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Uncorrelated Packet Loss Model for Networked Control Systems with $\mathcal{H}_\infty$ Design Constraint

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***Abstract*— Networked Controlled Systems (NCS) are control systems that rely on the performance of the communications to ensure a desired Quality-of-Control (QoC). However, the wireless link is imperfect; the packet has an intrinsic latency, and packets can be lost due to its stochastic nature. Although the newest generation of wireless networks (5G and beyond) can provide Ultra-Reliable Low Latency Communications (URLLC) to attempt to remove the problems caused by the wireless network. This methodology is expensive in terms of communications resources, and for NCS, the constant stream of data could be spared since some updates might contain similar information. Although there are multiple methodologies to design a NCS, not many methods attempt to develop the communications system based on reducing the consumption of communications resources. Therefore, this work finds the maximum transmission interval and delay to optimize the maximum peak Age-of-Information (AoI) while getting a model of the Maximum Allowable Packet Loss Probability (MAPLP) for the case that the $\mathcal{H}_\infty$ norm of the control system must be maintained lower than a specified threshold. Finally, the model is validated in the case of platooning using Cooperative Adaptive Cruise Control (CACC), showing a high accuracy compared to the results of the solution of the optimization problem.**

I. INTRODUCTION

The constant evolution of wireless networks has led to improved data rates and transmission reliability of packets [1]. Moreover, it has enabled wireless media as a resource for different applications. For example, NCS use the wireless media to communicate control information through multiple entities and provide flexibility in their setup compared to wired control systems [2]. However, these benefits come at a cost that the wireless link is unreliable, and packets can be delayed due to the wireless network. Then, a possibility to circumvent these difficulties is to increase the complexity of the communications system and use URLLC to have a seamless wireless link, which is a possibility for the newest generations of wireless networks (5G and beyond) [3]. Although it would reduce the uncertainty on the packet reception, it does not imply that the communications resources are being used effectively [4]. Moreover, while there has been some effort to understand the effects of packet losses on NCS and determine the average packet error rate of a communications system, there is not a definitive method to design both together, e.g., packet loss effect on the $\mathcal{H}_\infty$ norm [5] and the packet error rate of Modulation and Coding Schemes (MCSs) for 5G-NR [6] in platooning.

Conversely, the usefulness of most transmitted packets can be measured by their freshness, which the AoI can evaluate. Although a low AoI and the seamless wireless channel ensure the control performance of an NCS [7], the frequent transmission of packets requires a high demand of communications resources in terms of resource blocks in frequency and time alongside the increased consumption of energy from the transceivers [1]. Additionally, the constant transmission of control packets of each entity of the NCS, although small in content size, will occupy bandwidth other entities can not use if packet interference must be avoided. Therefore, the required bandwidth for a seamless communications link for NCSs increases depending on the number of entities limiting its maximum size.

Thus, this work proposes to maximize the average peak AoI to optimize the usage of communications resources depending on the periodic transmission interval, link delay, and packet loss probability while ensuring the stability and the $\mathcal{H}_\infty$ norm of the NCS. Also, it determines the limits of these communications parameters, i.e., it finds the Maximum Allowed Transmission Interval (MATI), Maximum Allowable Delay (MAD), and MAPLP of the NCS. Specifically, proposes a methodology to find a closed-form solution for the MAPLP in terms of the transmission interval and latency of the communications link.

This work is divided into five different sections, section II shows the AoI model, and section III shows the mathematical model of a wireless NCS using a Markov Jump Linear Systems (MJLS) approach. Moreover, section IV depicts a methodology to optimize the usage of communications resources while illustrating the proposed packet loss model. Also, section V demonstrates the application of the model into an NCS for vehicular communications known as CACC, followed by section VI, where the findings are discussed.

II. MATHEMATICAL DESCRIPTION OF AOI

Consider a NCS's communications link in which the transmission interval is periodic and equal to the sampling period of the controller noted as τ_s, and has a delay τ_d. Then, the AoI can provide a metric to calculate the freshness of the transmitted information, where Fig. 1 depicts the evolution of the AoI of different packets.

The AoI of a packet is zero at the time of generation and increases linearly with a slope of one. Since the wireless network produces a delay, the AoI at the reception of the packet is τ_d and will increase until the next packet is received.

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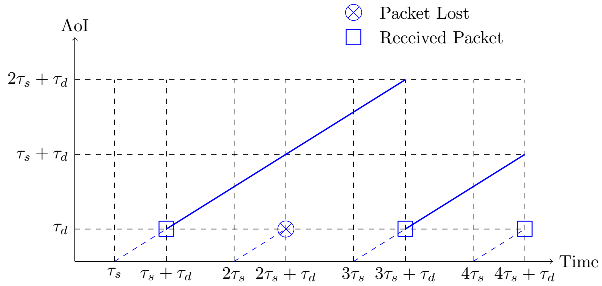

Fig. 1. AoI of packets transmitted through the wireless media. The dashed line represents the time the packet is transmitted through the wireless network. The NCS uses the received information until the next update.

Notice that some packets are lost due to the stochastic nature of the wireless channel. Therefore, the previously received packet's AoI will increase until the next packet is received. If ℓ is defined as the number of sampling periods between received packets, then the packet's peak AoI, i.e., the peak AoI, is given by:

$$AoI_{peak} = \ell\tau_s + \tau_d \tag{1}$$

Furthermore, multiple sources can affect the probability of receiving a packet, such as the distance between the objects, obstacles in the Line-Of-Sight (LOS), and random fluctuations produced by the multiple propagation paths of the signal due to the transmitter (or receiver) movement at a velocity v. The latter can be abstracted as a Markov chain shown in Fig. 2 where the states distinguish between the reception or not of the packet depending on the Signal-to-Noise Ratio (SNR) of the received signal [8].

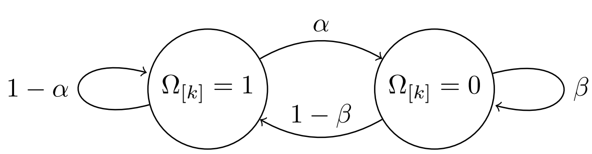

Fig. 2. Markov Chain structure of the wireless link.

Although this type of abstraction was used for symbols, it can be abstracted to packets by assuming that:

- All symbols are correlated and have the same SNR if the packet duration is less than the inverse of the Doppler frequency ($f_d = vf_c/c$, where c is the speed of light, and f_c is the carrier frequency).
- The used bandwidth is less than the inverse of the delay of the propagation paths.

In this case, the random variable $\Omega_{[k]} \in \{0, 1\}$ is used to describe the bad or good condition of the wireless channel at time $k\tau_s$, respectively. Then, the good condition can be mapped to the case where the received SNR is higher than a threshold determined by the communications system, e.g., [6]. Moreover, the transition probability of the Markov chain is defined as:

$$\alpha := P(\Omega_{[k+1]} = 0|\Omega_{[k]} = 1) \tag{2}$$

$$\beta := P(\Omega_{[k+1]} = 0|\Omega_{[k]} = 0) \tag{3}$$

and the transition matrix is as follows:

$$M_c = \begin{bmatrix} 1-\alpha & \alpha \\ 1-\beta & \beta \end{bmatrix} \tag{4}$$

Additionally, the Markov chain is assumed to be ergodic and has a steady state probability distribution $\pi = \{\pi_1 \quad \pi_0\}$ that fulfills the condition $\pi^T M_c = \pi^T$.

Then, the average peak AoI can be calculated by finding the distribution probability of ℓ. Thus, assuming the system just received a packet, the following packet is received with a probability $1-\alpha$. However, if another packet (or multiple) is (are) lost, the probability is $\alpha\beta(1-\beta)^{\ell-2}$ to receive the packets after waiting for $\tau_s\ell$ for $\ell \geq 2$. Thus, the probability distribution is given by:

$$f(\ell) = \begin{cases} 1-\alpha & \text{if} \quad \ell = 1 \\ \alpha\beta^{\ell-2}(1-\beta) & \text{if} \quad \ell \geq 2 \end{cases} \tag{5}$$

Furthermore, the expected value of ℓ, i.e., the expected number of multiples of τ_s that the control system has to wait to receive a packet, is given by:

$$\mathbb{E}[\ell] = \sum_{\ell=1}^{\infty} \ell f(\ell) = 1 - \alpha + \alpha\left(\frac{2-\beta}{1-\beta}\right) \tag{6}$$

Thus, the $\mathbb{E}[AoI_{peak}]$ for the wireless channel abstraction is given by:

$$\mathbb{E}[AoI_{peak}] = \tau_s\left(1 - \alpha + \alpha\left(\frac{2-\beta}{1-\beta}\right)\right) + \bar{\tau}_d \tag{7}$$

where $\bar{\tau}_d$ is the average of the delay. For practical considerations, this paper assumes that $\tau_d = \bar{\tau}_d$ as done previously in [9]. On the other hand, the wireless channel can be assumed to be uncorrelated, i.e., the consecutive packets' SNR is not correlated, thus, making $\alpha = \beta$. In [8], the correlation between samples in a wireless channel is given by $J_0(2\pi f_d\tau_s)$, where $J_0(\cdot)$ is the Bessel function of the first kind of order zero, $f_d = vf_c/c$ is the doppler frequency, f_c is the carrier frequency, and c is the speed of light. Then, the correlation goes to zero if $\tau_s \gg 0$ or $f_d \gg 0$. Moreover, notice that if $\alpha = \beta$, the distribution probability in (5) is the geometric distribution, the equation (6) transforms into $(1-\beta)^{-1}$, and equation (7) reduces to:

$$\mathbb{E}[AoI_{peak}] = \frac{\tau_s}{1-\beta} + \bar{\tau}_d \tag{8}$$

III. NCS MODEL WITH PERIODIC COMMUNICATIONS

The model of a NCS has to account for the stochastic nature of the wireless link. A simple abstraction of such a link is to assume periodic transmissions but with random packet losses. Then, suppose the NCS is considered to be affected by a perturbation $u_r(t)$, and the system is controlled by local $u_s(t)$ and transmitted $u_c(t)$ information. In that case, the NCS can be represented initially as a continuous system given by:

$$\dot{x}(t) = Ax(t) + B_s u_s(t) + B_c u_c(t-\tau_d) + B_r u_r(t) \quad (9)$$
$$y(t) = Cx(t) \quad (10)$$

where the transfer function of the ideal NCS is assumed to be strictly proper. Moreover, consider the discretization of the control and the transmitted information, assuming the same sampling period τ_s. Then, the delay can be decomposed into $\tau_d = \ell\tau_s + m\tau_s$ similar to the method shown in [10], where $\ell = \lfloor \frac{\tau_d}{\tau_s} \rfloor$ and $m \in [0,1)$. Therefore, assuming that $k \in \mathbb{N}_{\geq 0}$ and that the control signals $u_\iota(t)$ ($\iota \in \{s,c,r\}$) are piecewise continuous in the interval $t = [k\tau_s, (k+1)\tau_s)$, the discretization of the system, is given by:

$$\begin{aligned} x[k+1] = e^{A\tau_s}x[k] + \Sigma_{0,s}u_s[k] + \Omega_{[k]}\Sigma_{2,c}u_c[k-\ell] \\ + \Omega_{[k-1]}\Sigma_{1,c}u_c[k-\ell-1] + \Sigma_{0,r}u_r[k] \end{aligned} \quad (11)$$
$$y[k] = Cx[k] \quad (12)$$

where $x(k\tau_s) = x[k]$, and the Σ terms are given by:

$$\Sigma_{0,\iota} = \int_0^{\tau_s} e^{A\tau}B_\iota d\tau, \quad \iota \in \{s,r\} \quad (13)$$
$$\Sigma_{1,c} = \int_{(1-m)\tau_s}^{\tau_s} e^{A\tau}B_c d\tau \quad (14)$$
$$\Sigma_{2,c} = \int_0^{(1-m)\tau_s} e^{A\tau}B_c d\tau \quad (15)$$

However, equation (11) includes the dependency of the reception of the packets at k and $k-1$, which can be removed by including a buffer variable $u_{bf}[k-1]$ resulting in:

$$\begin{bmatrix} x[k+1] \\ u_{bf}[k] \end{bmatrix} = \begin{bmatrix} e^{A\tau_s} & \Sigma_{1,c} + (1-\Omega_{[k]})\Sigma_{2,c} \\ 0 & (1-\Omega_{[k]}) \end{bmatrix} \underbrace{\begin{bmatrix} x[k] \\ u_{bf}[k-1] \end{bmatrix}}_{\psi[k]}$$
$$+ \begin{bmatrix} \Sigma_{0,s} \\ 0 \end{bmatrix} u_s[k] + \begin{bmatrix} \Sigma_{0,r} \\ 0 \end{bmatrix} u_r[k] + \Omega_{[k]} \begin{bmatrix} \Sigma_{2,c} \\ 1 \end{bmatrix} u_c[k-\ell] \quad (16)$$
$$y[k] = Cx[k] \quad (17)$$

Moreover, the buffer can be used again if the newest update is not received. On the other hand, these equations, together with the Markov chain of Fig. 2, give rise to a MJLS [9], [11], where the combination of $\psi[k]$ and $\Omega_{[k]}$ is a random variable. Furthermore, assume that $u_s[k] = Kx[k]$ corresponds to a feedback gain and that $u_c[k] = u_r[k]$ can be treated as a feedforward term, and can be easily replaced in equation (16), if $\ell = 0$, into:

$$\psi[k+1] = \underbrace{\begin{bmatrix} e^{A\tau_s} + \Sigma_{0,s}K & \Sigma_{1,c} + (1-\Omega[k])\Sigma_{2,c} \\ 0 & (1-\Omega[k]) \end{bmatrix}}_{A^{(0)}_{\Omega[k]}} \psi[k]$$
$$+ \underbrace{\begin{bmatrix} \Sigma_{0,r} + \Omega[k]\Sigma_{2,c} \\ \Omega[k] \end{bmatrix}}_{B^{(0)}_{\Omega[k]}} u_r[k] \quad (18)$$
$$y[k] = \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{C^{(0)}} \psi^{(0)}[k] \quad (19)$$

where the superscript (0) in (18)-(19) is used to distinguish the extended state space for the case when $\ell = 0$. However, for $\ell > 0$, it is necessary to remove the dependency on the input $u_r[k-\ell]$. This state space can be extended to depend only on $u_r[k]$ as shown in (20). Although the input term $B^{(\ell)}_{\Omega_{[k]}}$ does not have a dependency on $\Omega_{[k]}$ in (20), the notation is kept to generalize the NCS's analysis on the stability and the $\mathcal{H}_\infty$ norm. Moreover, notice that the dependency on τ_s and τ_d is not explicitly noted in the matrices $A^{(\ell)}_{\Omega_{[k]}}$ and $B^{(\ell)}_{\Omega_{[k]}}$ to ease the notation. Finally, to condense both systems (equations (18)-(19) for $\ell = 0$, and equations (20)-(21) for $\ell > 0$), this work uses $\mathcal{G}(\tau_s, \tau_d, \beta)$ to represent the state space system together with the Markov chain probabilities given in (4), and the definition of the $\mathcal{H}_\infty$ norm is given by:

$$|\mathcal{G}|_\infty := \sup_{u_r[k]\neq 0} \frac{\left|\sum_k y^2[k]\right|^{\frac{1}{2}}}{\left|\sum_k u_r^2[k]\right|^{\frac{1}{2}}} \quad (22)$$

IV. COMMUNICATIONS REQUIREMENTS

Before analyzing the stability of the MJLS, it is crucial to understand the basic communications requirements of the ideal communications channel, i.e., $\mathcal{G}(\tau_s, \tau_d, 0)$, where calculating stability and the $\mathcal{H}_\infty$ norm is already well studied [12]. For this case, the communications system has to assign resources in frequency and time while ensuring these resources will be free for the NCS to use. Then, the scheduler has to reserve the spectrum depending on τ_s while τ_d is determined by the technology and configuration used by the wireless network. Although the scheduler could assign

$$\begin{bmatrix} x[k+1] \\ u_{bf}[k] \\ u_r[k-\ell+1] \\ \vdots \\ u_r[k-1] \\ u_r[k] \end{bmatrix} = \underbrace{\begin{bmatrix} e^{A\tau_s} + \Sigma_{0,s}K & \Sigma_{1,c} + (1-\Omega_{[k]})\Sigma_{2,c} & \Omega_{[k]}\Sigma_{2,c} & 0 & \cdots & 0 \\ 0 & (1-\Omega_{[k]}) & \Omega_{[k]} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A^{(\ell)}_{\Omega_{[k]}}} \underbrace{\begin{bmatrix} x[k] \\ u_{bf}[k-1] \\ u_r[k-\ell] \\ \vdots \\ u_r[k-2] \\ u_r[k-1] \end{bmatrix}}_{\psi^{(\ell)}[k]} + \underbrace{\begin{bmatrix} \Sigma_{0,r} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{B^{(\ell)}_{\Omega_{[k]}}} u_r[k] \quad (20)$$

$$y[k] = \underbrace{\begin{bmatrix} C & 0 & \ldots & 0 \end{bmatrix}}_{C^{(\ell)}} \psi^{(\ell)}[k] \quad (21)$$

resources to the wireless link to become almost indistinguishable from a wired link using URLLC, it implies a high consumption of communications resources regarding bandwidth and number of transmitted packets [3]. Moreover, using low latency in the wireless link implies that some packets will carry similar information that does not impact the control system's performance and could have been spared. Then, to impose a design constraint, this work assumes that the communications systems goal is to ensure that $\mathcal{G}(\tau_s, \tau_d, \beta)$ is stable, and $|\mathcal{G}(\tau_s, \tau_d, \beta)|_\infty \leq \gamma_t$, where it is assumed already that $\mathcal{G}(0,0,0)$ already fulfills both conditions and $\mathcal{G}(0,0,1)$ is stable but $|\mathcal{G}(0,0,1)|_\infty > \gamma_t$ and γ_t is a $\mathcal{H}_\infty$ norm constraint. Therefore, it is natural to ask if there is an optimal configuration to allocate the communications resources of an NCS system that still ensures the control performance of the system. In this manner, the communications bounds can be separated into two optimization problems:

$$\begin{aligned} \text{MATI} := \max \quad & \tau_s \\ \text{s.t.} \quad & |\mathcal{G}(\tau_x, 0, 0)|_\infty \leq \gamma_t \quad \forall \tau_x \in [0, \tau_s] \\ & |\lambda_i(A_1^{(0)})| < 1, \forall i \end{aligned} \tag{23}$$

$$\begin{aligned} \text{MAD} := \max \quad & \tau_d \\ \text{s.t.} \quad & |\mathcal{G}(0, \tau_y, \beta_d)|_\infty \leq \gamma_t \quad \forall \tau_y \in [0, \tau_d] \\ & \text{Re}\{\lambda_i(A)\} < 0, \forall i \end{aligned} \tag{24}$$

where $\lambda(\cdot)$ is the eigenvalue operator. Although the optimal communications parameters would be the Pareto optimal of the combination of (23) and (24) resulting in a set of (τ_s, τ_d) pairs, the MATI and MAD are used to name the bounds independently with the aim of finding a function that connects both and delimits the feasible region of transmission intervals and latency [13]. Furthermore, this condition intentionally allows the communications system to use a worse parametrization than the region delimited by MATI and MAD from a practical point of view. Additionally, since the case of the MAD considers that $\tau_s = 0$, the control and communications systems are represented as a continuous system where the communications is a pure delay. However, the packet loss β_d lacks meaning since a new sample is produced in an infinitely short time. Therefore, this work assumes that if $\tau_s = 0$, $\beta_d \in [0, 1]$ without affecting the result of the system and (9)-(10) can be used under similar assumptions instead.

Previous work has shown methodologies to find both parameters, in [14] uses the combination of a Lyapunov function and LMI constraint, and in [13] a Pade approximation of the Zero-Order Hold (ZOH) in the communications link was used to find the optimal communications requirements for the platooning use case. However, these two optimization problems can be reduced to only one with the following Lemma.

Lemma 1: Assuming that the communications and the control system have the same sampling period, the ideal wireless link ($\beta = 0$) can be assumed to be a ZOH with sampling $\tau_s > 0$ and a delay $\tau_d > 0$. Then, the pair (τ_s, τ_d) that belong to:

$$\frac{\text{MATI}}{2} = \frac{\tau_s}{2} + \tau_d \tag{25}$$

the communications requirements bound if the sampling bandwidth $\omega_{\text{bw}} = 2\pi/\tau_s$ is sufficiently large compared to the control system's bandwidth.

Proof: Recall that for the solution of MATI, the system fulfills both control performance conditions and belongs to (25) trivially ($\tau_s = \text{MATI}$). On the other hand, the communications link transfer function can be assumed to be a ZOH with a delay given by:

$$R(s) = \text{ZOH}(s)D(s) = (1 - e^{-s\tau_s})e^{-s\tau_d}/(s\tau_s) \tag{26}$$

Then, replacing $s = j\omega$, and the sampling bandwidth $\omega_{\text{bw}} = 2\pi/\tau_s$, the transfer function can be modified to:

$$\begin{aligned} R(j\omega) &= e^{-j\omega\left(\frac{\tau_s}{2}+\tau_d\right)} \left(e^{j\frac{\omega\tau_s}{2}} - e^{-j\frac{\omega\tau_s}{2}}\right) / (j\omega\tau_s) \\ &= e^{-j\omega\left(\frac{\tau_s}{2}+\tau_d\right)} \text{sinc}\left(\frac{\omega\tau_s}{2}\right) \\ &= e^{-j\omega\left(\frac{\text{MATI}}{2}\right)} \text{sinc}\left(\frac{\omega\pi}{\omega_{\text{bw}}}\right) \end{aligned} \tag{27}$$

From the assumption that ω_{bw} is larger than the control system's bandwidth, sinc$(\cdot)$ is close to 1. Thus, regardless of the pair (τ_s, τ_d) that fulfills (25), the communications link is invariant and produces the same delay to the system, which ensures the same $\mathcal{H}_\infty$ than $\tau_s = \text{MATI}$. ■

Therefore, using Lemma 1 gives that MAD = MATI/2 when $\tau_s = 0$. Furthermore, to find the MATI, it is possible to use the method in [13] as a first estimate and then correct it to the value that achieves the maximum given the numerical resolution. Another alternative is to use a bisection algorithm and find the MATI if $|\mathcal{G}(\tau_s, 0, 0)|_\infty$ is a monotonically increasing function.

A. Stability and the H_∞ norm of MJLS

The MJLS framework can produce different metrics to understand the behavior of the stochastic state space. It is possible to determine $\mathbb{E}\{y[k]\}$ using [15]:

$$\mathbb{E}\{\psi^{(\ell)}[k+1]\} = E\mathbb{E}\{\psi^{(\ell)}[k]\} + Hu_r[k] \tag{28}$$

$$\mathbb{E}\{y[k]\} = F\mathbb{E}\{\psi^{(\ell)}[k]\} \tag{29}$$

where

$$E = \left(M_c' \otimes I_{A_{\Omega_{[k]}}^{(\ell)}}\right) \text{diag}[A_{\Omega_{[k]}}^{(\ell)}] \tag{30}$$

$$H = \left(M_c' \otimes I_{A_{\Omega_{[k]}}^{(\ell)}}\right) \text{diag}[B_{\Omega_{[k]}}^{(\ell)}]\pi \tag{31}$$

$$F = [C^{(\ell)} \quad C^{(\ell)}] \tag{32}$$

$$\text{diag}[J_{\Omega_{[k]}}] = \begin{bmatrix} J_1 & 0 \\ 0 & J_0 \end{bmatrix}, J = \{A^{(\ell)}, B^{(\ell)}\} \tag{33}$$

and $I_{A_{\Omega_{[k]}}^{(\ell)}}$ is an identity matrix of size of $A_{\Omega_{[k]}}^{(\ell)}$. Then, we can assume a mean transfer function given by:

$$H_{\mathcal{G}(\tau_s,\tau_d,\beta)}(z) = \frac{\mathbb{E}\{y[z]\}}{u_r[z]} = F(zI_E - E)^{-1}H \tag{34}$$

where z is the variable of the $\mathcal{Z}$ transform in (28)-(29).

The stability of the MJLS is essential to understand the effects of the wireless network on the system's stability. Moreover, it could be intuitive to ask if (34) is stable, i.e., if $|\lambda(E)| < 1$. However, as shown in [9], [11], [15], this hypothesis would lead to a wrong interpretation of the system's performance since multiple realizations of the system could diverge, but the average converges to a finite value. Therefore, the variance becomes ideal for determining if all trajectories converge. Thus, if we assume that $u_r[k] = 0$, the evolution of the variance of the MJLS is given by:

$$\zeta[k+1] = \underbrace{(M_c^T \otimes I_{\text{diag}[A_1^{(\ell)}, A_0^{(\ell)}]})\text{diag}[(A_1^{(\ell)})^2, (A_0^{(\ell)})^2]}_{\bar{M}}\zeta[k] \quad (35)$$

$$\zeta[k] = \begin{bmatrix} \mathbb{E}\{\psi[k] \otimes \psi[k] 1\{\Omega[k] = 1\}\} \\ \mathbb{E}\{\psi[k] \otimes \psi[k] 1\{\Omega[k] = 0\}\} \end{bmatrix} \quad (36)$$

$$\left(A_i^{(\ell)}\right)^2 = A_i^{(\ell)} \otimes A_i^{(\ell)} \quad (37)$$

where $1\{\Omega_k\}$ indicates that the system is in the good or bad state of the Markov chain, correspondingly, and $\otimes$ is the Kronecker product. Moreover, if we assume that $\sum_k |u_r[k]|^2 < \infty$, there is a sample k from where all the variance samples of the system are not affected by $u_r[k]$ and equation (35) is valid. Then, the variance of the system will decrease, and the system is Mean Square Stable (MSS) if (see further details in [9], [11]):

$$r(\bar{M}) < 1 \quad (38)$$

where $r(\cdot)$ is the spectral radius operator and ensures that the absolute value of all eigenvalues is less than one, and $\bar{M}$ is the operator of the variance shown in (35). On the other hand, the performance of the system can also be evaluated by analyzing the maximum possible gain that the output $\mathbb{E}\{y[k]\}$ has with respect to $u_r[k]$. Then, assuming that the system in equation (28)-(29) is represented by $\bar{\mathcal{G}}(\tau_s, \tau_d, \beta)$ and is MSS, γ is an upper bound of the $\mathcal{H}_\infty$ norm defined by using an Linear Matrix Inequality (LMI) given by [5]:

$$\begin{aligned} L_{\bar{\mathcal{G}}}(\beta,\gamma) = &\sum_{i=0}^{1} \pi_i \begin{bmatrix} A_i^{(\ell)} & B_i^{(\ell)} \\ C^{(\ell)} & 0 \end{bmatrix}^T \begin{bmatrix} G & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_i^{(\ell)} & B_i^{(\ell)} \\ C^{(\ell)} & 0 \end{bmatrix} \\ &- \begin{bmatrix} G & 0 \\ 0 & \gamma^2 + \epsilon_1^2 \end{bmatrix} \le 0 \end{aligned} \quad (39)$$

where G is a Lyapunov function and ϵ_1 is a variable to make the LMI (39) semidefinite negative instead of definite negative and provide some slack for the numerical solution. Then, the closest value to the $\mathcal{H}_\infty$ norm of $\bar{\mathcal{G}}$ can be found by:

$$\begin{aligned} \gamma_{\bar{\mathcal{G}}(\tau_s,\tau_d)}(\beta) := \min \quad & \gamma \\ \text{s.t.} \quad & L_{\mathcal{G}}(\beta,\gamma) \le 0 \\ & r(\bar{M}(\beta)) < 1 \end{aligned} \quad (40)$$

Moreover, since the system has a design specification on $|\bar{\mathcal{G}}|_\infty \le \gamma_t$, we can define a β_{max} referred as to the MAPLP which requires requires that $\gamma_{\mathcal{G}}(\beta) \le \gamma_t, \forall \beta \in [0, \beta_{max}]$. Then, the optimization problem can be defined as:

$$\begin{aligned} \beta_{max}(\tau_s,\tau_d) := \max \quad & \beta \\ \text{s.t.} \quad & \gamma_{\bar{\mathcal{G}}(\tau_s,\tau_d)}(\beta_x) \le \gamma_t \\ & \forall \beta_x \in [0, \beta] \end{aligned} \quad (41)$$

However, this approach will give a conservative lower bound since it depends on the selection of the Lyapunov function. On the other hand, if the condition of (41) is replaced by the maximum gain of the mean transfer function (equation (34)), the new optimization problem is defined as:

$$\begin{aligned} \bar{\beta}_{max}(\tau_s,\tau_d) := \max \quad & \beta \\ \text{s.t.} \quad & \left|H_{\mathcal{G}(\tau_s,\tau_d,\beta_x)}\right|_\infty \le \gamma_t + \epsilon_2 \\ & r(\bar{M}(\beta)) \le 1 \\ & \forall \beta_x \in [0, \beta] \end{aligned} \quad (42)$$

Moreover, both problems give a different idea of the system's behavior. The LMI approach evaluates the gain concerning $\mathbb{E}\{y^2[k]\}$. In contrast, the transfer function approach uses $\mathbb{E}\{y[k]\}^2$. The solution of both optimization problems gives an intuition on the packet loss requirements that the communications system has to ensure (lower and upper bound) and determine its design complexity. Moreover, the selection of ϵ_1 and ϵ_2 will determine how strict is the solution of the optimization problem. If the value of $|\mathcal{G}(0,0,0)|_\infty \approx \gamma_t$, it will give some slack to numerical fluctuations from the numerical algorithms that solve either of the optimization problems.

B. Packet loss model

Although the previous section proposes methodologies to find the MAPLP, it does not provide an analytical solution to the problem. The idea is to combine the constraints of the communications resources with the $\mathbb{E}\{AoI_{peak}\}$ to find a model of $\bar{\beta}_{\max}(\tau_s, \tau_d)$. This selection is made since it does not depend on the choice of a Lyapunov function at the cost of being an upper bound.

Therefore, combining the models of the previous section, the objective is to use the bounds of the communications resources to find the configuration of (τ_s, τ_d) that maximizes:

$$\mathbb{E}\{AoI_{peak}\} = \frac{\tau_s}{1 - \bar{\beta}_{max}(\tau_s, \tau_d)} + \tau_d \quad (43)$$

assuming that:

Assumption 1: The function AoI_{peak} increases linearly in the direction of τ_s and τ_d with constant c_s and c_d, accordingly.

Then, using Assumption 1 and that $\tau_d = 0$, a first packet loss model can be generated, shown in the following Lemma:

Lemma 2: The packet loss model without delay is given by:

$$\bar{\beta}_{\max}(\tau_s, 0) = \Gamma \frac{\tau_s - \text{MATI}}{\tau_s - \Gamma\text{MATI}} \quad (44)$$

where $\Gamma = \left(1 - \frac{1}{c_s}\right)$ for $c_s \neq 1$.

Proof: Applying the partial derivative of equation (43) with respect to τ_s with $\tau_d = 0$, gives the following differential equation:

$$\frac{\partial \bar{\beta}_{\max}(\tau_s, 0)}{\partial \tau_s} = \frac{c_s(1-\bar{\beta}_{\max}(\tau_s,0))^2 - (1-\bar{\beta}_{\max}(\tau_s,0))}{\tau_s} \tag{45}$$

which is solved by:

$$\bar{\beta}_{\max}(\tau_s, 0) = \frac{\Gamma\tau_s - d}{\tau_s - d} \tag{46}$$

where d is an integration constant. Then, replacing on the condition from the (23), $\bar{\beta}_{\max}(\text{MATI}, 0) = 0$ gives: $d = \Gamma\text{MATI}$. Finally, this condition can be replaced in (46) to obtain (44). Moreover, notice that if $c_s = 1$, it results in $d = 0$ and $\bar{\beta}_{\max}(\tau_s, 0) = 0, \forall \tau_s$ which is an undesired behavior for the model. ■

Lemma 3: The packet loss model using the second condition of assumption 1 is:

$$\bar{\beta}_{\max}(\tau_s, \tau_d) = \frac{(c_d - 1)\tau_d + \frac{\Gamma}{1-\Gamma}\tau_s - \frac{\Gamma}{1-\Gamma}\text{MATI}}{(c_d-1)\tau_d + \frac{\tau_s}{1-\Gamma} - \frac{\Gamma}{1-\Gamma}\text{MATI}} \tag{47}$$

if Lemma 2 is fulfilled.

Proof: Using Lemma 2 into (43) gives:

$$AoI_{peak}(\tau_s, 0) = \frac{\tau_s - \Gamma\text{MATI}}{1-\Gamma} \tag{48}$$

Then, using assumption 1, c_d has to be given by:

$$c_d = \frac{AoI_{peak}(\tau_s, \tau_d) - AoI_{peak}(\tau_s, 0)}{\tau_d} \tag{49}$$

which can be reorganized into equation (47) by using (43), and (48). ■

Finally, using Lemma 2 and 3, and replacing $\Gamma = (c_s - 1)/c_s$, the average peak of information of the NCS is given by:

$$\mathbb{E}\{AoI_{peak}\}(\tau_s, \tau_d) = c_d\tau_d + c_s\tau_s - (c_s - 1)\text{MATI} \tag{50}$$

Then, the maximization of AoI_{peak} will depend on c_s, c_d and the MATI from the NCS system. Moreover, finding the optimal parameters would guide the design of the communications system, e.g., determining the modulation and coding scheme, and transmission power, among others.

V. Packet Loss model for CACC

An application that needs the use of wireless media is platooning. In this use case, a platoon of vehicles attempts to maintain a desired intervehicle distance at a constant speed while ensuring that if there is a change in the acceleration from the leader, the platoon will not present increasing oscillations throughout the platoon. Moreover, it is possible to assume that the wireless channel is fairly uncorrelated due to the continuous fast movement of vehicles on a highway.

A simple model of the platooning system is to use a Proportional-Derivative (PD) controller that follows a velocity-dependent spacing policy and a feedforward system that receives the control information to anticipate the vehicle's movement as shown in [13], [15], [16]. These specifications can be written as follows:

$$\dot{e}_i(t) = v_{i-1}(t) - v_i(t) - \tau_h a_i(t) \tag{51}$$

$$\dot{a}_i(t) = -\frac{a_i(t)}{\eta} + \frac{u_i(t)}{\eta} \tag{52}$$

$$\dot{u}_{ff,i}(t) = -\frac{a_i(t)}{\tau_h} + \Omega\frac{u_{i-1}(t-\tau_d)}{\tau_h} \tag{53}$$

$$u_i(t) = u_{ff,i}(t) + k_p e_i(t) + k_d\dot{e}_i(t) \tag{54}$$

and the control performance condition is to ensure the string stability of the platoon, i.e., no amplified oscillations, given by:

$$\left|\frac{u_i(t)}{u_{i-1}(t)}\right|_\infty \leq \gamma_t = 1 \tag{55}$$

where η is the time constant of the vehicle, τ_h is the headway time, k_p is the proportional constant, k_d is the derivative constant, i is the index of the vehicle in the platoon, $e_i(t)$ is the intervehicle distance error, $v_i(t)$ is the velocity, $a_i(t)$ is the acceleration, $u_{ff,i}(t)$ is the feedforward filter output and $i = 0$ is the platoon leader. In this work, we are interested in the communications between adjacent vehicles and their effects on the control performance. Thus, the system can be written in a state space $x(t) = [v_{i-1}(t), a_{i-1}(t), e_i(t), v_i(t), a_i(t), u_{ff,i}(t)]^T$ as:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\eta} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -\tau_h & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\eta} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\tau_h} \end{bmatrix} \quad B_s = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\eta} \\ 0 \end{bmatrix} \tag{56}$$

$$B_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\tau_h} \end{bmatrix} \quad B_r = \begin{bmatrix} 0 \\ \frac{1}{\eta} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{57}$$

$$C = K = \begin{bmatrix} k_d & 0 & k_p & -k_d & -k_d\tau_h & 1 \end{bmatrix} \tag{58}$$

Then, using equations (56)-(58) into the discretized model (16)-(17), the MAPLP from both methods (41)-(42) can be evaluated. Moreover, a bisection algorithm is used to find the MATI from (23) for different τ_s. This methodology returns similar results for the CACC case compared to the methodology used in [13]. Moreover, the MAPLP provides a methodology to verify the calculated MATI since it either gives $\bar{\beta}_{max}(\tau_s, 0) \to 0$ or $\bar{\beta}_{\max}(\tau_s, 0) = \emptyset$, then, the closer $bar\beta_{max}$ gets to zero the better the estimate. Additionally, Table I shows the parameters used in the simulations.

Thus, Fig. 3 depicts the solutions of the MAPLP for $\tau_d = 0$ obtained from (41) and (42) for two different headway times, and the τ_s is normalized to compare both approaches. It is possible to see that $\bar{\beta}_{max}$ gives a more optimistic value of the MAPLP and β_{max} is a lower bound, but both values get close together as $\tau_s \to$ MATI. Moreover, Fig. 3 shows that the MAPLP is independent of the headway

Parameter	Value	Unit
η	0.1	$[s]$
k_p	0.5	$[-]$
k_d	0.25	$[-]$
MATI $t_h = 0.2$ [s]	20	[ms]
MATI $t_h = 0.3$ [s]	48	[ms]
ϵ_1	1e-4	[-]
ϵ_2	1e-4	[-]

TABLE I
SIMULATION PARAMETERS

time if normalized by the MATI, which seems to compress the information of the control system.

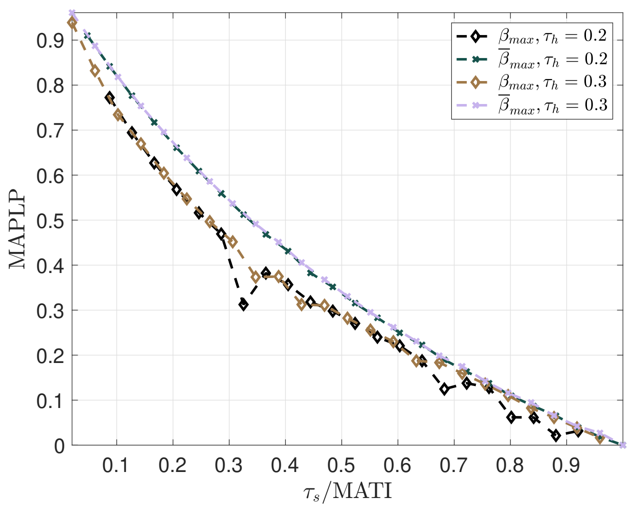

Fig. 3. Comparison of MAPLP for the optimization problems using the Lyapunov condition (diamond marker) and the Mean transfer function (cross marker) using different headway times and $\tau_d = 0$.

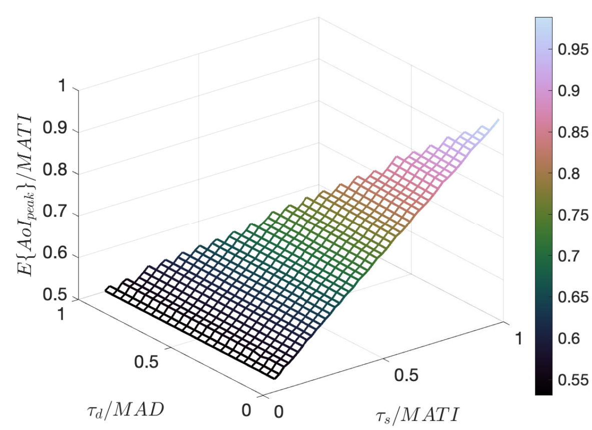

Fig. 4. Mean peak AoI using $\tau_h = 0.2$ and normalizing τ_s with MATI, τ_d with MAD, and mean peak AoI with MATI.

On the other hand, the average peak AoI can be calculated to obtain the constants for assumption 1. Fig. 4 shows that $c_s = 1/2$ and $c_d = 0$. Therefore, $\bar{\beta}_{max}$ has to change at a specific rate to compensate for the changes on τ_d when τ_s is left constant. Moreover, the maximum $\mathbb{E}\{AoI_{peak}\} = \text{MATI}$ while the minimum is MATI/2. The minimum illustrates the effect of including the buffer and using it to compensate for the packet losses, where the stored information has a non-zero age and can be correlated to the worst performance seen in [15] when the system does not use the buffer. Thus, using Lemma 2 and 3 the MAPLP and the average AoI_{peak} can be written as:

$$\bar{\beta}_{model}(\tau_s, \tau_d) = \frac{1 - \left(\frac{\tau_s}{\text{MATI}} + \frac{\tau_d}{\text{MAD}}\right)}{1 + \left(\frac{\tau_s}{\text{MATI}} - \frac{\tau_d}{\text{MAD}}\right)} \tag{59}$$

$$\frac{\mathbb{E}\{AoI_{peak}\}_{model}(\tau_s)}{\text{MATI}} = \frac{\tau_s}{\text{MATI}} + \frac{1}{2} \tag{60}$$

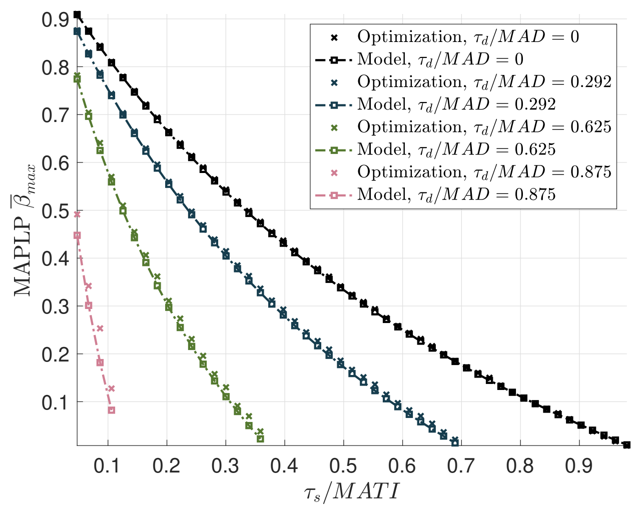

Fig. 5. Comparison between the solution of the optimization problem (42) and the model (59) for different values of τ_d/MAD and $\tau_h = 0.2$.

The correctness of the MAPLP model can be verified in Fig. 5 and 6 for different values of the delay. Although Fig. 5 does not include the values of $\tau_h = 0.3$, the results are similar to the depicted ones. Moreover, the value of $|\bar{\beta}_{max} - \bar{\beta}_{model}|^2$ is shown in Fig. 6, where the model is quite similar for $\tau_d \to 0$, but becomes worse as $\tau_d \to \text{MAD}$ and $\tau_s \to 0$. However, as discussed previously, this is the case in which the model becomes unrealistic.

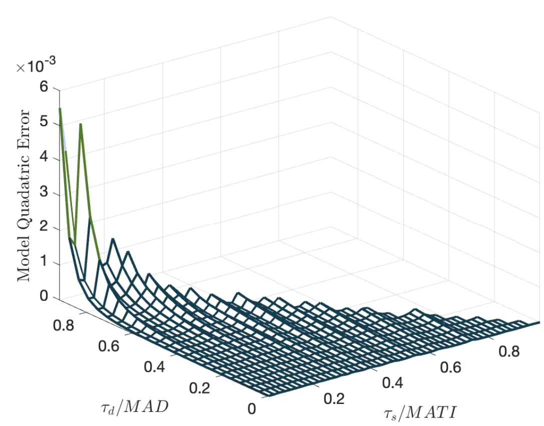

Fig. 6. Model quadratic error between the obtained model (59) and the solution of the optimization problem (42).

Then, the probability model obtained in (59) could be combined with the packet reception rate shown in [6] to determine the communications systems that guarantee the string stability of the platoon.

VI. Conclusions and discussion

This work illustrates a methodology to find a function to represent the MAPLP using a MJLS state space and based on the optimal communications requirements such MATI, MAD and peak AoI. Furthermore, it showed that the error between the MAPLP obtained by the proposed theoretical model and the solution of the optimization problem using the mean transfer function is low, a maximum of 0.006 for the platooning case shown, which could be used for a more efficient design of communications systems for NCS with $\mathcal{H}_\infty$ constraint. Also, it showed that using the Lyapunov function gives a lower bound to the MAPLP and that the result is similar to the mean transfer function for sampling values close to the MATI. Thus, the closed-form solution could be used for more general inputs if $\tau_s \rightarrow$ MATI.

Moreover, this work shows that the MATI is a more general concept than just an optimal transmission interval since it directly affects both MAD and peak AoI. Besides, it abstracts the parameters of the control system such that normalizing τ_s by MATI results in the same MAPLP for different τ_h, at least for the periodic case with uncorrelated packet losses.

Additionally, the optimal communications resources for the communications system in a NCS must be determined by a Communications-Control Co-Design (CoCoCo) approach. For example, the selection of different parameters of the wireless network, such as data rates, edge computing nodes, and network size, among others, could determine the latency of the wireless link imposing a limit on the transmission interval that has to be lower than the MATI. On the other hand, the selection of the transmission power or the MCS will determine a packet loss probability that must be lower than the MAPLP to ensure the control requirements of the NCS. Also, for low latencies, the Medium Access Control (MAC) layer can assign higher transmission intervals to minimize the packets needed to be transmitted but require high-reliability mechanisms to ensure its reception. Therefore, the solution of a joint design optimization problem that incorporates all of these parameters must be proposed to fully enable a CoCoCo approach, ensuring the control performance and efficient usage of communication resources. The packet loss model obtained in this work provides a first step to efficiently relate control and communications parameters without solving multiple optimization problems and requiring only the MATI.

Although the approach used in this paper is general, it remains a question if it works for any control system or if there are some conditions on how the $\mathcal{H}_\infty$ norm is affected by the sampling of the system, e.g., is $\mathcal{H}_\infty$ a monotonic increasing function with respect to τ_s? or how will the difference between the controller's sampling period and transmission interval affect the $\mathcal{H}_\infty$ condition? Moreover, the control system could receive information from multiple sources, requiring the extension of the number of the Markov chain states, and different packet loss probabilities, which could lead to the extension of the $\mathcal{H}_\infty$ constraint to a Multiple-Input Multiple-Output (MIMO) case. The study of these questions will provide a better insight for a better CoCoCo.

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