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Finite-gain $\mathcal{L}_1$ interval impulsive observer design under denial-of-service attacks

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***Abstract*— The design of a robust observer under denial-of-service attacks is addressed for linear time invariant systems in the bounded-error framework. The cyber-attacks occur between the output of the sensors localized on the physical plant and the cyber part embedding the observer. The data required by the observer are thus available at sporadic measurement time instants. In this setting, an interval impulsive observer is synthesized. The stability analysis of the dynamics of the state estimation error is done leveraging finite-gain $\mathcal{L}_1$ stability theory for hybrid systems. The observer $\mathcal{L}_1$ gain is computed by combining interval analysis and the resolution of algebraic inequalities that greatly reduces the synthesis complexity when compared to the state-of-the-art approaches that usually rely on solving many bilinear matrix inequalities. A numerical example illustrates the approach and the performance of the designed robust observer.**

I. INTRODUCTION

Cyber-Physical Systems (CPS) are increasingly interconnected, thus necessitate the development of reliable and durable mechanisms to ensure security in case of malevolent attacks. The study of networked CPS, which encompasses physical entities such as observers, sensors, and actuators communicating through networks, is a rapidly expanding research area [1]. Indeed, these systems have an extensive range of potential applications in fields such as telesurgery, mobile sensor networks, intelligent transportation, and autonomous mobile robots. Security issues have been studied from different angles in recent years due to the increasing number of cyber-attacks and the increasing vulnerability of networked CPS. Much effort has been devoted to analyzing the influence of specific malicious attacks such as denial-of-service (DoS) attacks, replay attacks, and data injection attacks [2], [3], [4]. The DoS attack, which aims to prevent communication between system components, has been widely investigated because this kind of attack is one of the most feasible attack and can lead to disastrous consequences [5]. Regarding the CPS modelling, hybrid models, and in particular impulsive models, are ideally suited to describe various evolutionary processes where states undergo abrupt changes at certain times. Over the past few decades, research on impulsive systems has received a great deal of attention in control design as well as in state estimation. At the same time, interval observers which guarantee the existence of a solution between the bounds of reconstructed tubes of state variables trajectories have extensively been studied. Initially introduced by Gouzé et al. [6] for linear biological systems, many applications for discrete, impulsive, and switched systems have been reported in the literature [7], [8], [9], [10], [11]. The state estimation with aperiodic measurements has first been addressed in [12] where a impulsive observer was established. This work was taken up in [11] where an interval impulsive observer was extended to address the bounded-error framework. They analysed the $\mathcal{L}_2$ stability of the state estimation error while making some strong restrictions on the system. Furthermore, the complexity of this method requires the resolution of an infinite number of Bilinear Matrix Inequalities (BMI).

In this paper, we address the interval state estimation of a linear-time invariant (LTI) system under DoS attacks. We exploit the impulsive observer framework proposed in [12] and relax the strong restrictions on the system needed to ensure the observer convergence. The main contributions of this paper lie in the particularities listed below:

- The design of an interval impulsive observer that is robust to DoS attacks. The DoS attack occurs on the channel between the sensor of the physical installation and the observer on the ”cyber” part in charge of the numerical computations. The observer subject to a DoS attack can be considered as an observer that receives sporadic measurements.
- The $\mathcal{L}_1$-gain stability analysis of the observation error is carried out leveraging the relaxed version of the stability theorem for hybrid systems presented in [13].
- The synthesis of the observer $\mathcal{L}_1$-gain is simplified by using interval analysis and boils down to resolving few algebraic inequalities instead of resolving an infinity of BMIs as proposed in [11].
- The restriction made in [11] to non-singular matrices is no longer needed, thus extending the applicability of the approach to a larger class of systems.

This work is organized as follows. Essential definitions and preliminaries are first introduced in Section II. The structure of the interval impulsive observer under DoS attacks is described in Section III. The stability analysis is

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carried out in Section IV. The methodology for designing the observer is given in section V. A numerical example illustrates the performances of the proposed observer in section VI. Concluding remarks are given in section VII.

II. Preliminaries and definitions

A. Nomenclature

Let $x \in \mathbb{R}^n$ be a vector, the notation $|.|_p$ is the p-norm whose expression is $|x|_p = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}$. The p-norm of x to a closed set $\mathcal{S} \subset \mathbb{R}^n$ is denoted $|x|_{\mathcal{S}}$ and is defined by $|x|_{\mathcal{S}} = \inf_{y \in \mathcal{S}} |x-y|_p$. $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix. From a matrix point of view, the symbols $\leq, \geq, <$, and $>$ should be understood element-wise; the "max" operator is an element-wise comparison operator such that $\forall A, B \in \mathbb{R}^{n\times n}, M = \max\{A, B\}$ implies that $m_{i,j} = \max\{a_{i,j}, b_{i,j}\}$. The element-wise absolute value $|A| = A^+ - A^-$ where $A^+ = \max\{A, 0\}$ and $A^- = A^+ - A$. $\mathbb{R}_{>0}$ (resp. $\mathbb{R}_{\geq 0}$) represents the set $\mathbb{R} \setminus (-\infty, 0]$ (resp. $\mathbb{R} \setminus (-\infty, 0)$).

B. Definitions

Definition 1: A continuous (respectively discrete) dynamical system whose state dynamics follows $\dot{x}(t) = Ax(t)$ (resp. $x(t_{k+1}) = Ax(t_k)$), is said to be cooperative if A is Metzler (resp. non-negative).

Remark 1: In the sequel, we use the notation of hybrid time domains: $z \equiv z(t,j)$, and $z^+ \equiv z(t_j, j)$. In hybrid systems, the solutions are parameterized by both t, the continuous time, and j, the number of jumps. When a jump occurs at time t_j, the hybrid system state jumps from $z \equiv z(t_j, j-1)$ to $z^+ \equiv z(t_j, j)$ [14].
The concept of $\mathcal{L}_p$ norm is essential when talking about $\mathcal{L}_p$ stability in the input-output sense [11].

Definition 2: $\mathcal{L}_p$ norm. Let $T \in \mathbb{R}_{\geq 0}$ be a scalar, z be a hybrid signal, and $dom(z) \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$, the domain of z, the T-truncated $\mathcal{L}_p$ norm of z is

$$\left\|z_{[T]}\right\|_p = \left(\sum_{i=1}^{j(T)} |z(t_i, i)|^p + \sum_{i=0}^{j(T)} \int_{t_i}^{\sigma_i} |z(s, i)|^p ds \right)^{1/p}, \quad (1)$$

with $t_0 = 0$, $j(T) = \max\{k : (t, k) \in dom(z), t + k \leq T\}$, and $\sigma_i = \min(t_{k+1}, T - i)$.

The expression (1) is a generalization that considers both the continuous (second term) and discrete (first term) part of the signal. The $\mathcal{L}_p$ norm of z is

$$\|z\|_p = \lim_{T \to T^*} \left\|z_{[T]}\right\|_p, \quad (2)$$

where $T^* = \sup\{t + j : (t, j) \in dom(z)\}$. If the above limit exists and is finite, one writes $z \in \mathcal{L}_p$ (or $z \in \mathcal{L}_p^{n_z}$ if z is multidimensional).

An input-output system supposes a direct link between input and output without knowledge of the internal structure described by the states equations. The $\mathcal{L}_p$ stability evaluates the input-to-output stability of a system. Let us now introduce the system under consideration. A hybrid system consists of two main parts: a differential equation that governs the continuous dynamics when the system flows, and a difference equation that governs the discrete dynamics when the system jumps. We consider the hybrid system written as follows:

$$\begin{cases} \dot{x} = f(x, d), \forall (x, d) \in \mathcal{C}_x, \\ x^+ = g(x, d), \forall (x, d) \in \mathcal{D}_x, \\ y = h(x, d), \end{cases} \quad (3)$$

where $\mathcal{C}_x$ is the flow set, $\mathcal{D}_x$ the jump set, x the state vector, d the input vector, and y the output vector.

Definition 3: $\mathcal{L}_p$ Stability. System (3) is finite-gain $\mathcal{L}_p$ stable from d to y with gain upper bounded by γ_p if there exists a scalar $\beta \geq 0$ such that the following condition is satisfied by any solutions of system (3)

$$\|y\|_p \leq \beta |x(0,0)|_p + \gamma_p \|d\|_p, \forall d \in \mathcal{L}_p^{n_d}. \quad (4)$$

The strict case: Stability analysis using the $\mathcal{L}_p$ method can be done through Lyapunov's (storage) functions [13]. For this aim, let us consider a positive semi-definite continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that:

$$0 \leq V(x) \leq c_2 |x|^p, \forall (x, d) \in \mathcal{C}_x \cup \mathcal{D}_x, \quad (5a)$$

$$\langle \nabla V(x), f(x, d) \rangle \leq -\gamma_{yf} |h(x, d)|_p^p + \gamma_{df} |d|_p^p, \forall (x, d) \in \mathcal{C}_x, \quad (5b)$$

$$V(g(x, d)) - V(x) \leq -\gamma_{yg} |h(x, d)|_p^p + \gamma_{dg} |d|_p^p, \forall (x, d) \in \mathcal{D}_x. \quad (5c)$$

The constants c_2, γ_{yf} and γ_{yg} are strictly positive and γ_{df} and γ_{dg} are non-negative ones. If conditions (5) are satisfied, $V(x)$ is a finite-gain $\mathcal{L}_p$ storage function for the system (3), the system (3) is finite-gain $\mathcal{L}_p$ stable, and the $\mathcal{L}_p$ gain is upper bounded by $\gamma_p = \sqrt[p]{\gamma_d/\gamma_y}$ with $\gamma_d = \max\{\gamma_{df}, \gamma_{dg}\}$ and $\gamma_y = \max\{\gamma_{yf}, \gamma_{yg}\}$. The conditions (5) are commonly encountered in the literature. However, in the study of particular observers such as the impulsive observer with sporadic data [11], these conditions induce a strong restriction on the LTI class considered: the state matrice "A" has to be non-singular. A relaxed version of this definition is proposed in [13].

The relaxed case: Consider two positive scalars c_1 and c_2 and a bounded Lyapunov function:

$$c_1 |x|_p^p \leq V(x) \leq c_2 |x|_p^p, \forall x \in \mathcal{C}_x \cup \mathcal{D}_x. \quad (6)$$

Considering the assertions made in *Proposition 3* and *Corollary 3* in [13], if $d(t)$ is a bounded input with known bounds, and inequalities (5) are satisfied, with $\gamma_{yf} = 0$, a relaxation of *Definition 3* can be admitted. This relaxation consists in introducing the "reverse" dwell-time dynamics. The notion of "reverse" dwell-time is the idea of forcing impulses, i.e., preventing overly long intervals between impulses [15]. The introduction of a "reverse" dwell-time allows to ensure that there are sufficiently regular pulses for the system to remain stable. It is of interest when the considered impulsive system has a continuous dynamics which tends to destabilize while the discrete dynamics tends to stabilize. The dynamics of the "reverse" dwell-time τ_1 is defined as follows [13]:

$$\begin{cases} \dot{\tau}_1 = 1, \\ \tau_1^+ = \max\{0, \tau_1 - \delta\}, \end{cases} , \tau_1 \in [0, T], \quad (7)$$

where δ and T are positive constants characterizing the dwell-time dynamics. Combining the dynamics (7) with the hybrid dynamics (3) gives:

$$\begin{cases} \begin{pmatrix} \dot{x} \\ \dot{\tau}_1 \end{pmatrix} = \begin{pmatrix} f(x,d) \\ 1 \end{pmatrix}, \quad \forall (x,d,\tau_1) \in \mathcal{C}_x \times [0,T], \\ \begin{pmatrix} x^+ \\ \tau_1^+ \end{pmatrix} = \begin{pmatrix} g(x,d) \\ \max\{0, \tau_1 - \delta\} \end{pmatrix}, \forall (x,d,\tau_1) \in \mathcal{D}_x \times [0,T], \\ y = h(x,d). \end{cases} \tag{8}$$

As specified above, in the relaxed case where $\gamma_{yf} = 0$, for any (T, δ) there exists a sufficiently small positive scalar λ_0 close to zero, satisfying:

$$c_2 \lambda_0 e^{-\lambda_0 T} > 0, \tag{9a}$$

$$e^{(\lambda_0 (T-\delta))}(e^{(\lambda\delta)} - 1)c_2 < \gamma_{dg}. \tag{9b}$$

such that the scalars defined in (10) are all positive.

$$\bar{c}_1 = c_1 e^{-\lambda_0 T}, \tag{10a}$$

$$c_2 = c_2, \tag{10b}$$

$$\overline{\gamma}_{df} = \gamma_{df}, \tag{10c}$$

$$\overline{\gamma}_{dg} = \gamma_{dg} e^{\lambda_0 \delta}, \tag{10d}$$

$$\overline{\gamma}_{yf} = min(\gamma_{yf}, e^{-\lambda_0 T}\gamma_{yf}) + \lambda_0 e^{(-\lambda_0 T)} c_2, \tag{10e}$$

$$\overline{\gamma}_{yg} = e^{-\lambda_0 (T-\delta)}\gamma_{yg} - (e^{\lambda_0 \delta} - 1)c_2. \tag{10f}$$

Inequalities (9) establish upper and lower bounds for T and λ_0. Thus, the system (8) is $\mathcal{L}_p$ stable with gain $\gamma_p = \sqrt[p]{\frac{\max\{\overline{\gamma}_{df}, \overline{\gamma}_{dg}\}}{\min\{\overline{\gamma}_{yf}, \overline{\gamma}_{yg}\}}}$.

III. Interval impulsive observer under DoS Attacks

The cyber-physical system under study can be modelled as a LTI system as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ed(t), \\ y(t_k) = Cx(t_k) + Fd(t_k), \end{cases} \tag{11}$$

where $x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}, y \in \mathbb{R}^{n_y}$ are respectively the state, the control input and the output vectors with $t \in \mathbb{R}, k \in \mathbb{N}$, $t \in [t_k, t_{k+1}[$. A, B, C, E, and F are known matrices with appropriate dimensions. $d \in \mathbb{R}^{n_d}$ represents an unknown but bounded disturbance. Afterwards, considering the usual notation of interval analysis, we assume that there exist known vectors $\underline{d}(t)$ and $\overline{d}(t)$ such that

$$\underline{d}(t) \leq d(t) \leq \overline{d}(t). \tag{12}$$

In the sequel, $\overline{x}(t)$ and $\underline{x}(t)$ are the estimates of the upper and lower bounds of the state vector so that $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$. In this work, we propose to design an interval impulsive observer subject to DoS attacks using a new method inspired by the work of [11], which dealt with an interval impulsive observer under sporadic measurements. The goal here is to compute a guaranteed enclosure for the state vector of the system (11) by estimating $\overline{x}(t)$ and $\underline{x}(t)$ taking into account the DoS attacks.

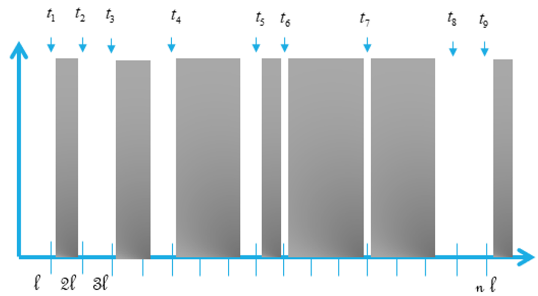

Fig. 1: Illustration of time-dependant DoS attacks. ↓ represents the sampling instants where the measurements are available and denoted by $t_i, i \in \mathbb{N}$.

A. DoS Sequences

A DoS attack aims to interfere with data transmission by making the network unavailable. An illustration is given Fig. 1. The plant output is regularly sampled, at each sampling instant nl, but the measurements are transmitted to the observer over a faulty network. The random attack is represented by the grey areas where the attack is effective. It is essential to formalize the constraints of the attack in order to get closer to reality. Indeed, if it is true that the attacker does not have infinite energy to carry out a long duration attack in continuous time, it also remains obvious that such an attack would make any state vector estimation impossible. So, let $\{h_n\}$, $n \in \mathbb{N}$, be the time instants at which on/off transitions of the network availability occur (obviously $h_0 \geq 0$). These time instants also correspond to the switching from "0" (transmission succeeded) to "1" (transmission failed). The time intervals of each DoS sequence is defined as $H_n = [h_n, h_n + \tau_n)$ where τ_n is the interval length. For any interval $[t_1, t_2]$, $0 \leq t_1 \leq t_2$, let us denote the two following sets respectively, for the DoS and healthy status of the network:

$$\begin{array}{l} D(t_1, t_2) = \cup_{n \in \mathbb{N}} H_n \cap [t_1, t_2] \\ H(t_1, t_2) = [t_1, t_2] \backslash D(t_1, t_2) \end{array}$$

The number of DoS on/off transitions over $[t_1, t_2]$ is denoted $n(t_1, t_2)$. To characterize the DoS attacks, the "duration-frequency" model initially studied in [16] is considered. As already assumed in [17], [18], [19], the frequency and the duration of DoS attacks are subject to limitations.

*Assumption 1 (**DoS Frequency limitation**):* For any time interval $[t_1, t_2]$, there exists $(\eta, \tau_D) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ such that: $n(t_1, t_2) \leq \eta + (t_2 - t_1)/\tau_D$.

*Assumption 2 (**DoS Duration limitation**):* For any time interval $[t_1, t_2]$, there exists $(\varsigma, T_d) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 1}$ respectively known as chatter bound and average duration ratio such that the DoS duration verifies: $a(t_1, t_2) \leq \varsigma + (t_2 - t_1)/T_d$.

B. Interval impulsive observer design

The dynamics of an impulsive observer for system (11) consists of two different behaviors. The first behaviour is an open-loop prediction, relying on the continuous dynamics and without using any measurement. The second behavior is a closed-loop behavior, updated at discrete-time instants when a measurement is available.

We first recall that the product between a matrix and a vector in interval analysis can be framed as follows:

$$\underline{x} \leq x \leq \overline{x} \Rightarrow A^{+}\underline{x} - A^{-}\overline{x} \leq Ax \leq A^{+}\overline{x} - A^{-}\underline{x}. \quad (13)$$

Let us now formalize the interval impulsive observer.

The **prediction step** (open-loop): Combining the first equation of (11) with (12) and (13) gives:

$$\begin{cases} \dot{\underline{x}} = A_M\underline{x} - A_N\overline{x} + Bu + E^{+}\underline{d} - E^{-}\overline{d}, \\ \dot{\overline{x}} = A_M\overline{x} - A_N\underline{x} + Bu + E^{+}\overline{d} - E^{-}\underline{d}, \end{cases} \quad (14)$$

with $A_M = D_A + (A - D_A)^{+}$, $A_N = A_M - A$, and D_A the diagonal of A. The initial state interval satisfies :

$$\underline{x}(0,0) \leq x(0,0) \leq \overline{x}(0,0). \quad (15)$$

The **correction step** (closed-loop) occurs at discrete-time instants $t_i, i \in \mathbb{N}$ when the measurements are available. At these correction instants, the equality $y(t_i) - Cx(t_i) - Fd(t_i) = 0$ is satisfied. Denoting $x^{+} \equiv x(t_i)^{+}$, i.e., the state vector after the jump, we can write for all $i \in \mathbb{N}$:

$$x^{+} = x + L(Cx + Fd - y) \quad (16)$$

where $L \in \mathbb{R}^{n_x \times n_y}$ is the correction gain matrix to be tuned. Considering the interval approach on the vector x^{+}, we get:

$$\begin{cases} \underline{x}^{+} = (I_n + LC)^{+}\underline{x} - (I_n + LC)^{-}\overline{x} + (LF)^{+}\underline{d} - (LF)^{-}\overline{d} - Ly, \\ \overline{x}^{+} = (I_n + LC)^{+}\overline{x} - (I_n + LC)^{-}\underline{x} + (LF)^{+}\overline{d} - (LF)^{-}\underline{d} - Ly. \end{cases} \quad (17)$$

The upper and lower observer error bounds are given by:

$$\begin{cases} \underline{e}(t,j) = x(t,j) - \underline{x}(t,j), \\ \overline{e}(t,j) = \overline{x}(t,j) - x(t,j). \end{cases} \quad (18)$$

Let us define the state vector for the the hybrid system (20):

$$z = [\xi, \tau]^{\top}, \quad (19)$$

with $\xi = [\underline{e}, \overline{e}]^{\top}$, and τ the timer between two jumps. Equations (14) and (17) allow us to define the hybrid system that models the dynamic behavior of observation error vector (18):

$$\begin{cases} \dot{z} = \begin{bmatrix} \overline{A}\xi + \tilde{E}\psi \\ -1 \end{bmatrix}, \forall z \in \mathcal{C}_{\xi}, \\ z^{+} = \begin{bmatrix} \Gamma(L)\xi + \tilde{F}(L)\psi \\ \mu \end{bmatrix} \in \begin{bmatrix} \Gamma(L)\xi + \tilde{F}(L)\psi \\ [\tau_{\min}, \tau_{\max}] \end{bmatrix}, \forall z \in \mathcal{D}_{\xi}, \end{cases} \quad (20)$$

with $\overline{A} = \begin{bmatrix} A_M & A_N \\ A_N & A_M \end{bmatrix}$, $\tilde{E} = \begin{bmatrix} E^{+} & E^{-} \\ E^{-} & E^{+} \end{bmatrix}$,
$\tilde{F}(L) = \begin{bmatrix} (LF)^{+} & (LF)^{-} \\ (LF)^{-} & (LF)^{+} \end{bmatrix}$, $\psi(t,j) = \begin{bmatrix} \overline{d} - \underline{d} \\ \overline{d} - d \end{bmatrix}$, and
$\Gamma(L) = \begin{bmatrix} (I_n + LC)^{+} & (I_n + LC)^{-} \\ (I_n + LC)^{-} & (I_n + LC)^{+} \end{bmatrix}$.
$\mu \in [\tau_{\min}, \tau_{\max}]$ is the value of the timer τ after jump. $\mathcal{C}_{\xi}$ and $\mathcal{D}_{\xi}$ are respectively the flow and jump sets defined as :

$$\begin{aligned} \mathcal{C}_{\xi} &= \{(\xi,\tau) \in \mathbb{R}^{n} \times \mathbb{R}_{\geq}0 : \tau \in [0, \tau_{\max}]\}, \\ \mathcal{D}_{\xi} &= \{(\xi,\tau) \in \mathbb{R}^{n} \times \mathbb{R}_{\geq}0 : \tau = 0\}. \end{aligned} \quad (21)$$

Remark 2: A_M and A_N satisfy $A = A_M - A_N$ and are constructed according to the Muller's existence theorem in order to guarantee the Metzler property for $\overline{A}$. Thus, knowing that $\Gamma(L)$, $\tilde{F}(L)$, $\tilde{E}$ and ψ are non-negative, the dynamics of system (20) is consequently non-negative.

We define the set Θ to be used in the sequel:

$$\Theta = \left\{(\xi,\tau) \in \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0} \,|\, \xi = 0, \tau \in [0, \tau_{\max}]\right\}. \quad (22)$$

IV. Stability analysis

The aim of this section is to discuss the stability conditions of the error dynamics (20). Let $L \in \mathbb{R}^{n_x \times n_x}$ be a matrix and $\lambda \in \mathbb{R}^{2n}_{>0}$ a positive vector.

Theorem 1: Let us combine the dwell-time dynamics (7) with error dynamics (20) as done in (8). Recall that $T \in [\tau_{\min}, \tau_{\max}]$ and $\mu \in [\tau_{\min}, \tau_{\max}]$. For a given matrix $L \in \mathbb{R}^{n_x \times n_y}$, if there exist positive scalars c_1, c_2, λ_0, δ, $\gamma_{yg}, \gamma_{dg}, \gamma_{df}$,$\overline{\gamma}_{yf}$, $\overline{\gamma}_{df}$, $\overline{\gamma}_{yg}$, $\overline{\gamma}_{dg}$ and $\gamma_{yf} = 0$ satisfying inequalities (23)-(24):

$$\begin{cases} \lambda^{\top} e^{\overline{A}\tau}\tilde{E} - \gamma_{df} 1^{\top}_{2n_d} \leq 0^{\top}_{2n_d}, \\ \lambda^{\top}\left(e^{\overline{A}\mu}\Gamma(L) - I_{2n}\right) + \gamma_{yg} 1^{\top}_{2n} \leq 0^{\top}_{2n}, \\ \lambda^{\top} e^{\overline{A}\mu}\tilde{F}(L) - \gamma_{dg} {1_{2n_d}}^{\top} \leq 0^{\top}_{2n_d}. \end{cases} \quad (23)$$

$$\begin{cases} c_2\lambda_0 e^{-\lambda_0 T} > 0, \\ e^{(\lambda_0(T-\delta))}(e^{(\lambda\delta)} - 1)c_2 < \gamma_{dg}. \end{cases} \quad (24)$$

Then Eqs. (14)-(15)-(17) form a finite $\mathcal{L}_1$-gain interval impulsive observer for system (11). And systems (7)-(20) are finite-gain $\mathcal{L}_1$ stable from ψ to ξ with $\mathcal{L}_1$ gain:

$$\gamma_1 = \max\left\{\overline{\gamma}_{df}, \overline{\gamma}_{dg}\right\} / \min\left\{\overline{\gamma}_{yf}, \overline{\gamma}_{yg}\right\}. \quad (25)$$

The proof of this stability theorem follows the main lines of *Definition 3* in the relaxed case.

Proof 1: Equation (23) is the direct consequence of the stability conditions of *Definition 3* applied to the observation error system (20) in the relaxed case. The proof is divided into two steps:

Step 1: Non-negativity of the observation error

Let the ordery condition (15) be true. Knowing that $\overline{A}$ is a Metzler matrix, $\tilde{E}$ and ψ are both non negative, the continuous dynamics of (20) is non-negative $\forall t \in [(t_j, j), (t_{j+1}, j)]$. Furthermore, the relations (18) and the non-negativity of $\Gamma(L)$ ensure the non-negativity of the error dynamics at reset time. Consequently, the non-negativity of the observation error is guaranteed by construction: $\xi(0,0) \geq 0 \Rightarrow \forall(t,j)\; \xi(t,j) \geq 0$.

Step 2 : $\mathcal{L}_1$ stability of the observation error

The conditions (10) of *Definition 3* are used to analyze the $\mathcal{L}_1$ stability of the augmented error dynamics. We first analyze the continuous component of the error dynamics. Let $V(\xi,\tau)$ define the continuous Lyapunov function which will be used in the remainder of the proof:

$$V(\xi,\tau) = \lambda^{\top} e^{\overline{A}\tau}\xi. \quad (26)$$

It is important to note that this function is bounded by two positive scalars c_1 and c_2, as follows:

$$c_1 |z|_{\Theta} \leq V(z) \leq c_2 |z|_{\Theta}, \forall z \in \mathcal{C}_{\xi} \cup \mathcal{D}_{\xi} \quad (27)$$

with $c_1 = \min(\min_\tau(\lambda^\top e^{\overline{A}\tau}))$, and $c_2 = \max(\max_\tau(\lambda^\top e^{\overline{A}\tau}))$, while $\tau \in [0, \tau_{\max}]$. $|z|_\Theta$ is the norm to the set Θ defined by $|z|_\Theta = |\xi|_1$. From (26), we can deduce:

$$\nabla V(\xi,\tau) = \left(\lambda^\top e^{\overline{A}\tau}, \lambda^\top \overline{A} e^{\overline{A}\tau}\xi\right)^\top, \tag{28}$$

then

$$\left\langle \nabla V(\xi,\tau), \begin{bmatrix} \overline{A}\xi + \tilde{E}\psi \\ -1 \end{bmatrix} \right\rangle = \lambda^\top e^{\overline{A}\tau}\overline{A}\xi + \lambda^\top e^{\overline{A}\tau}\tilde{E}\psi - \lambda^\top \overline{A} e^{\overline{A}\tau}\xi. \tag{29}$$

Since $e^{\overline{A}\tau}$ and $\overline{A}$ commutes, (29) boils down to

$$\left\langle \nabla V(\xi,\tau), \begin{bmatrix} \overline{A}\xi + \tilde{E}\psi \\ -1 \end{bmatrix} \right\rangle = \lambda^\top e^{\overline{A}\tau}\tilde{E}\psi. \tag{30}$$

Knowing that $|\xi|_1 = 1_{2n}^\top \xi$, $|\psi|_1 = 1_{2n_d}^\top \psi$ and by designing an upper bound of the $\mathcal{L}_1$-gain for the operator $\psi \to \xi$ as shown in Eq.(5b), we have the relation:

$$\left\langle \nabla V(\xi,\tau), \begin{bmatrix} \overline{A} + \tilde{E}\psi \\ -1 \end{bmatrix} \right\rangle \leq -\gamma_{yf} 1_{2n}^\top \xi + \gamma_{df} 1_{2n_d}^\top \psi, \tag{31}$$

which is also equivalent to:

$$\lambda^\top e^{\overline{A}\tau}\tilde{E}\psi \leq -\gamma_{yf} 1_{2n}^\top \xi + \gamma_{df} 1_{2n}^\top \psi. \tag{32}$$

From this expression, we can deduce the value $\gamma_{yf} = 0$. Then, the inequality (32) can be rewritten as follows:

$$\lambda^\top e^{\overline{A}\tau}\tilde{E} - \gamma_{df} 1_{2n_d}^\top \leq 0_{2n_d}^\top. \tag{33}$$

By applying the same principle to the discrete part of the impulsive system (5c) and using the same Lyapunov function, we have:

$$\lambda^\top e^{\overline{A}\tau}\left[\Gamma(L)\xi + \tilde{F}(L)\psi\right] - \lambda^\top \xi \leq -\gamma_{yg} 1_{2n}^\top \xi + \gamma_{dg} 1_{2n_d}^\top \psi. \tag{34}$$

Which can be rewritten as:

$$\begin{bmatrix} \left(\lambda^\top e^{\overline{A}\mu}\Gamma(L) - \lambda^\top\right) + \gamma_{yg} 1_{2n}^\top \\ \lambda^\top e^{\overline{A}\mu}\tilde{F}(L) - \gamma_{dg} 1_{2n_d}^\top \end{bmatrix}^\top \begin{bmatrix} \xi \\ \psi \end{bmatrix} \leq 0. \tag{35}$$

The first part of the proof shows that $\xi(t,j) \geq 0, \forall (t,j) \in dom(\xi)$. So, knowing that ψ is also positive, the inequalities (33) and (35) are equivalent to (23). In the specific case where $\gamma_{yf} = 0$, the relaxed version of the stability constraints of *Definition 3* can be used. So, there exists a sufficiently small positive scalar λ_0, such that the two inequalities (9 a,b) are satisfied. The positive scalars (10) exist and are all positive. This concludes the proof. ■

V. Synthesis method

We propose in this section to synthesize the observer gain using interval analysis. For this aim, let us denote $G = I + LC$ which is equivalent to $G = G^+ - G^-$ where G^+ and G^- are respectively positive and negative parts of G. We consider that $\forall G_1, G_2 \in \mathbb{R}_{\geq 0}^{n\times n}$, $G = G_1 - G_2$, there exists a corresponding matrix $\Delta \in \mathbb{R}_{\geq 0}^{n\times n}$ such that $G = (G^+ + \Delta) - (G^- + \Delta)$. So, $\Gamma(L)$ can be rewritten as:

$$\Gamma(L) = \Gamma(G_1, G_2) = \begin{bmatrix} G_1 & G_2 \\ G_2 & G_1 \end{bmatrix}. \tag{36}$$

The main idea of this method is to evaluate numerically the matrices G_1 and G_2 that satisfy the stability conditions (23) in order to deduce the matrices G^+ and G^-. The gain synthesis can be performed by solving simultaneously equations (23) and (36). The same principle is applied to the matrix $\tilde{F}(L)$ with $LF = R_1 - R_2$.

$$\tilde{F}(L) = \tilde{F}(R_1, R_2) = \begin{bmatrix} R_1 & R_2 \\ R_2 & R_1 \end{bmatrix}. \tag{37}$$

Given that τ and μ belong to known intervals, we propose a solution by leveraging interval analysis. In fact, $\forall a \in [a_1, a_2], a_2 \leq 0 \Rightarrow a \leq 0$. The goal is to verify the satisfaction of the inequalities (23) by using the upper bound of each first member.

Let f be a function and $f([x])$ be the smallest interval that includes image of $[x]$ through f.

$$f([x]) = [\{f(x) \,|x \in [x]\}]. \tag{38}$$

The notation $\sup(f([x]))$ is the upper bound of the interval box $f([x])$. Further details on interval analysis can be found in [20].

Considering equations (36) and (38), and the stability conditions (23), we could rewrite using interval analysis the expression (39) which, added to (9) is a sufficient stability condition:

$$\begin{cases} \lambda^\top \Xi_1 \tilde{E} - \gamma_{df} 1_{2n_d}^\top \leq 0_{2n_d}^\top, \\ \lambda^\top (\Xi_2 \Gamma(G_1, G_2) - I_{2n}) + \gamma_{yg} 1_{2n}^\top \leq 0_{2n}^\top, \\ \lambda^\top \Xi_2 \tilde{F}(R_1, R_2) - \gamma_{dg} {1_{2n_d}}^\top \leq 0_{2n_d}^\top \end{cases} \tag{39}$$

$$\begin{cases} c_2 \lambda_0 e^{-\lambda_0 \tau_{max}} > 0 \\ e^{(\lambda_0(\tau_{max} - \delta))}(e^{(\lambda\delta)} - 1)c_2 < \gamma_{dg} \end{cases} \tag{40}$$

where $\Xi_1 = \sup\left(e^{\overline{A}[\tau]}\right)$, $\Xi_2 = \sup\left(e^{\overline{A}[\mu]}\right)$, $[\tau] = [0, \tau_{\max}]$, $[\mu] = [\tau_{\min}, \tau_{\max}]$.

The gain synthesis is achieved by solving the algebraic inequalities (39)-(40). Given that optimization algorithms cannot guarantee strict positivity, all strict inequalities were converted to non-strict inequalities by adding a bias $\epsilon = 10^{-6}$, i.e. $u - \epsilon \geq 0 \Rightarrow u > 0$.

In order to minimize the impact of noise on the estimated state, it is necessary to minimize the $\mathcal{L}_1$-gain (25). To this end, we have chosen in the synthesis algorithm to minimize the expression $\overline{\gamma}_{df} + \overline{\gamma}_{dg} - \overline{\gamma}_{yf} - \overline{\gamma}_{yg}$, where the positive scalars $\overline{\gamma}_{df}$, $\overline{\gamma}_{dg}$, $\overline{\gamma}_{yf}$, and $\overline{\gamma}_{yg}$ are defined in (10).

VI. Illustrative Example

The double-spring-mass-damper system is considered as a case study. It consists of two masses connected by three springs and three dampers. Two forcing terms (external forces) are applied to each of the masses. The state vector is made up of the positions and speeds $[x_1, \dot{x}_1, x_2, \dot{x}_2]$ for each mass. The inputs are the multi-periodic forcing terms $[f_1, f_2]$ with $f_1 = 14[1 + 2\sin(10t) + \cos(40t)]$, $f_2 = 10[2\sin(15t) + \sin(30t)]$. The matrices describing system

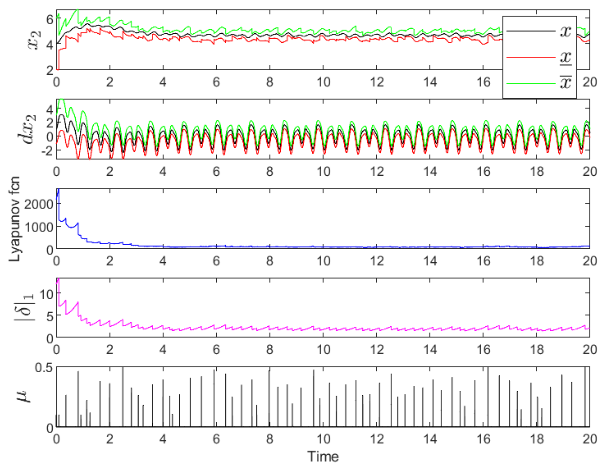

Fig. 2: The estimation bounds for the state variables x_2 and $\dot{x}_2$, the Lyapunov function (26), the norm of estimation error width $\delta = \overline{e} + \underline{e}$, and the inter-measurement time $\mu \in [0.1, 0.5]$.

dynamics are:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3.3 & -5.6 & 1.6 & 2.3 \\ 0 & 0 & 0 & 1 \\ 1 & 1.4 & -2 & -2.6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1.6 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$E = \begin{bmatrix} 0.1 & -0.2 \\ -0.7 & 0.6 \\ 0.2 & -0.2 \\ -0.5 & 0.6 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0.6 & -0.8 \\ -0.4 & 0.5 \end{bmatrix}.$$

$d(t)$ is a 2-dimensional vector bounded by $\overline{d}$ and $-\overline{d}$ with $\overline{d} = [-0.5, 0.5]$. The calculation of the gain L is carried out by the means of MATLAB's YALMIP toolbox, based on FMINCON solver. The synthesis provides the optimal gain

$$L = \begin{bmatrix} -0.4885 & -0.0007 \\ 0 & 0 \\ 0.0032 & -0.4895 \\ 0 & 0 \end{bmatrix}.$$

The other computed variables are $\lambda_0 = 0.00015333$, $\lambda = 10^{-6}(1, 1, 1, 1, 1, 1, 1, 1)$, $c_2 = 0.017685$, $c_1 = 1.2845e{-}05$. The state vector is initialised at $[2, 2, 1, 1]$ and the observer at $[8; -4; 2; -1; 12; 0; 6; 3]$. Simulations are carried out for a sufficient long time to eliminate transient phase. The sporadic measurement time-step μ is taken randomly within the interval $[\tau_{\min}, \tau_{\max}]$, for $\tau_{\min} = 0.1$ and $\tau_{\max} = 0.5$, as shown on Fig.2. Our synthesis method was successful for values of τ_{max} up to $\tau_{max} = 1.5$.

VII. Conclusion

In this work, an interval impulsive observer under DoS attacks was designed for a LTI system by taking advantage of the existing framework for state estimation under sporadic measurements in the bounded-error framework. The stability of the estimation error has been studied taking into account the relaxed version of the $\mathcal{L}_1$ stability theory for hybrid systems. We have extended the relevant system class that can be addressed by our observer, and simplified the design procedure of the observer gain by combining interval analysis and algebraic inequalities solving, instead of the state-of-the-art approaches that solve several BMIs.

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