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Robust Control of a Nonsmooth or Switched Euler-Lagrange Dynamic System Using ARISE Control

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***Abstract*—Research has been performed for many years regarding the development of nonlinear control techniques. One commonly researched concept in nonlinear control theory is the development of methods that can address unknown disturbances or unmodeled effects in a dynamic system when the terms have constant upper bounds. Continuous and discontinuous nonlinear controllers have evolved over many years to address these unknown terms bounded by constants. Unfortunately, these evolved controllers have either been confined to some classes of nonlinear systems or have shown to be susceptible to the chattering effect. The focus of this work is to construct a control framework that addresses unknown dynamic terms with constant bounds while simultaneously minimizing chatter for nonsmooth or switched Euler-Lagrange dynamic systems. The proposed control framework includes a filtered error signal designed to compensate for the unmodeled effects bounded by constants and an adaptive control law designed to address uncertainties in the Euler-Lagrange dynamic system's control effectiveness matrix. To ensure the effectiveness of the proposed control law for a nonsmooth Euler-Lagrange dynamic model, a nonsmooth Lyapunov-based stability analysis is performed that proves semi-global exponential tracking to an ultimate bound.**

***Index Terms*—Lyapunov methods, Nonlinear control, integral sliding mode, switched systems, robust and adaptive control.**

I. Introduction

Control system structures for uncertain nonlinear systems have been studied for many years (cf. [1], [2], [9], [12]). A typical control system structure for uncertain nonlinear systems is a high-gain robust controller. This controller can ensure that uncertain dynamic systems remains within a stable region by having gains large enough to compensate for uncertainties in the system. It is a common practice to implement robust controllers with adaptive terms to improve control efficiency [21]. However, adaptive approach are unable to approximate unstructured terms in a dynamic model. Consequently, one approach to address uncertain terms in the dynamics that are upper bounded by constants is to combine high-gain robust controllers with discontinuous controllers such as sliding mode (SM) controllers [11], [16], [25], [27]. Though SM control produces an excellent stability outcome, its implementation is problematic since SM controllers have instantaneous jumps in the control effort, which requires the actuator to act with an infinite frequency. Consequently, SM controllers (and controllers with similar discontinuous control structures) produce a chattering effect [21].

In an effort to develop a continuous controller that is able to account for dynamic model terms with constant bounds and to reap the tracking performance of SM controllers without the limitations, a category of robust controllers called Robust Integral of the Sign of the Error (RISE) were developed. Unlike SM, RISE implements the integral of the SM term into the control law, creating a continuous controller which alleviates chattering, eliminates the need for infinite frequency, and ensures asymptotic tracking. Moreover, with some alterations, in [19], a RISE control structure was developed that achieved exponential error tracking. RISE has previously been implemented for applications such as estimation, control, and optimization (cf. [7], [8], [15], [20], [22], [23], [28]–[30]). However, RISE is limited to a class of continuous dynamic systems since RISE control requires the derivative of the unknown model dynamics, the control input, and the uncertain disturbances must exist and be bounded.

Beyond RISE control, higher-order SM controllers were developed to alleviate the limitations of SM control (cf., [5], [6], [10], [14], [17]). Critically, it was shown that the chattering effect was reduced as a result of the implementation of higher-order SM control. [5]. In order to implement higher-order SM controllers, prior works have required the dynamic system to have a constant upper bound and to be sufficiently smooth (cf. [13], [14], [17]). Thus, higher-order SM controllers have limitations similar to RISE control, such as the dynamic model being restricted to smooth and continuous systems, and constraints being placed on the control objective. Also, if a higher-order SM controller is first order, it will be equivalent to the previously discussed SM controller that is chatter-prone [14].

Preliminary efforts by the authors have been made to develop a class of feedback controllers for a wide range of systems, including nonlinear systems that are continuous as well as discontinuous (e.g., switched), that are capable of compensating for unstructured disturbances that are bounded by constants without creating chatter [3], [26]. These recent control developments by the authors will henceforth be called Auxiliary-injected Robust Integral of the Sliding modE (ARISE) controllers. The early results from the ARISE controllers have been promising; however, to date the ARISE controller has only been implemented on control-affine nonlinear dynamic models [3], [26]. The study in this paper is motivated to develop an ARISE controller for a general, uncertain, and switched Euler-Lagrange dynamic model. However, the extension of ARISE control to Euler-Lagrange dynamic models is particularly complicated by the existence of the inertia matrix, which requires modifications to the error system development and stability analysis. To demonstrate the generality of the proposed approach, a switched and uncertain control effectiveness matrix has also been assumed.In this work, a SM term is injected through a filtered auxiliary error signal into the closed-loop dynamics (i.e., an ARISE controller is developed). The control law is designed in a way such that it has the integral of the SM term in it, which will help in reducing the chattering effect. Furthermore, an adaptive update law is defined to address the unknown terms in the control effectiveness matrix. In addition, through a Lyapunov-like stability analysis, a semi-global result with exponential tracking towards an ultimate bound is achieved, provided that the conditions on the gains and initial values are satisfied. It should be stated that the controller that is developed in this paper can also be extended to continuous systems.

II. Dynamics

To facilitate the design of the proposed ARISE controller, an uncertain, nonsmooth, and nonlinear Euler-Lagrange dynamic model is considered that is modeled as follows:

$$M_\sigma(x)\ddot{x} + C_\sigma(x,\dot{x})\dot{x} + G_\sigma(x) + F_\sigma\dot{x} + d_\sigma(t) = \underbrace{\tau_\phi(x,\dot{x},t)}_{g_\phi(x,\dot{x},t)u(t)}, \quad (1)$$

where $x:\mathbb{R}_{\geq 0}\to\mathbb{R}^n$ is the generalized position coordinate, $\dot{x}:\mathbb{R}_{\geq 0}\to\mathbb{R}^n$ is the generalized velocity coordinate, $\ddot{x}:\mathbb{R}_{\geq 0}\to\mathbb{R}^n$ is the generalized acceleration coordinate, $M_\sigma:\mathbb{R}^n\to\mathbb{R}^{n\times n}$ is an unknown non-smooth but continuous inertia matrix, $C_\sigma:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{n\times n}$ is an unknown switched centripetal-Coriolis matrix, $G_\sigma:\mathbb{R}^n\to\mathbb{R}^n$ is an unknown switched gravitational effects vector, $F_\sigma\in\mathbb{R}^{n\times n}$ is an unknown switched coefficient damping matrix comprised of constants, and $d_\sigma:\mathbb{R}_{\geq 0}\to\mathbb{R}^n$ is an unknown disturbance vector. The signal $\sigma(x,\dot{x},t):\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_{\geq 0}\to\mathcal{I}=\{1,2,\ldots,\sigma_{max}\}$, signifies the index of M_σ, C_σ, G_σ, F_σ, and d_σ, where $\sigma_{max}\in\mathbb{N}$ is the finite index set for all involved subsystems. In (1), $g_\phi:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_{\geq 0}\to\mathbb{R}^{n\times m}$ is the nonsmooth and unknown control effectiveness matrix, $u:\mathbb{R}_{\geq 0}\to\mathbb{R}^m$ is the control input, and $\tau_\phi:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_{\geq 0}\to\mathbb{R}^n$ is the generalized input torque or force. The signal $\phi(x,\dot{x},t):\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_{\geq 0}\to\mathcal{B}=\{1,2,\ldots,b\}$, is a set of switching signals and signifies the index of $\tau_\phi(x,\dot{x},t)$ and $g_\phi(x,\dot{x},t)$, where $b\in\mathbb{N}$ is the finite index set for all the distinctive forms of $\tau_\phi(x,\dot{x},t)$, and $g_\phi(x,\dot{x},t)$.

Remark 1. Every index $\sigma\in\mathcal{I}$ switches with every discrete jump or non-smooth occurrence that takes place in dynamic parameters of the system in (1). Similarly, for every change in the form of $\tau_\phi(x,\dot{x},t)$ and $g_\phi(x,\dot{x},t)$, a discrete jump in $\phi\in\mathcal{B}$ takes place. Thus, when $\phi\in\mathcal{B}$ and $\sigma\in\mathcal{I}$ are constants, the dynamics are continuous in (1).

Moreover, the common assumptions made below must be satisfied by the dynamical system in (1).

Assumption 1. For all $\sigma\in\mathcal{I}$, $M_\sigma(x)$ is a positive definite and symmetric matrix such that its satisfies $c_m\|\Upsilon\|^2\leq\Upsilon^T M_\sigma(x)\Upsilon\leq c_M\|\Upsilon\|^2$, $\forall\Upsilon\in\mathbb{R}^n$, $\forall t\geq t_0$ (i.e., $c_m\leq\|M_\sigma(x)\|\leq c_M$), where $c_m\in\mathbb{R}_{>0}$ and $c_M\in\mathbb{R}_{>0}$ are positive known constants.

Assumption 2. For all $\sigma\in\mathcal{I}$, $C_\sigma(x,\dot{x})$ is bounded as $\|C_\sigma(x,\dot{x})\|\leq c_C\|\dot{x}\|,\forall t\geq t_0$ in which $c_C\in\mathbb{R}_{>0}$ is a known constant. Moreover, $C_\sigma(x,\dot{x})$ also satisfies $C_\sigma(x,\Upsilon)v=C_\sigma(x,v)\Upsilon$, $\forall\Upsilon$, $\forall\sigma\in\mathcal{I}$, $\forall v\in\mathbb{R}^n$.

Assumption 3. For all $\sigma\in\mathcal{I}$, $G_\sigma(x)$ is bounded as $\|G_\sigma(x)\|\leq c_G,\forall t\geq t_0$, in which $c_G\in\mathbb{R}_{>0}$ is a known constant.

Assumption 4. For all $\sigma\in\mathcal{I}$, F_σ is bounded as $\|F_\sigma\|\leq c_F,\forall t\geq t_0$, in which $c_F\in\mathbb{R}_{>0}$ is a known constant.

Assumption 5. For all $\sigma\in\mathcal{I}$, $d_\sigma(t)$ is bounded as $\|d_\sigma(t)\|\leq c_D,\forall t\geq t_0$, in which $c_D\in\mathbb{R}_{>0}$ is a known constant.

Assumption 6. The generalized force/torque input, $\tau_\phi(x,\dot{x},t)=g_\phi(x,\dot{x},t)u(t)$, can be parameterized linearly $\forall t\geq t_0$ and $\forall\phi\in\mathcal{B}$ as

$$g_\phi(x,\dot{x},t)u(t)=Y(x,\dot{x},t,u)\theta, \quad (2)$$

where $\theta\in\mathbb{R}^q$ is a vector of bounded unknown constant parameters and $Y:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_{\geq 0}\times\mathbb{R}^m\to\mathbb{R}^{n\times q}$ are measurable regression matrices. The vector θ can be upper and lower bounded as

$$\underline{\theta}_i\leq\theta_i\leq\bar{\theta}_i,\forall i\in\{1,\ldots,q\}, \quad (3)$$

where θ_i is the i^{th} parameter of θ. Additionally, the unknown control effectiveness matrix $g(x,\dot{x},t)$ is assumed to be bounded if x and $\dot{x}$ are bounded.

III. Control Development

In (1), One of the control aim of the dynamic system in (1) is that the generalized states track a smooth desired path $x_d:\mathbb{R}_{\geq 0}\to\mathbb{R}^n$. The control objective can be quantified by a measurable error signal represented by $e:\mathbb{R}_{\geq 0}\to\mathbb{R}^n$ and defined as

$$e(t)\triangleq x_d(t)-x(t). \quad (4)$$

The measurable auxiliary tracking error, indicated as $r:\mathbb{R}_{\geq 0}\to\mathbb{R}^n$, is defined as

$$r\triangleq\dot{e}+\alpha_1 e+\alpha_2 e_f, \quad (5)$$

where $\alpha_1,\alpha_2\in\mathbb{R}_{>0}$ are user-defined control gains, and $e_f:\mathbb{R}_{\geq 0}\to\mathbb{R}^n$ is a filtered auxiliary error signal determined by solving the following auxiliary filter dynamics:

$$\dot{e}_f\triangleq -k_1\operatorname{sgn}(r)-k_2 r+\alpha_2 e-\beta_1 e_f, \quad (6)$$

where $k_1,k_2,\beta_1\in\mathbb{R}_{>0}$ are user-defined control gains, $\operatorname{sgn}(\cdot)$ is a sliding mode function (i.e., a signum function). The other control aims in this study is to implement a SM term, which is $-k_1\operatorname{sgn}(r(t))$, in the closed-loop error dynamics without actually implementing the SM term in the controller itself. This control aim motivated the inclusion of $-k_1\text{sgn}(r(t))$ in $\dot{e}_f$ and the auxiliary signal $e_f(t)$ in (5).

Taking the time derivative of (5) and then multiplying throughout by M_σ, and using (1), (4), and (6) results in

$$\begin{aligned} M_\sigma \dot{r} &= \chi_\sigma - e - C_\sigma r - g_\phi u - M_\sigma \alpha_2 k_1 \text{sgn}(r) \\ &\quad - M_\sigma \alpha_2 k_2 r - M_\sigma \alpha_2 \beta_1 e_f, \end{aligned} \tag{7}$$

where $\chi_\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}^n, \forall \sigma \in \mathcal{I}$ is an auxiliary term determined as

$$\begin{aligned} \chi_\sigma &\triangleq M_\sigma \ddot{x}_d + C_\sigma (\dot{x}_d + \alpha_1 e + \alpha_2 e_f) + G_\sigma \\ &\quad + F_\sigma \dot{x} + d_\sigma + M_\sigma \alpha_1 \dot{e} + \left(M_\sigma \alpha_2^2 + 1\right) e. \end{aligned} \tag{8}$$

Using Assumptions 1-5, (4), and (5), χ_σ can be upper bounded with respect to the generalized states such that

$$\|\chi_\sigma\| \leq \Phi + \rho(\|z\|)\|z\|, \forall \sigma \in \mathcal{I} \tag{9}$$

where $\Phi \in \mathbb{R}_{>0}$ is a known constant, $\rho(\cdot)$ is radially unbounded, positive, and strictly increasing function, and $z : \mathbb{R}_{\geq 0} \to \mathbb{R}^{3n}$ is a composite error vector determined as

$$z(t) \triangleq \begin{bmatrix} e^T & r^T & e_f^T \end{bmatrix}^T. \tag{10}$$

One complication in the development of the control system is that $g_\phi(x, \dot{x}, t)$, the control effectiveness matrix, is not known in (7). Nevertheless, the generalized input torque/force, $\tau_\phi(x, \dot{x}, t) = g_\phi(x, \dot{x}, t)\, u(t)$, can be estimated and expressed as

$$\hat{g}_\phi(x, \dot{x}, t)\, u(t) = Y(x, \dot{x}, t, u)\, \hat{\theta}(t), \tag{11}$$

$\forall \phi \in \mathcal{B}$, where $\hat{g}_\phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{m \times n}$, is the estimation of $g_\phi(x, \dot{x}, t)$ and $\hat{\theta} : \mathbb{R}_{\geq 0} \to \mathbb{R}^q$ is the estimation of θ . The error in the parameter estimation, $\tilde{\theta} : \mathbb{R}_{\geq 0} \to \mathbb{R}^q$, is defined as

$$\tilde{\theta}(t) \triangleq \theta - \hat{\theta}(t). \tag{12}$$

Assumption 7. The control effectiveness estimation matrix $\hat{g}_\phi(x, \dot{x}, t)$ is a full row rank matrix for $t \geq t_0$ and $\forall \phi \in \mathcal{B}$. The pseudo inverse of $\hat{g}_\phi(x, \dot{x}, t)$ is given by $\hat{g}_\phi^+ : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{m \times n}$, $\forall \phi \in \mathcal{B}$, in which $\hat{g}_\phi^+(\cdot) \triangleq \hat{g}_\phi^T(\cdot)\left(\hat{g}_\phi(\cdot)\hat{g}_\phi^T(\cdot)\right)^{-1}$. Moreover, $\hat{g}_\phi(x, \dot{x}, t) \in \mathcal{L}_\infty\ \forall \phi \in \mathcal{B}$ given that $\hat{\theta}, x \in \mathcal{L}_\infty$.

Implementing $g_\phi u = Y\theta$ and adding/subtracting $Y\hat{\theta}$ to (7), and using (11), and (12) results in the open-loop error dynamics below:

$$\begin{aligned} M_\sigma \dot{r} &= \chi_\sigma - e - C_\sigma r - Y\tilde{\theta} - \hat{g_\phi} u - M_\sigma \alpha_2 k_1 \text{sgn}(r) \\ &\quad - M_\sigma \alpha_2 k_2 r - M_\sigma \alpha_2 \beta_1 e_f. \end{aligned} \tag{13}$$

Based on the subsequent analysis and by examining the open-loop error system in (13) the control input has been designed as,

$$u \triangleq \hat{g}_\phi^+\left(k_3 r - \left(k_2 + \hat{M}\alpha_2\beta_1\right) e_f\right), \tag{14}$$

where $k_3 \in \mathbb{R}_{>0}$ are user defined control gains, and $\hat{M} \triangleq \frac{1}{2}(c_m + c_M) I_{n\times n}$ is an estimate for M, $I_{n\times n}$ is an identity matrix of $n \times n$ dimension. For all $\sigma \in \mathcal{I}$, the estimation error for the inertia of the system is defined as

$$\tilde{M}_\sigma \triangleq \hat{M} - M_\sigma. \tag{15}$$

Moreover, (15) can be bounded using Assumption 1 as $\left\|\tilde{M}_\sigma\right\| \leq \frac{1}{2}(c_M - c_m) = c_{\tilde{M}}$, where $c_{\tilde{M}} \in \mathbb{R}_{>0}$ is a positive constant. Similarly, the adaptive law is formulated as

$$\dot{\hat{\theta}}(t) \triangleq \text{proj}\left(-\Gamma Y^T r\right), \tag{16}$$

where $\text{proj}(\cdot)$ is the smooth projection algorithm from [4] to ensure that the adaptive law remains bounded as stated in (3), and $\Gamma \in \mathbb{R}^{q\times q}$ is a positive definite diagonal gain matrix. By implementing (14) into (13), the closed-loop error dynamics can be acquired and simplified to yield

$$\begin{aligned} M_\sigma \dot{r} &= \chi_\sigma - e - C_\sigma r - Y\tilde{\theta} + k_2 e_f - M_\sigma \alpha_2 k_1 \text{sgn}(r) \\ &\quad - (k_3 + M_\sigma \alpha_2 k_2) r + \tilde{M}_\sigma \alpha_2 \beta_1 e_f. \end{aligned} \tag{17}$$

It can be noticed from the equations (14) and (6), that the integral of the SM term is present in the controller. This was made possible by including the SM term in (6) which further appeared in the closed-loop error dynamics in (17), which was then used to compensate for the unknown/uncertain terms in the system. Moreover, the designed controller remains continuous if g_ϕ and thus $\hat{g}_\phi^+$ are continuous.

IV. Stability Analysis

To prove stability, a family of Lyapunov candidate functions that are continuously differentiable and positive definite, given by $V_\sigma : \mathcal{D} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, are defined $\forall \sigma \in \mathcal{I}$ as

$$V_\sigma \triangleq \frac{1}{2}e^T e + \frac{1}{2}r^T M_\sigma r + \frac{1}{2}e_f^T e_f + \frac{1}{2}\tilde{\theta}^T \Gamma^{-1}\tilde{\theta}, \tag{18}$$

where $\mathcal{D} \subseteq \mathbb{R}^{3n+q}$ is given as $\mathcal{D} \triangleq \left\{y \in \mathbb{R}^{3n+q} \mid \|y\| < \gamma\right\}$, $\gamma \in \mathbb{R}_{>0}$ is a known constant and also meets the inequality $\gamma \leq \inf\left\{\rho^{-1}\left(\left(\sqrt{2c_m\alpha_2 k_2 \delta_1}, \infty\right)\right)\right\}$, $\delta_1 \in \mathbb{R}$ is a known constant and is defined as $\delta_1 \triangleq \min\left(\alpha_1,\ k_3 - \frac{\alpha_2\beta_1 c_{\tilde{M}}\varepsilon_1}{2},\ \beta_1 - \frac{\alpha_2\beta_1 c_{\tilde{M}}}{2\varepsilon_1} - \frac{k_1\varepsilon_2}{2}\right)$, and $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_{>0}$ are choosable constants. Furthermore, lets define the set of acceptable initial conditions as

$$\mathcal{S}_\mathcal{D} \triangleq \left\{y \in \mathbb{R}^{3n+q} \,\middle|\, \|y\| < \sqrt{\frac{\lambda_1}{\lambda_2}}\gamma\right\}. \tag{19}$$

Moreover, V_σ could be bounded $\forall \sigma \in \mathcal{I}$ as

$$\lambda_1 \|y\|^2 \leq V_\sigma \leq \lambda_2 \|y\|^2, \tag{20}$$

where $\lambda_1 \triangleq \frac{1}{2}\min\left(1, c_m, \lambda_{\min}\left\{\Gamma^{-1}\right\}\right)$ and $\lambda_2 \triangleq \frac{1}{2}\max\left(1, c_M, \lambda_{\max}\left\{\Gamma^{-1}\right\}\right)$. Also, $\lambda_{\min}\{\cdot\}$ is the minimum eigenvalue of $\{\cdot\}$ and $\lambda_{\max}\{\cdot\}$ is the maximum eigenvalue of $\{\cdot\}$. Also, $y : \mathbb{R}_{\geq 0} \to \mathbb{R}^{3n+q}$ is a vector determined as

$$y \triangleq \begin{bmatrix} z^T & \tilde{\theta}^T \end{bmatrix}^T. \tag{21}$$

Another way to bound V_σ $\forall\sigma\in\mathcal{I}$ is

$$\lambda_1\left\|z\right\|^2\leq V_\sigma\leq\lambda_2\left\|z\right\|^2+v_1, \tag{22}$$

where v_1 is determined as

$$v_1\triangleq\frac{1}{2}\lambda_{\text{max}}\left\{\Gamma^{-1}\right\}\left\|\tilde{\theta}\right\|_{\text{max}}^2, \tag{23}$$

where $\left\|\tilde{\theta}(t)\right\|\leq\left\|\tilde{\theta}\right\|_{max}, \forall t\geq t_0$ for some $\left\|\tilde{\theta}\right\|_{max}\in\mathbb{R}_{>0}$ because of the proj $(\cdot)$ operator in (16) and the bounds for θ in (3).

Theorem 1. *For the uncertain, nonsmooth, and nonlinear Euler-Lagrange dynamic model described in (1) with all the assumptions discussed in Assumptions 1-7, the control input discussed in (14) and the update law discussed in (16) ensure semi-global exponential tracking to an ultimate bound in the sense that*

$$\begin{aligned}\left\|y(t)\right\|^2 \quad\leq\quad & \tfrac{\lambda_2}{\lambda_1}\left\|y(t_0)\right\|^2\exp\left(-\delta\left(t-t_0\right)\right)\\ &+\tfrac{v}{\lambda_1\delta}\left(1-\exp\left(-\delta\left(t-t_0\right)\right)\right),\end{aligned} \tag{24}$$

$\forall t\in[t_0,\infty)$*, where* $\delta\triangleq\frac{\delta_1}{2\lambda_2}$ *and* $v\triangleq\delta v_1+\frac{k_1 n}{2\varepsilon_2}$*, provided that* $y(t_0)\in\mathcal{S}_\mathcal{D}$ *and the following conditions are satisfied*

$$\frac{1}{2}\alpha_2k_1\left(c_m+c_M\right)>\Phi+\sqrt{n}c_{\tilde{M}}\alpha_2k_1, \tag{25}$$

$$k_3>\frac{\alpha_2\beta_1c_{\tilde{M}}\varepsilon_1}{2},\quad \beta_1>\frac{\alpha_2\beta_1c_{\tilde{M}}}{2\varepsilon_1}+\frac{k_1\varepsilon_2}{2}, \tag{26}$$

$$\sqrt{\frac{v}{\delta\lambda_1}}<\gamma. \tag{27}$$

Proof: To facilitate the proof, the times when σ switches to and from $\sigma=q\in\mathcal{I}$ are denoted by $\{t_i^{\text{on},q}\},\left\{t_i^{\text{off},q}\right\}$, $i\in\{0,1,2,...\}$, respectively, and the times when ϕ switches to and from $\phi=p\in\mathcal{B}$ are denoted by $\left\{t_j^{p,\text{on}}\right\},\left\{t_j^{p,\text{off}}\right\}$, $j\in\{0,1,2,...\}$, respectively. Consider the case when $\sigma=q$ for an arbitrary $q\in\mathcal{I}$ (i.e., $t\in\left[t_i^{\text{on},q},t_i^{\text{off},q}\right)$ for some , $i\in\{0,1,2,...\}$) such that the terms described by σ are continuous and differentiable. Note that the times in $t\in\left[t_i^{\text{on},q},t_i^{\text{off},q}\right)$ correspond to V_q from (18).

For $t\in\left[t_i^{\text{on},q},t_i^{\text{off},q}\right),\forall i$, let $y(t)$ be a Filippov solution to the differential inclusion $\dot{y}\in K[h](y)$, where $h:\mathbb{R}^{3n+q}\to\mathbb{R}^{3n+q}$ is defined as $h(y)\triangleq\left[\begin{array}{cccc}\dot{e}^T, & \dot{r}^T, & \dot{e}_f^T, & \dot{\tilde{\theta}}^T\end{array}\right]^T$, and K is the differential inclusion operator stated in [18]. Because of the existence of discontinuities in the closed-loop error dynamics in 17, the derivative of V_q with respect to time exists almost everywhere (a.e), for almost all $t\in\left[t_i^{\text{on},q},t_i^{\text{off},q}\right),\forall i$, in the sense that $\dot{V}_q(y)\overset{\text{a.e.}}{\in}\dot{\tilde{V}}_q(y)$, where $\dot{\tilde{V}}_q(y)$ is the generalized derivative of (18) with respect to time along along $\dot{y}=h(y)$, and $\dot{\tilde{V}}_q\subseteq\bigcap_{\xi\in\partial V_q(y)}\xi^T\left[K[h]^T(y),1\right]^T$ (according to [24, Equation 13]). Since V_q is continuously differential with respect to y, $\partial V_q(y)=\nabla V_q(y)$, where ∇ denotes the gradient operator. Finding the generalized derivative of (18) with respect to time, then substituting the adaptive law in (16), the parameter estimation error in (12), the auxiliary error in (5), the closed loop error system in (17), and the auxiliary filtered error in (6) results in

$$\begin{aligned}\dot{\tilde{V}}_q\quad\subseteq\quad & r^T(\chi_q-e-C_qr-K\left[Y\tilde{\theta}\right]+k_2e_f\\ &-M_q\alpha_2k_1K\left[\text{sgn}\left(r\right)\right]-\left(k_3+M_q\alpha_2k_2\right)r\\ &+\tilde{M}_q\alpha_2\beta_1e_f-\tilde{\theta}^T\Gamma^{-1}K\left[\text{Proj}\left(-\Gamma Y^Tr\right)\right]\\ &+e_f^T\left(-k_1K\left[\text{sgn}\left(r\right)\right]-k_2r+\alpha_2e-\beta_1e_f\right)\\ &+e^T\left(r-\alpha_1e-\alpha_2e_f\right)+\tfrac{1}{2}r^T\dot{M}_qr.\end{aligned} \tag{28}$$

Now, for $t\in\left[t_i^{\text{on},q},t_i^{\text{off},q}\right),\forall i$, lets consider the narrower case where $\phi=p$ for an arbitrary $p\in\mathcal{B}$ (i.e., times when the set $\left[t_i^{\text{on},q},t_i^{\text{off},q}\right)\cap\left[t_j^{p,\text{on}},t_j^{p,\text{off}}\right)$ is non-empty for any $i,j\in\{0,1,2,...\}$). During these times, ϕ is constant and thus $K[Y]=Y$. Using (9) to upper bound χ_q, canceling common terms, noting that $-r^TK\left[\text{sgn}\left(r\right)\right]\leq-\left\|r\right\|$ and $\left\|K\left[\text{sgn}\left(r\right)\right]\right\|\leq\sqrt{n}$ (dimension of r is defined by n), using Assumption 1, the estimate for $\hat{M}$ over (15), (15), and the bound for $\left\|\tilde{M}_\sigma\right\|$ under (15), and implementing the gain condition from (25), the Lyapunov function can be simplified and upper bounded as

$$\begin{aligned}\dot{V}_q\quad\overset{\text{a.e.}}{\leq}\quad & -\alpha_1\left\|e\right\|^2-\beta_1\left\|e_f\right\|^2-\left(k_3+c_m\alpha_2k_2\right)\left\|r\right\|^2\\ &+\left\|r\right\|\rho\left(\left\|z\right\|\right)\left\|z\right\|+\alpha_2\beta_1c_{\tilde{M}}\left\|r\right\|\left\|e_f\right\|\\ &+k_1\left\|e_f\right\|\left\|K\left[\text{sgn}\left(r\right)\right]\right\|.\end{aligned} \tag{29}$$

Using Young's Inequality and completing the squares, allows for (29) to be bounded as

$$\begin{aligned}\dot{V}_q\quad\overset{\text{a.e.}}{\leq}\quad & -\alpha_1\left\|e\right\|^2-\left(\beta_1-\tfrac{\alpha_2\beta_1c_{\tilde{M}}}{2\varepsilon_1}-\tfrac{k_1\varepsilon_2}{2}\right)\left\|e_f\right\|^2\\ &-\left(k_3-\tfrac{\alpha_2\beta_1c_{\tilde{M}}\varepsilon_1}{2}\right)\left\|r\right\|^2\\ &+\tfrac{1}{4c_m\alpha_2k_2}\rho^2\left(\left\|z\right\|\right)\left\|z\right\|^2+\tfrac{k_1n}{2\varepsilon_2}.\end{aligned} \tag{30}$$

Using the definition of v and δ in the theorem statement, the gain conditions in (26), the bound in (22), the fact that $\left\|y\right\|\geq\left\|z\right\|$, and the definition of $\mathcal{D}$ below (18), (30) can be further bounded as

$$\dot{V}_q\quad\overset{\text{a.e.}}{\leq}\quad-\delta V_q+v, \tag{31}$$

$\forall t\in\left[t_i^{\text{on},q},t_i^{\text{off},q}\right)\cap\left[t_j^{p,\text{on}},t_j^{p,\text{off}}\right)$ given that $y(t)\in\mathcal{D}$, $\forall t\in\left[t_i^{\text{on},q},t_i^{\text{off},q}\right)\cap\left[t_j^{p,\text{on}},t_j^{p,\text{off}}\right)$. Since the bound for $\dot{V}_q$ in (31) holds for any arbitrary $\phi\in\mathcal{B}$, the bound holds for all $\phi\in\mathcal{B}$. Therefore, the result in (31) holds for $\forall t\in\left[t_i^{\text{on},q},t_i^{\text{off},q}\right)$ given that $y(t)\in\mathcal{D}$, $\forall t\in\left[t_i^{\text{on},q},t_i^{\text{off},q}\right)$. Following a multiple Lyapunov function approach and based on (31) and the fact that M_σ is a continuous function, it can be proven that

$$\begin{aligned}V_q(t)\quad\leq\quad & V_q\left(t_0^{\text{on},q}\right)\exp\left(-\delta\left(t-t_0^{\text{on},q}\right)\right)\\ &+\tfrac{v}{\delta}\left[1-\exp\left(-\delta\left(t-t_0^{\text{on},q}\right)\right)\right],\end{aligned} \tag{32}$$

$\forall t\in\left[t_i^{\text{on},q},t_i^{\text{off},q}\right),\forall i,\forall q\in\mathcal{I}$, provided that $y\in\mathcal{D},\forall t\in[t_0,\infty)$. Note that $t_0^{\text{on},q}$ represents the first time when σ

switches to any $q \in \mathcal{I}$. Using (20) and defining $t_0 \triangleq \min\left\{t_0^{\text{on},1}, \ldots t_0^{\text{on},\sigma_{max}}\right\}$, the results in (24) can be obtained, provided that $y \in \mathcal{D}, \forall t \in [t_0, \infty)$. A sufficient condition for $y \in \mathcal{D}, \forall t \in [t_0, \infty)$ is that $y(t_0) \in S_D$ and the gain condition in (27) is satisfied. Selecting the control gains k_1, k_2, k_3, α_1, α_2, and β_1, appropriately will guarantee that $y \in \mathcal{S_D}$ and that all of the gain conditions in (25)-(27) are met. Moreover, using (10), (21), and (24), along with (4), (5), (12), it can be determined that $x, \dot{e}, \dot{x}, \hat{\theta} \in \mathcal{L}_\infty$. From Assumption 7 and (16), it can be determined that $\hat{g}_\phi^+, \hat{g}_\phi \in \mathcal{L}_\infty$. Given the fact that $x_d(t)$ is bounded along with (14), it can be shown that $u \in \mathcal{L}_\infty$. ∎

V. Results/Discussion

The compensation of a discontinuous or switched system with unstructured disturbances bounded by constants through the design of an ARISE controller (which has an integral SM term in the controller itself) is the major contribution of this work. In contrast, a RISE control also implements an integral SM term in it, but the drawback of RISE controllers is that they can be implemented only in smooth systems. The proposed ARISE controller is implementable for a much wider class of dynamic systems compared to RISE control.

VI. Conclusion

In this study, an ARISE controller was designed for a class of nonsmooth and uncertain Euler-Lagrange nonlinear dynamic systems with an uncertain control effectiveness matrix. A SM term was injected in the error dynamics through a filtered error signal rather than from the controller. In addition, the controller was designed in a way that it has an integral of a SM term in it to address the disturbances of the system that have constant bounds. In addition, an adaptation law was used to develop an estimate for the control effectiveness matrix. To ensure stability, a Lyapunov-based stability analysis was completed, which showed semi-global exponential tracking to an ultimate bound. The focus of future efforts will be to investigate modifications to the stability analysis to remove the ultimate bound from this stability result.

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