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Robust simultaneous stabilization via minimax adaptive control

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***Abstract*— The paper explores the usage of minimax adaptive controllers to guarantee finite $\mathcal{L}_2$-gain simultaneous stabilization of linear time-invariant (LTI) plants. It is shown that a minimax adaptive controller simultaneously stabilizes any two multiple-input multiple-output (MIMO) P-stabilizable LTI plants when no LTI controller can achieve that, and the worst attained $\mathcal{L}_2$-gain bound for the transient dynamics is readily computable.**

I. INTRODUCTION

Recently, there has been a surge of research interest at the intersection of machine learning, system identification and adaptive control; see, for example, [14] for a review. Most work concerns the stochastic setting, but recently works relating to worst-case disturbances started to appear, see [1], [19]. Based on a game theory formulation of the H_∞ optimal control introduced in [2], the minimax adaptive control approach was presented in [7], [17]. It focuses on worst-case models of disturbances and uncertain parameters and assumes no prior knowledge of a stabilizing controller. The exploration-exploitation strategy exhibited by these adaptive controllers ensure stability when faced with large uncertainties that can not be tackled using robust LTI controllers.

The relevance of the problem of simultaneous stabilization of a finite set of plants could be justified in the sense that besides stabilizing a nominal plant of the underlying system, it is also desirable to stabilize discrete perturbations of this nominal plant induced by structural changes or component failure [20]. It could also be motivated within the scope of nonlinear systems where the concern is to design controllers around several operating points of interest [10].

The problem of simultaneous stabilization of linear time-invariant plants has been studied extensively in the literature. It was considered in [9], [10], [18], [21] with the restriction that the controller is linear time-invariant. However, as discussed in [22], there are counterexamples where an LTI controller fails to stabilize even a pair of single-input single-output systems simultaneously. This then calls for the need to employ a more complex controller architecture in attempting a solution to this problem.

The works in [11]–[13] show that using periodic time-varying compensators, it is possible to $\mathcal{L}_2$ stabilize any finite collection of discrete-time LTI plants. The same conclusion was reached in [16]. The use of periodic time-varying controllers for the simultaneous placement of the closed-loop poles of N SISO LTI plants was also reported in [6]. On the other hand, the work in [8] explores using switching deadbeat controllers to stabilize a finite set of scalar systems. This was generalized to the MIMO setting in [3] under the condition that the systems are controllable. However, one difficulty of these works is that extracting $\mathcal{L}_2$-gain bounds, which reflects the performance of these controllers, is cumbersome and complicated, especially for MIMO models.

The current work concerns using minimax adaptive controllers to provide guarantees on finite $\mathcal{L}_2$-gain stabilization of LTI plants, possibly MIMO, in the presence of adversarial disturbances. In doing that, We relax the controllability assumption needed in [3] to only stabilizability. We also provide an explicit easily computable $\mathcal{L}_2$-gain bound achieved by the adaptive controller.

The outline of the paper is as follows: Section 2 presents the framework of minimax adaptive control synthesis and formulates the problem; section 3, on the other hand, establishes the main results of the paper. Section 4 is dedicated for numerical examples and section 5 concludes the paper.

A. Notation

The set of $n\times m$ matrices with real coefficients is denoted by $\mathbb{R}^{n\times m}$. For a symmetric matrix $A\in\mathbb{R}^{n\times n}$, we indicate that A is positive definite by writing $A\succ 0$ and positive semi-definite by writing $A\succeq 0$. I_n denotes the identity matrix of dimension $n\times n$. Given $x\in\mathbb{R}^n$ and $A\in\mathbb{R}^{n\times n}$, the notation $|x|_A^2$ means x^TAx. Given a positive definite matrix $P\prec\gamma^2 I_n$ and a scalar $\gamma>0$, we define the positive definite matrix $S_{P,\gamma^2} := (P^{-1}-\gamma^{-2}I_n)^{-1}$. We let $\|A\|$ denote the spectral radius of a symmetric matrix $A\in\mathbb{R}^{n\times n}$, i.e., $\|A\| := \max_j |\lambda_j(A)|$, where $\lambda_j(A)$, $j=1,\dots,n$ are the eigenvalues of A.

II. PROBLEM FORMULATION

Minimax adaptive control synthesis was presented in [17]. Given a compact set $\mathcal{M}\subset\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times m}$, an initial state $x_0\in\mathbb{R}^n$ and a scalar $\gamma>0$, we seek a solution to the following optimization problem

$$\begin{aligned}\inf_{\mu}\sup_{w,A,B,N}&\sum_{t=0}^{N}(|x_t|_Q^2+|u_t|_R^2-\gamma^2|w_t|^2)\\ \text{s.t. } & x_{t+1}=Ax_t+Bu_t+w_t,\quad t\geq 0\\ & u_t=\mu_t(x_0,\dots,x_t,u_0\dots u_{t-1}),\\ & (A,B)\in\mathcal{M}.\end{aligned}\tag{1}$$

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The problem is a zero sum dynamic game where the μ-player minimizes the cost and the (w, A, B)-player maximizes it. Note that the adversary not only is allowed to choose the disturbance w but also the model pair (A, B) [2].

The following result from [17] presents an explicit solution to (1) in terms of an adaptive controller that guarantees a pre-specified $\mathcal{L}_2$-gain bound from disturbances to errors for model sets $\mathcal{M}$ of the type $\mathcal{M} := \{(A_1, B_1), \dots, (A_s, B_s)\}$.

Proposition 1: Given $A_1, \dots, A_s \in \mathbb{R}^{n\times n}$, $B_1, \dots, B_s \in \mathbb{R}^{n\times m}$ and positive definite matrices $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$, suppose there exist $K_1, \dots, K_s \in \mathbb{R}^{m\times n}$ and $P_{ij} \in \mathbb{R}^{n\times n}$ with $0 \prec P_{ij} = P_{ji} \prec \gamma^2 I$ and

$$\begin{aligned} |x|^2_{P_{jk}} \geq |x|^2_Q + |K_k x|^2_R \\ + |(A_i - B_iK_k + A_j - B_jK_k)x/2|^2_{S_{P_{ij},\gamma^2}} \\ - \gamma^2 |(A_i - B_iK_k - A_j + B_jK_k)x/2|^2 \end{aligned} \tag{2}$$

for $x \in \mathbb{R}^n$ and $i, j, k \in \{1, \dots, s\}$, excluding the case $i \neq j = k$. Then, the input-output gain is bounded by γ for the adaptive control law given by

$$\begin{aligned} u_t &= -K_{k_t}x_t, \\ k_t &= \arg\min_i \sum_{\tau=1}^{t-1} |A_i x_\tau + B_i u_\tau - x_{\tau+1}|^2 . \end{aligned} \tag{3}$$

The gain bound γ provides robustness guarantees to unmodelled (possibly nonlinear and infinite-dimensional) dynamics following the small gain theorem [23]. Therefore, in solving (2) for γ, K_i and P_{ij} for $i, j \in \{1, \cdots, s\}$, one wishes to acquire as small as possible values of γ. We specialize Proposition 1 to the case $\mathcal{M} := \{(A_1, B_1), (A_2, B_2)\}$, i.e., the simultaneous stabilization of two LTI plants, possibly MIMO.

Corollary 1: Consider the model set $\mathcal{M} := \{(A_1, B_1), (A_2, B_2)\}$. It suffices for the existence of the adaptive minimax controller that the following set of inequalities hold

$$\begin{aligned} P_i &\succeq Q + K_i^T RK_i + (A_i - B_iK_i)^T S_{P_i,\gamma^2}(A_i - B_iK_i), \\ T &\succeq Q + K_k^T RK_k + (A_i - B_iK_k)^T S_{P_i,\gamma^2}(A_i - B_iK_k), \\ T &\succeq Q + K_k^T RK_k + \frac{1}{4} |(A_k - B_kK_k + A_i - B_iK_k)|^2_{S_{T,\gamma^2}} \\ &\quad - \frac{\gamma^2}{4} |(A_k - B_kK_k - A_i + B_iK_k)|^2 , \end{aligned}$$

where $i, k \in \{1, 2\}$ and $i \neq k$, and the quadruplet (P_1, P_2, T, γ^2) adheres to the construction $0 \prec P_i \prec T$ and $0 \prec T \prec \gamma^2 I_n$.

Proof. The first inequality is obtained by selecting $i = j = k$ in (2). The second one, on the other hand, is obtained by selecting $i = j \neq k$. Lastly, to obtain the third inequality, we define $T := P_{12} = P_{21}$ and we select $i = k \neq j$. By consequence, we have demonstrated that the set of matrix inequalities in (2) collapses to the matrix inequalities mentioned in the statement of the Corollary when $\mathcal{M} := \{(A_1, B_1), (A_2, B_2)\}$. □

The natural question that arises then is if the set of inequalities in (2) are feasible for any $\mathcal{M}$ of the type $\mathcal{M} := \{(A_1, B_1), (A_2, B_2)\}$ and with a finite $\mathcal{L}_2$-gain γ under the condition that each pair is stabilizable. This question is the center of this paper. We capture this question in the following problem.

Problem 1: Suppose the two pairs $(A_1, B_1), (A_2, B_2)$ are stabilizable. Show that a minimax adaptive controller finite $\mathcal{L}_2$-gain stabilizes the two pairs in the set simultaneously. The main results of the paper are presented in the sequel.

III. MAIN RESULTS

A. Guarantees on stabilization of two P-stabilizable MIMO LTI plants

We first introduce the notion of P-stabilization.

Definition 1: Call the model set $\mathcal{M} := \{(A_1, B_1), (A_2, B_2)\}$ P-stabilizable if $\exists\, P \succ 0$ and $\gamma > 0$ such that the following holds

$$P \succeq Q + K_i^T RK_i + (A_i - B_iK_i)^T S_{P,\gamma^2}(A_i - B_iK_i),$$

for $i \in \{1, 2\}$, for some stabilizing controllers K_1 and K_2.

Remark 1: Two systems are P-stabilizable if they accept the same solution $P \succ 0$ to the H_∞ discrete algebraic Riccati inequality (DARI). Another way to look at it is that they are stabilizable with the same quadratic Lyapunov function $V(x) = x^T Px$.

Example 1: Consider two plants $(A_1, B_1), (A_2, B_2)$ in the controllability canonical form. It is then possible to place the closed loop poles of the two plants inside the unit circle such that it holds that $A_1 - B_1K_1 = A_2 - B_2K_2$ for some K_1 and K_2; hence, $\exists\, P \succ 0$ common among the two plants for some large enough $\gamma > 0$ and therefore they are P-stabilizable.

The main result of the paper is stated next.

Theorem 1: Suppose (A_1, B_1) and (A_2, B_2) are P-stabilizable. Then, there exists a minimax adaptive controller that stabilizes both plants simultaneously in the $\mathcal{L}_2$-gain sense.

Proof. The proof is constructive and is achieved by showing the feasibility of the set of inequalities given in Corollary 1 for all P-stabilizable pairs $(A_1, B_1), (A_2, B_2)$ for the choices $P_1 = P_2 = P$ and $T := \alpha P$ for some $\alpha > 1$, i.e., showing the feasibility of

$$P \succeq Q + K_i^T RK_i + (A_i - B_iK_i)^T S_{P,\gamma^2}(A_i - B_iK_i) \tag{4}$$

$$\alpha P \succeq Q + K_k^T RK_k + (A_i - B_iK_k)^T S_{P,\gamma^2}(A_i - B_iK_k) \tag{5}$$

$$\begin{aligned} \alpha P \succeq Q + K_k^T RK_k \\ + \frac{1}{4} |(A_k - B_kK_k + A_i - B_iK_k)|^2_{S_{\alpha P,\gamma^2}} \\ - \frac{\gamma^2}{4} |(A_k - B_kK_k - A_i + B_iK_k)|^2 \end{aligned} \tag{6}$$

Where $i, k \in \{1, 2\}$, $i \neq k$, and the triplet (P, T, γ^2) is selected such that $0 \prec P \prec T$, and $0 \prec T \prec \gamma^2 I_n$. The two pairs being P-stabilizable means that (4) accepts a feasible solution pair (P, γ) for $i \in \{1, 2\}$. Fix γ and select α large enough such that (5) holds, i.e., the following holds

$$\alpha P \succeq Q + K_k^T R K_k + (A_i - B_i K_k)^T S_{P,\,\gamma^2}(A_i - B_i K_k), \tag{7}$$

where $i \neq k$. It remains to show the feasibility of (6) for some $\bar{\gamma} > \gamma$. Define the matrix variables

$$M_k = A_k - B_k K_k,$$
$$\Delta_{ik} = \frac{(A_i - B_i K_k) - (A_k - B_k K_k)}{2},$$

then (6) could be rewritten succinctly as

$$\alpha P \succeq Q + K_k^T R K_k + (M_k + \Delta_{ik})^T S_{\alpha P,\,\bar{\gamma}^2} (M_k + \Delta_{ik}) - \bar{\gamma}^2 \Delta_{ik}^T \Delta_{ik}. \tag{8}$$

Assuming that (4) holds, multiplying by α yields the following

$$T = \alpha P \succeq Q + K_k^T R K_k + M_k^T S_{\alpha P,\,\alpha\gamma^2} M_k.$$

Then, (8) holds if the following holds

$$(M_k + \Delta_{ik})^T S_{\alpha P,\,\bar{\gamma}^2} (M_k + \Delta_{ik}) \preceq \bar{\gamma}^2 \Delta_{ik}^T \Delta_{ik} + M_k^T S_{\alpha P,\,\alpha\gamma^2} M_k. \tag{9}$$

Expanding the left hand side of (9) yields

$$(M_k + \Delta_{ik})^T S_{\alpha P,\,\bar{\gamma}^2} (M_k + \Delta_{ik}) = M_k^T S_{\alpha P,\,\bar{\gamma}^2} M_k + \Delta_{ik}^T S_{\alpha P,\,\bar{\gamma}^2} \Delta_{ik} + M_k^T S_{\alpha P,\,\bar{\gamma}^2} \Delta_{ik} + \Delta_{ik}^T S_{\alpha P,\,\bar{\gamma}^2} M_k. \tag{10}$$

Using Lemma 1 at the Appendix by taking $E = M_k$, $F = \Delta_{ik}$, $M = S_{\alpha P,\,\bar{\gamma}^2}$ and a non-zero scalar β yields the inequality

$$M_k^T S_{\alpha P,\,\bar{\gamma}^2} \Delta_{ik} + \Delta_{ik}^T S_{\alpha P,\,\bar{\gamma}^2} M_k \preceq \beta^2 M_k^T S_{\alpha P,\,\bar{\gamma}^2} M_k + \beta^{-2} \Delta_{ik}^T S_{\alpha P,\,\bar{\gamma}^2} \Delta_{ik}. \tag{11}$$

Combining (9-11) we obtain the sufficient condition

$$M_k^T S_{\alpha P,\,\bar{\gamma}^2} M_k + \Delta_{ik}^T S_{\alpha P,\,\bar{\gamma}^2} \Delta_{ik} + \beta^2 M_k^T S_{\alpha P,\,\bar{\gamma}^2} M_k + \beta^{-2} \Delta_{ik}^T S_{\alpha P,\,\bar{\gamma}^2} \Delta_{ik} \preceq \bar{\gamma}^2 \Delta_{ik}^T \Delta_{ik} + M_k^T S_{\alpha P,\,\alpha\gamma^2} M_k.$$

Re-arranging gives

$$M_k^T \left(S_{\alpha P,\,\alpha\gamma^2} - \left(1 + \beta^2\right) S_{\alpha P,\,\bar{\gamma}^2}\right) M_k + \Delta_{ik}^T \left(\bar{\gamma}^2 I_n - \left(1 + \beta^{-2}\right) S_{\alpha P,\,\bar{\gamma}^2}\right) \Delta_{ik} \succeq 0.$$

Hence, it is sufficient to show the existence of a $\bar{\gamma}^2 > \alpha\gamma^2$ such that the next two inequalities hold

$$S_{\alpha P,\,\alpha\gamma^2} - \left(1 + \beta^2\right) S_{\alpha P,\,\bar{\gamma}^2} \succeq 0, \tag{12}$$
$$\bar{\gamma}^2 I_n - \left(1 + \beta^{-2}\right) S_{\alpha P,\,\bar{\gamma}^2} \succeq 0. \tag{13}$$

(12-13) could be satisfied by selecting a small enough β and a large enough $\bar{\gamma}$ as will be demonstrated in the next section.

To conclude, we have shown that (4-6) could be made feasible for any two P-stabilizable pairs (A_1, B_1), (A_2, B_2) and therefore the theorem is proven. □

B. Explicit value of the achievable $\mathcal{L}_2$-gain bound

We provide an explicit value for $\bar{\gamma}$ that the adaptive controller achieves. This is done by solving the inequalities in (12-13). This bound is provided in the next Theorem.

Theorem 2: Let (A_1, B_1), (A_2, B_2) be stabilizable with a common tuple (P, γ). Then, a minimax adaptive controller stabilizes both plants with an $\mathcal{L}_2$-gain bound given by

$$\bar{\gamma}^2 = \alpha(2 + \beta^{-2})\|P\|,$$

where

$$\beta^2 = -\frac{\|P\|^{-1} + \|P^{-1}\| - 3\gamma^{-2}}{4\left(\|P^{-1}\| - \gamma^{-2}\right)} + \frac{1}{4}\sqrt{\Lambda},$$
$$\Lambda = \left(\frac{\|P\|^{-1} + \|P^{-1}\| - 3\gamma^{-2}}{\|P^{-1}\| - \gamma^{-2}}\right)^2 + \frac{8\gamma^{-2}}{\|P^{-1}\| - \gamma^{-2}},$$

and $\alpha > 1$ is selected such that

$$\alpha P \succeq Q + K_k^T R K_k + (A_i - B_i K_k)^T S_{P,\,\gamma^2}(A_i - B_i K_k),$$

where $i,\, k \in \{1, 2\}$, $i \neq k$.

Proof. Consider the condition in (13). Written explicitly

$$\bar{\gamma}^2 I_n - \left(1 + \beta^{-2}\right) \left((\alpha P)^{-1} - \bar{\gamma}^{-2} I_n\right)^{-1} \succeq 0.$$

Solving for $\bar{\gamma}$ yields

$$\bar{\gamma}^2 I_n \succeq \alpha(2 + \beta^{-2})P.$$

Therefore, the lowest achievable $\bar{\gamma}^2$ is

$$\bar{\gamma}^2 = \alpha(2 + \beta^{-2})\|P\|. \tag{14}$$

To get a feasible value for β we plug (14) in (12). This leads to

$$\left((\alpha P)^{-1} - \left(\alpha\gamma^2\right)^{-1} I_n\right)^{-1} - \left(1 + \beta^2\right) \left((\alpha P)^{-1} - \left(\alpha\left(1 + \beta^{-2}\right)\|P\|\right)^{-1} I_n\right)^{-1} \succeq 0,$$

canceling out the α term and simplifying further yields

$$\left(\left(3\beta^2 + 2\beta^4 + 1\right)\gamma^{-2} - \beta^2\|P\|^{-1}\right) I_n \succeq \left(\beta^2 + 2\beta^4\right) P^{-1}.$$

Therefore, β has to satisfy

$$\left(3\beta^2 + 2\beta^4 + 1\right)\gamma^{-2} - \beta^2\|P\|^{-1} \succeq \left(\beta^2 + 2\beta^4\right)\|P^{-1}\|.$$

Rearranging yields the scalar inequality in β

$$2\beta^4 + \frac{\|P\|^{-1} + \|P^{-1}\| - 3\gamma^{-2}}{\|P^{-1}\| - \gamma^{-2}}\beta^2 - \frac{\gamma^{-2}}{\|P^{-1}\| - \gamma^{-2}} \geq 0.$$

Since we are interested in the smallest value of β, we solve for the equality instead

$$2\beta^4 + \frac{\|P\|^{-1} + \|P^{-1}\| - 3\gamma^{-2}}{\|P^{-1}\| - \gamma^{-2}}\beta^2 - \frac{\gamma^{-2}}{\|P^{-1}\| - \gamma^{-2}} = 0.$$

First, we must make sure the existence of a $\beta^2 \in \mathbb{R}$ such that the equation holds. Secondly, such β^2 has to be positive. The first condition is met since the discriminant, which we denote by Λ, is positive, that is

$$\Lambda = \left(\frac{\|P\|^{-1} + \|P^{-1}\| - 3\gamma^{-2}}{\|P^{-1}\| - \gamma^{-2}}\right)^2 + \frac{8\gamma^{-2}}{\|P^{-1}\| - \gamma^{-2}} > 0.$$

This is the case since $\|P^{-1}\| - \gamma^{-2} > 0$. This can be concluded since by construction $\lambda_{\min}(P) \leq \lambda_{\max}(P) < \gamma^2 \iff \|P^{-1}\| = \frac{1}{\lambda_{\min}(P)} > \gamma^{-2} \iff \|P^{-1}\| - \gamma^{-2} > 0$. The next step is to show that one of the solutions

$$\beta_{1,2}^2 = -\frac{\|P\|^{-1} + \|P^{-1}\| - 3\gamma^{-2}}{4\left(\|P^{-1}\| - \gamma^{-2}\right)} \pm \frac{1}{4}\sqrt{\Lambda}$$

is positive. β_2^2 is obviously rejected since it yields a negative solution. However, β_1^2 is clearly admissible since $\frac{1}{4}\sqrt{\Lambda} > \frac{\|P\|^{-1} + \|P^{-1}\| - 3\gamma^{-2}}{4(\|P^{-1}\| - \gamma^{-2})}$. Therefore we take

$$\beta^2 = -\frac{\|P\|^{-1} + \|P^{-1}\| - 3\gamma^{-2}}{4\left(\|P^{-1}\| - \gamma^{-2}\right)} + \frac{1}{4}\sqrt{\Lambda}.$$

To get the lowest achievable $\bar{\gamma}$, we plug this value of β in (14). This concludes the proof Theorem. □

C. A specialized $\mathcal{L}_2$-gain bound for the single-input single-output (SISO) case

in the SISO case we have $\|P\|^{-1} = \|P^{-1}\| = P^{-1}$, and therefore the discriminant from before simplifies to

$$\Lambda = \left(\frac{2P^{-1} - 3\gamma^{-2}}{P^{-1} - \gamma^{-2}}\right)^2 + \frac{8\gamma^{-2}}{P^{-1} - \gamma^{-2}} = \left(\frac{2P^{-1} - \gamma^{-2}}{P^{-1} - \gamma^{-2}}\right)^2,$$

and β^2 simplifies to

$$\beta^2 = \frac{\gamma^{-2}}{2\left(P^{-1} - \gamma^{-2}\right)} = \frac{P}{2\left(\gamma^2 - P\right)} > 0,$$

and hence the achievable $\bar{\gamma}^2$ is given by

$$\bar{\gamma}^2 = \alpha(2 + \beta^{-2})P = \alpha P\left(2 + \frac{2\left(\gamma^2 - P\right)}{P}\right) \iff \bar{\gamma} = 2\alpha\gamma^2.$$

IV. Numerical Examples

A. systems with unknown input direction

Consider the unstable SISO system with unknown input direction

$$x_{t+1} = a x_t \pm u_k + w_t, \quad a > 1.$$

We attempt to stabilize the system and extract an $\mathcal{L}_2$-gain bound we call it $\bar{\gamma}$ and compare it to the optimal bound given in [22]. Stated in our context, we are trying to simultaneously stabilize the two pairs $(a, 1)$ and $(a, -1)$. To establish ground for comparison, we follow [22] by taking $Q = 1$ and $R = 0$, which corresponds to bounding the $\mathcal{L}_2$-gain from w to x.

We start by finding a pair (P, γ) that commonly solves the H_∞ DARI for both systems. We pick $K_1 = -K_2 = K$ where K is the H_∞ controller given by

$$K = \frac{B S_{P,\gamma^2}}{R + B^2 S_{P,\gamma^2}} A = a,$$

which corresponds to a deadbeat controller. Plugging $K = a$ in the H_∞ DARI yields the solution $P = Q = 1$ and the condition $\gamma^2 > P = 1 \iff \gamma > 1$. Let's consider the pair $(1, \gamma)$ to be a feasible pair, where $\gamma > 1$. Next, we select α to be

$$T := \alpha P = \alpha = 1 + \frac{4a^2}{1 - \gamma^{-2}} = 1 + \frac{4a^2\gamma^2}{\gamma^2 - 1}.$$

Last step would be to solve the inequalities (12-13) for a feasible pair $(\bar{\gamma}, \beta)$. From section (III-C) and having $P = 1$, the smallest achievable $\bar{\gamma}$ is given by

$$\bar{\gamma} = \sqrt{2\gamma^2 + \frac{8\gamma^4 a^2}{\gamma^2 - 1}}, \quad \gamma > 1.$$

In [22], it is conjectured that no controller can achieve a better $\mathcal{L}_2$-gain bound $a + \sqrt{a^2 + 1}$. We compare our obtained bound to this optimal bound for different values of a and for a fixed value of $\gamma = 1.4$; this comparison is given in figure 1. We justify the non-tightness by the fact that unlike [22], our bound account for a much larger uncertainty set as compared to that of only unknown input sign, and also works in the case $R \neq 0$, and therefore when applied to special cases such as this it tends to be conservative. However, we believe that the techniques used in the constructive proof could be refined to produce much tighter bounds when the model set $\mathcal{M}$ is more accurately defined a priori.

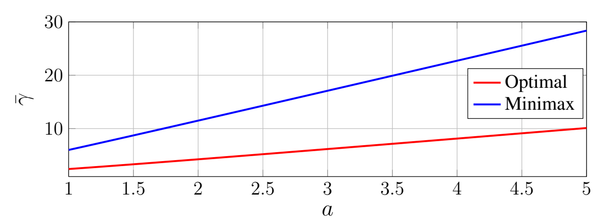

Fig. 1: A comparison between The optimal $\mathcal{L}_2$-gain bound and the one achieved by the minimax controller for the system given in example IV-A.

B. Inverted pendulum with uncertain actuator gain

Consider the unstable state-space model corresponding to an inverted pendulum with uncertain input matrix

$$x_{t+1} = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} x_t + \begin{bmatrix} b \\ 0 \end{bmatrix} u_t + w_t,$$

where b takes two values, $b_1 = 1$ or $b_2 = 3$. By taking $K = [k_1\ k_2]$, it could be checked by an application of Jury's test for stability [15] that such system can not be stabilized by a static linear state feedback law. To see this, we plug in K and obtain the following characteristic polynomial in the variable z

$$D(z) = z^2 + z(bk_1 - 3) + bk_2 + 1.$$

We attempt to ensure that the polynomial $D(z)$ has roots inside the unit circle through the design variables k_1 and k_2. Following Jury's Stability test, the following necessary conditions must first hold

(1) $D(1) > 0$;

(2) $D(-1) > 0$;
(3) $|a_0| < a_2$;
where $D(z)$ is considered in the form $D(z) = a_2 z^2 + a_1 z + a_0$. These conditions translate to the following
(1) $b(k_1 + k_2) > 1$;
(2) $b(k_2 - k_1) > -5$;
(3) $|bk_2 + 1| < 1$;
solving the inequalities for $b = 1$ and $b = 3$ yields the solution given in table I.

TABLE I

$b = 1$	$b = 3$
$3 < k_1 < 5$	$1 < k_1 < \frac{5}{3}$
$-2 < k_2 < 0$	$-\frac{2}{3} < k_2 < 0$

Notice that there exist no value of k_1 for which we could place the poles of the characteristic polynomial inside the unit circle for both values of b. This asserts that our design objectives are not met using a static state feedback control law.

The goal now is to find an adaptive feedback law that stabilizes the system for both values of b. We select $Q = I$ and $R = 1$. We first find a feasible pair (P, γ) that solves the DARI in (4) for both pairs. We get

$$P = \begin{bmatrix} 9.62 & -2.68 \\ -2.68 & 2 \end{bmatrix}, \gamma = 10.$$

Next, we fix $T := \alpha P$ where α is found as in (7); we get $\alpha = 35$ and therefore

$$T = \begin{bmatrix} 336.75 & -93.91 \\ -93.91 & 70 \end{bmatrix};$$

this choice of T renders (5) feasible. Last step is to find a $\bar{\gamma} > \gamma$ that will make (6) feasible for the same choice of T. This problem breaks down to finding a pair $(\bar{\gamma}, \beta)$ that renders (12-13) feasible. One way is to follow the explicit solutions given in theorem 2. Another alternative is to bisect over $\bar{\gamma}$ until (6) is made feasible. Following the second approach yields $\bar{\gamma} = 27$.

We run the adaptive control law given in (3) on this system in the presence of a stochastic disturbance w under the assumption that the true system corresponds to an input matrix with $b = b_2 = 3$. We compare the performance of the minimax adaptive controller to the optimal H_∞ controller which knows the actuator gain and a $2n$-periodic switching deadbeat controller. The 4-periodic switching deadbeat controller is given by

$$\bar{u}_t = \begin{cases} -\bar{K}_1 x_t, & t \bmod 4 = 0 \\ -\bar{K}_1 x_t, & t \bmod 4 = 1 \\ -\bar{K}_2 x_t, & t \bmod 4 = 2 \\ -\bar{K}_2 x_t, & t \bmod 4 = 3 \end{cases}$$

where $\bar{K}_1$ and $\bar{K}_2$ are the deadbeat controller gains corresponding to the two systems respectively and the mod operator is the remainder of division. The results are presented in figure 2. As can be seen, the minimax adaptive controller exhibits a nearly optimal performance and outperforms the switching deadbeat controller. The minimax controller increases the control activity in the beginning for the sake of exploration. Once the true dyanmics is learned, it behaves similar to the H_∞ controller.

TABLE II: $\mathcal{L}_2$-gain from disturbance to state

Optimal H_∞	Minimax	Periodic deadbeat
1.88	8.8	59.1

C. Performance comparison

We compare the $\mathcal{L}_2$-gain achieved by the adaptive controller to that of the switching deadbeat controller and the optimal H_∞ controller for the system in IV-B. To obtain a tighter bound on the $\mathcal{L}_2$-gain of the minimax controller we resort to the work in [4] which treats the synthesis of minimax adaptive controllers. The approach presented in [4] proved to deliver tighter lower bounds on the $\mathcal{L}_2$-gain. To obtain the $\mathcal{L}_2$-gain of the system driven by the 4-periodic switching deadbeat controller, note that in such case the closed loop system is periodic with period $2n = 4$. This suggest representing it via an LTI system according to [11]. To this end, we define the new state and disturbance sequence

$$X_t = \begin{bmatrix} x_{4t} \\ \vdots \\ x_{4t+3} \end{bmatrix}, \eta_t = \begin{bmatrix} w_{4t} \\ \vdots \\ w_{4t+6} \end{bmatrix},$$

this yields the LTI representation of our periodic systems given by

$$X_{t+1} = A_p X_t + B_p \eta_t,$$

where

$$A_p = \begin{bmatrix} \bar{A}_{22}^2 \bar{A}_{21}^2 & 0 & 0 & 0 \\ 0 & \bar{A}_{21} \bar{A}_{22}^2 \bar{A}_{21} & 0 & 0 \\ 0 & 0 & \bar{A}_{21}^2 \bar{A}_{22}^2 & 0 \\ 0 & 0 & 0 & \bar{A}_{22} \bar{A}_{21}^2 \bar{A}_{22} \end{bmatrix},$$

$$B_p = \begin{bmatrix} \bar{A}_{22}^2 \bar{A}_{21} & A_{22}^2 & A_{22} & I_n & 0 & 0 & 0 \\ 0 & \bar{A}_{21} \bar{A}_{22}^2 & \bar{A}_{21} \bar{A}_{22} & \bar{A}_{21} & I_n & 0 & 0 \\ 0 & 0 & \bar{A}_{21}^2 \bar{A}_{21} & \bar{A}_{21}^2 & \bar{A}_{21} & I_n & 0 \\ 0 & 0 & 0 & \bar{A}_{22} \bar{A}_{21}^2 & \bar{A}_{22} \bar{A}_{21} & \bar{A}_{22} & I_n \end{bmatrix},$$

and

$$\bar{A}_{22} = A - B_2 \bar{K}_2,$$
$$\bar{A}_{21} = A - B_2 \bar{K}_1,$$
$$B_2 = \begin{bmatrix} b_2 \\ 0 \end{bmatrix}.$$

The $\mathcal{L}_2$-gain of this system is the H_∞ norm of the transfer function from η to X. Let's denote such transfer function by $G(z)$. Then, the $\mathcal{L}_2$-gain is

$$\|G(z)\|_\infty = \|(zI_{4n} - A_p) B_p\|_\infty.$$

The obtained numerical results are given in table II.

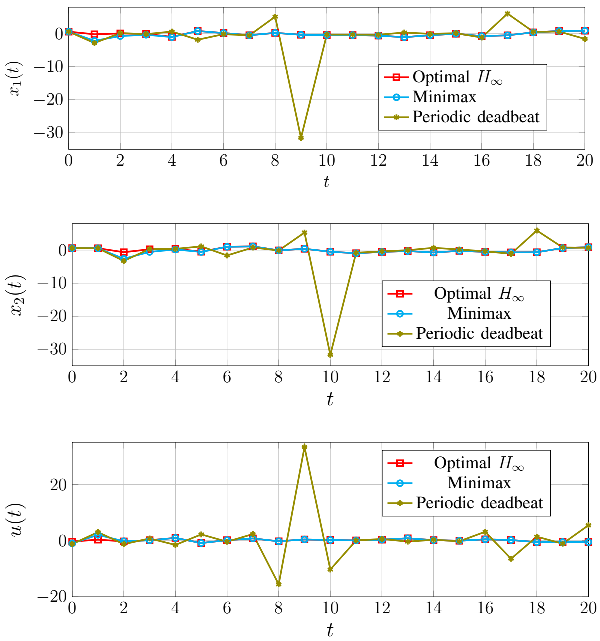

Fig. 2: The states and the control input trajectories when using the minimax adaptive controller, the optimal H_∞ controller and a 4-periodic switching deadbeat controller.

V. Conclusions and Future Works

The paper discussed the simultaneous stabilization of LTI plants via minimax adaptive control. We have established that any two P-stabilizable LTI plants could be stabilized via a minimax adaptive controller when no static state-feedback controller could accomplish that. We as well provided an explicit $\mathcal{L}_2$-gain bound on the achievable worst-case performance. Future work concerns scaling our guarantees to cover stabilization of more than two systems and providing tighter lower bound on the achievable $\mathcal{L}_2$-gain performance.

Appendix

A. Preliminary Lemmata

Lemma 1: Let E, $F \in \mathbb{R}^{n\times n}$ be two matrices and $M \in \mathbb{R}^{n\times n}$ be a positive definite matrix. Then, for any scalar $\beta \neq 0$, it holds that

$$E^T MF + F^T ME \preceq \beta^2 E^T ME + \beta^{-2} F^T MF.$$

Proof. Start by noting that for any two matrices E, $F \in \mathbb{R}^{n\times n}$ and any nonzero constant β it holds that

$$(\beta E - \beta^{-1}F)^T M(\beta E - \beta^{-1}F) \succeq 0,$$

expanding the left hand side of the inequality gives

$$\beta^2 E^T ME + \beta^{-2} F^T MF - E^T MF - F^T ME \succeq 0.$$

Then, it follows immediately that

$$E^T MF + F^T ME \preceq \beta^2 E^T ME + \beta^{-2} F^T MF.$$

Note that in the event $E = F$ along with the selection $\beta = 1$, we get equality. This shows that the upper bound provided in the lemma is tight. □

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