldsymbol{\Gamma} - \boldsymbol{\Pi}\boldsymbol{\Gamma}\mathbf{E} - \mathbf{D}\boldsymbol{\Gamma}]\mathbf{w} + \mathbf{G}\boldsymbol{\nu} \\ &= \mathbf{A}_\mathrm{L}\mathbf{e} + \mathbf{G}\boldsymbol{\nu}. \end{aligned} \tag{31}$$

Substituting (21) into (6a) gives

$$\tilde{\boldsymbol{\sigma}} = \mathbf{H}_\mathrm{N}(\mathbf{e} + \boldsymbol{\Pi}\boldsymbol{v}). \tag{32}$$

By hypothesis there exists $\mathbf{N}_\sigma$ such that

$$\mathbf{H}\boldsymbol{\Pi} - \mathbf{N}_\sigma\mathbf{C}\boldsymbol{\Pi} = \mathbf{H}_\mathrm{N}\boldsymbol{\Pi} = \mathbf{0}. \tag{33}$$

Therefore,

$$\boldsymbol{\nu} = \tilde{\boldsymbol{\psi}}(\tilde{\boldsymbol{\sigma}}), \quad \tilde{\boldsymbol{\sigma}} = \mathbf{H}_\mathrm{N}\mathbf{e}. \tag{34}$$

This implies that the dynamics of $\mathbf{e}$ is equivalent to (6). By assumption it holds that there exists $\mathbf{P} = \mathbf{P}^\mathrm{T} \succ 0$ such that

$$\begin{aligned} &\begin{bmatrix}\mathbf{e}\\ \boldsymbol{\nu}\end{bmatrix}^\mathrm{T} \begin{bmatrix}\mathbf{A}_\mathrm{L}^\mathrm{T}\mathbf{P} + \mathbf{P}\mathbf{A}_\mathrm{L} + \kappa\mathbf{I} & \mathbf{P}\mathbf{G}\\ \mathbf{G}^\mathrm{T}\mathbf{P} & \mathbf{0}\end{bmatrix} \begin{bmatrix}\mathbf{e}\\ \boldsymbol{\nu}\end{bmatrix} \\ \leq &\begin{bmatrix}\mathbf{e}\\ \boldsymbol{\nu}\end{bmatrix}^\mathrm{T} \begin{bmatrix}-\mathbf{H}_\mathrm{N}^\mathrm{T}\mathbf{R}\mathbf{H}_\mathrm{N} & -\mathbf{H}_\mathrm{N}^\mathrm{T}\mathbf{S}^\mathrm{T}\\ -\mathbf{S}\mathbf{H}_\mathrm{N} & -\mathbf{Q}\end{bmatrix} \begin{bmatrix}\mathbf{e}\\ \boldsymbol{\nu}\end{bmatrix} \\ = &\begin{bmatrix}\tilde{\boldsymbol{\sigma}}\\ \tilde{\boldsymbol{\psi}}(\tilde{\boldsymbol{\sigma}})\end{bmatrix}^\mathrm{T} \begin{bmatrix}-\mathbf{R} & -\mathbf{S}^\mathrm{T}\\ -\mathbf{S} & -\mathbf{Q}\end{bmatrix} \begin{bmatrix}\tilde{\boldsymbol{\sigma}}\\ \tilde{\boldsymbol{\psi}}(\tilde{\boldsymbol{\sigma}})\end{bmatrix} \\ = -&\begin{bmatrix}\tilde{\boldsymbol{\psi}}(\tilde{\boldsymbol{\sigma}})\\ \tilde{\boldsymbol{\sigma}}\end{bmatrix}^\mathrm{T} \begin{bmatrix}\mathbf{Q} & \mathbf{S}\\ \mathbf{S}^\mathrm{T} & \mathbf{R}\end{bmatrix} \begin{bmatrix}\tilde{\boldsymbol{\psi}}(\tilde{\boldsymbol{\sigma}})\\ \tilde{\boldsymbol{\sigma}}\end{bmatrix} \leq 0. \end{aligned} \tag{35}$$

This further implies that $V(\mathbf{e}) = \mathbf{e}^\mathrm{T}\mathbf{P}\mathbf{e}$ is a Lyapunov function, with $\mathbf{P}$ chosen such that the above matrix inequalities hold true, and, therefore

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leq -\kappa\|\mathbf{e}\|^2 \tag{36}$$

which in turns implies the exponential stability of $\mathbf{e} \equiv \mathbf{0}$. Consequently, $\tilde{\mathbf{x}} \to \boldsymbol{\Pi}\mathbf{w}$ as t tends to infinity and thus,

$$\tilde{\mathbf{y}}_\mathrm{ss}(t) \triangleq \lim_{t\to\infty} \mathbf{C}\tilde{\mathbf{x}}(t) = \mathbf{C}\boldsymbol{\Pi}\mathbf{w}(t) \tag{37}$$

in steady state. For the difference to the corrected estimate $\bar{\mathbf{x}}$, one obtains

$$\begin{aligned}\mathbf{x} - \bar{\mathbf{x}} &= \mathbf{x} - (\hat{\mathbf{x}} - \mathbf{N}_y\mathbf{C}(\hat{\mathbf{x}} - \mathbf{x})) \\ &= -\tilde{\mathbf{x}} + \mathbf{N}_y\mathbf{C}\tilde{\mathbf{x}} = -(\boldsymbol{\Pi} - \mathbf{N}_y\mathbf{C}\boldsymbol{\Pi})\mathbf{w}\end{aligned} \tag{38}$$

so that for

$$\boldsymbol{\Pi} = \mathbf{N}_y\mathbf{C}\boldsymbol{\Pi} \tag{39}$$

one eventually has

$$\lim_{t\to\infty}[\bar{\mathbf{x}}(t) - \mathbf{x}(t)] = \mathbf{0}. \tag{40}$$

which completes the proof. ■

It remains to clarify under which conditions such corrections $\mathbf{N}_\sigma$ and $\mathbf{N}_y$ exist.

Proposition 5: Let the assumptions of Theorem 4 hold true and $\mathbf{C}\boldsymbol{\Pi}$ be left invertible. Let

$$(\mathbf{C}\boldsymbol{\Pi})^\dagger := [(\mathbf{C}\boldsymbol{\Pi})^\mathrm{T}\mathbf{C}\boldsymbol{\Pi}]^{-1}(\mathbf{C}\boldsymbol{\Pi})^\mathrm{T} \tag{41}$$

denote its Moore-Penrose pseudo inverse. Then the choice

$$\mathbf{N}_\sigma = \mathbf{H}\boldsymbol{\Pi}(\mathbf{C}\boldsymbol{\Pi})^\dagger, \qquad \mathbf{N}_y = \boldsymbol{\Pi}(\mathbf{C}\boldsymbol{\Pi})^\dagger, \tag{42}$$

ensures that the observer (5) converges to the state $\mathbf{x}$ exponentially.

Proof: By assumption $\mathbf{C}\boldsymbol{\Pi}$ is left invertible, and choosing $\mathbf{N}_\sigma$ and $\mathbf{N}_y$ as stated ensures that (33) and (39) holds providing for the convergence of the estimate $\bar{\mathbf{x}}$ to $\mathbf{x}$. ■

Remark: Note that the rank of a unique solution $\boldsymbol{\Pi}$ of the Sylvester equation (24) depends on the involved matrices and, in particular, on $\mathbf{D}$ and $\boldsymbol{\Gamma}$. Various cases that may appear are, e.g., discussed in [27]. If $d = n$, $\mathbf{D} = \mathbf{d}$ and $\boldsymbol{\Gamma} = \boldsymbol{\gamma}^\mathrm{T}$ are vectors the unique solution $\boldsymbol{\Pi}$ is invertible if and only if $(\mathbf{A}_\mathrm{L}, \mathbf{d})$ is controllable and $(\mathbf{E}, \boldsymbol{\gamma}^\mathrm{T})$ is observable. In the multivariable case the controllability of $(\mathbf{A}_\mathrm{L}, \mathbf{D})$ and observability of $(\mathbf{E}, \boldsymbol{\Gamma})$ are only necessary conditions.

Consider a particular input $\boldsymbol{v} = \boldsymbol{\Gamma}\mathbf{v}_i\mathrm{e}^{j\Omega_i t}$. The corresponding output estimation error is obtained from the transfer function of the linear part of the system, i.e.,

$$\bar{\tilde{\mathbf{y}}}_\mathrm{ss}(s) = \mathbf{M}(s)\bar{\boldsymbol{v}}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A}_\mathrm{L})^{-1}\mathbf{D}\bar{\boldsymbol{v}}(s) \tag{43}$$

where $\bar{\tilde{\mathbf{y}}}_\mathrm{ss}(s)$ and $\bar{\boldsymbol{v}}(s)$ denote the Laplace transform of $\tilde{\mathbf{y}}_\mathrm{ss}$ and $\boldsymbol{v}$, as

$$\tilde{\mathbf{y}}_\mathrm{ss}(t) = \mathbf{C}\boldsymbol{\Pi}\mathbf{v}_i\mathrm{e}^{j\Omega_i t} = \mathbf{M}(j\Omega)\boldsymbol{\Gamma}\mathbf{v}_i\mathrm{e}^{j\Omega_i t}. \tag{44}$$

Therefore, in general

$$\tilde{\mathbf{y}}_\mathrm{ss}(t) = \tilde{\mathbf{M}}\mathrm{e}^{\boldsymbol{\Lambda}t}\mathbf{V}^{-1}\mathbf{w}(0), \quad \tilde{\mathbf{M}} = \begin{bmatrix}\tilde{\mathbf{m}}_1 & \dots & \tilde{\mathbf{m}}_d\end{bmatrix} \tag{45}$$

with the vectors $\tilde{\mathbf{m}}_i = \mathbf{M}(j\Omega_i)\boldsymbol{\Gamma}\mathbf{v}_i$, $i = 1, \dots, d$. If $\tilde{\mathbf{M}}$ has rank d then there is no $\tilde{\mathbf{m}}_i = \mathbf{0}$. Then, a necessary condition for the left invertibility of $\mathbf{C}\boldsymbol{\Pi}$ is that $\mathbf{M}(s)$ does not possess a zero $s = j\Omega_i$ in the direction of $\boldsymbol{\Gamma}\mathbf{v}_i$. According to Lemma 3 this is ensured by the observability of the composite system (i.e. the interconnected linear part of the system).

Note further that, even in case that $\mathbf{C}\boldsymbol{\Pi}$ is not left-invertible it might still be possible to design observer gains such that the estimation error converges to zero. In that case, conditions need to be imposed on $\mathbf{H}$ (see Example 2 in Section V)

C. Disturbance reconstruction

In certain scenarios it might be of interest to reconstruct also the states of the exosystem or the disturbance $\boldsymbol{v}$. From (29) it is obtained that the trajectories of the system converge to the d-dimensional positively invariant subset

$$\Omega = \left\{ \begin{bmatrix}\tilde{\mathbf{x}}\\ \mathbf{w}\end{bmatrix} \in \mathbb{R}^{n+d} \mid \tilde{\mathbf{x}} = \boldsymbol{\Pi}\mathbf{w} \right\} \subset \mathbb{R}^{n+d} \tag{46}$$

and thus for $t \to \infty$ the output error satisfies $\tilde{\mathbf{y}}(t) = \mathbf{C\Pi w}(t)$. In view of this, if $(\mathbf{C\Pi})^\dagger$ exists the disturbance can be reconstructed from

$$\hat{\mathbf{w}}(t) = (\mathbf{C\Pi})^\dagger \tilde{\mathbf{y}}(t) = (\mathbf{C\Pi})^\dagger \left[\mathbf{C}\hat{\mathbf{x}}(t) - \mathbf{y}(t)\right]. \tag{47}$$

Otherwise, the disturbance might be estimated by an additional observer in a cascaded fashion. Assuming the series connection of the linear part of the system and the exosystem is observable, i.e, the matrix

$$\begin{bmatrix} \lambda\mathbf{I} - \mathbf{A} & -\mathbf{D} \\ \mathbf{0} & \lambda\mathbf{I} - \mathbf{E} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \tag{48}$$

has rank $n+q$ for all $\lambda \in \mathbb{C}$ which is readily obtained from the Hautus lemma. Therefore, it also holds that

$$\begin{bmatrix} \lambda\mathbf{I} - \mathbf{A} + \mathbf{LC} & -\mathbf{D} \\ \mathbf{0} & \lambda\mathbf{I} - \mathbf{E} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \tag{49}$$

has rank $n+q$ for all $\lambda \in \mathbb{C}$, i.e., the observed system is also observable. On Ω the dynamics are described by

$$\dot{\mathbf{w}} = \mathbf{Ew} \tag{50}$$

$$\tilde{\mathbf{y}} = \mathbf{C\Pi w} \tag{51}$$

and $\mathbf{w}$ can be estimated by

$$\dot{\hat{\mathbf{w}}} = (\mathbf{E} - \mathbf{L}_{\mathrm{w}}\mathbf{C\Pi})\hat{\mathbf{w}} + \mathbf{L}_{\mathrm{w}}\left[\mathbf{C}\hat{\mathbf{x}}(t) - \mathbf{y}(t)\right] \tag{52}$$

with the observer gain $\mathbf{L}_{\mathrm{w}}$ suitably tuned.

V. Case Example

To demonstrate the effectiveness of the proposed observer consider a simple system of the form (1) with $\boldsymbol{\varphi} \equiv \mathbf{0}$ and

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\mathbf{H} = \begin{bmatrix} 4 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \boldsymbol{\Gamma} = 2, \tag{53a}$$

with nonlinearity

$$\psi(\sigma) = \frac{\sigma(1-\sigma)}{1+\sigma^2} \tag{53b}$$

for $t > 0$ with initial state $\mathbf{x}(0) = \mathbf{x}_0 = [1\ 1]^{\mathrm{T}}$, as well as the exosystem (2) with dynamics

$$\mathbf{E} = 0 \tag{53c}$$

for $t > 0$, $\mathbf{w}(0) = \mathbf{w}_0 = [1.5]^{\mathrm{T}}$, i.e, the system is affected by a constant disturbance with unknown magnitude. The nonlinearity (53b) satisfies a sector condition of the form

$$(K_1\sigma - \psi(\sigma))(\psi(\sigma) - K_2\sigma) \geq 0$$

with $K_1 = 1.5, K_2 = -0.5$ (conservatively chosen), and which can be directly interpreted by expansion as a $(\mathbf{Q}, \mathbf{S}, \mathbf{R})$ dissipativity property with $\mathbf{Q} = -1 < 0$, $\mathbf{S} = K_1 + K_2 = 2$, and $\mathbf{R} = -K_1K_2 = 0.75$.

The following observer gains can be verified to satisfy the design constraints

$$\mathbf{N}_\sigma = 3.1, \quad \mathbf{N}_y = \begin{bmatrix} 1 \\ -0.93 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} -0.5 \\ 2 \end{bmatrix}. \tag{54}$$

$$\boldsymbol{\Pi} = \begin{bmatrix} -9.3 \\ 8.6 \end{bmatrix}, \quad \text{and} \quad \mathbf{P} = \begin{bmatrix} 50.51 & 39.16 \\ 39.16 & 30.94 \end{bmatrix} \tag{55a}$$

with $\kappa = 0.35$. Figure 1 shows the corresponding simulation result as well as the estimation of the disturbance. The observer is initialized with $\hat{\mathbf{x}}_0 = -\mathbf{x}_0$.

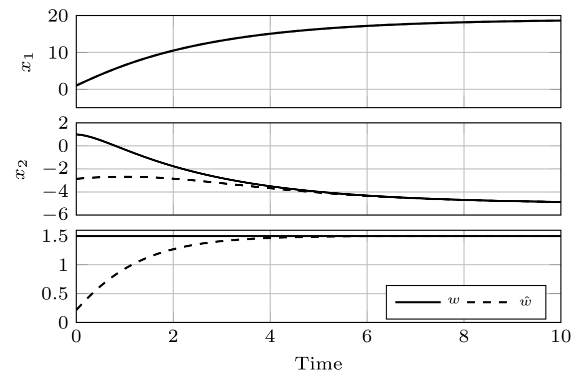

Fig. 1. Evolution of the states (——) and the corresponding estimates (- - -) provided by the observer.

In a second example the case where $\boldsymbol{\Pi}$ does not have full rank is illustrated. This is demonstrated using the system

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -0.5 & 1 \\ -1 & 0 & -1 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$\mathbf{H} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad \boldsymbol{\Gamma} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \tag{56a}$$

with the same nonlinearity $\psi(\sigma)$ as in (53b). for $t > 0$ with initial state $\mathbf{x}(0) = \mathbf{x}_0 = [1\ 1\ 1]^{\mathrm{T}}$, as well as the exosystem (2) with dynamic matrix

$$\mathbf{E} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \tag{56b}$$

for $t > 0$ with $\omega = 2$, $\mathbf{w}(0) = \mathbf{w}_0 = [1.5\ \ -1.5]^{\mathrm{T}}$, i.e, the system is affected by a sinusoidal disturbance $\mathbf{w}$ with known frequency ω and unknown phase and amplitude.

To obtain the observer gains and corrections one has to solve (19) with the constraints (42), (24), $\kappa > 0$ and $\mathbf{P} \succeq 0$. The optimization problem is solved in MATLAB with the help of Yalmip, see [28]. This approach yields for the observer gain

$$\mathbf{L} = \begin{bmatrix} 4 & 4.37 \\ 1 & -6.27 \\ -1 & 6.66 \end{bmatrix} \tag{57a}$$

and for the corrections

$$\mathbf{N}_\sigma = \begin{bmatrix} 0 & 2.52 \end{bmatrix}, \quad \mathbf{N}_y = \begin{bmatrix} 1 & 36.22 \\ 0 & -10.19 \\ 0 & -17.72 \end{bmatrix}. \tag{57b}$$

The corresponding solution of the Sylvester equation reads

$$\boldsymbol{\Pi} = \begin{bmatrix} -0.15 & 0.03 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{P} = \begin{bmatrix} 97.54 & 34.10 & -65.31 \\ 34.10 & 26.40 & -11.93 \\ -65.31 & -11.93 & 57.69 \end{bmatrix} \tag{58a}$$

with $\kappa = 6.35$. Figure 2 shows the simulation result with this observer where the intial state of the observer are set to $\hat{\mathbf{x}}(0) = -\hat{\mathbf{x}}_0$. It can be seen, that the estimates converge asymptotically to the real states despite the quasi unknown input. In the particular example $\mathbf{\Pi}$ has rank one and therefore

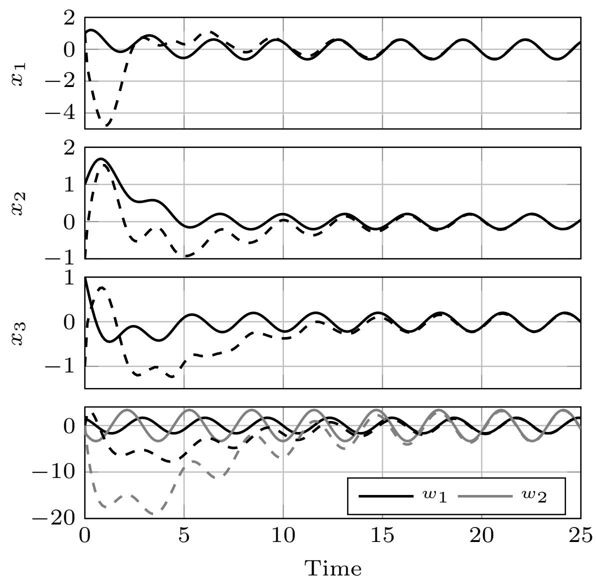

Fig. 2. Evolution of the states (—) and the corresponding estimates (- - -) provided by the observer.

$\mathbf{C\Pi}$ is not left-invertible. It is noteworthy that the pair $(\mathbf{A}_{\mathrm{L}}, \mathbf{D})$ is *not* controllable.

VI. CONCLUSION

The problem of designing a state observer for a nonlinear system with quasi-unknown input is addressed by extending the dissipativity-based observer design approach for the given setup. It is shown that under suitably chosen correction matrices asymptotic convergence can be ensured if the unknown input is periodic with a known dynamics (provided for the design as exosystem) and unknown initial value, and certain assumptions on the spectra as well as observability of the combination of the original system and the exosystem are satisfied. The correction matrices are solutions of matrix dissipativity inequalities and a Sylvester equation. The approach is illustrated with academic simulation examples showing the main design steps and performance of the approach.

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