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A Control Leonov Function Guaranteeing Global ISS of Two Coupled Synchronverters

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Abstract— **The synchronverter control algorithm is a highly promising option for operating AC power inverters in future low-inertia power systems. Yet, as with conventional synchronous generators, the standard synchronverter algorithm only provides limited robustness guarantees. Therefore, we propose an additional control that confers the closed-loop system with *global* robustness in the Input-to-State Stability (ISS) sense. In this paper, such a control law is derived for the case of two identical synchronverters interconnected over dynamic power lines by using the Control Leonov Function (CLeF) framework. The control is illustrated by simulations.**

I. Introduction

Due to energy shortage, air pollution and greenhouse gas emissions, the relevance of renewable energy sources has steadily increased in the last decades [15], [17]. Most of the renewable-energy generators comprise variable-frequency sources, high frequency AC sources or DC sources. Therefore, they need inverters to be connected with the electrical grid. So, the use of inverters in power systems will be more and more common in power generation [15], [17]. While inverters exhibit several advantages [11], the decreasing amount of Synchronous Generators (SGs) connected with the electrical grid implies that the inertia of power grids is reduced and produces higher power fluctuations [15], [17].

Therefore, designing high-performance control strategies for inverters is a highly important task to satisfy the requirements of future climate-neutral power systems [15], [17]. A highly attractive control concept is to control the inverter, such that its closed-loop behavior mimics a traditional SG. Such strategy is commonly known as *Virtual* SG [2] or *Synchronverter* [26]. A key advantage of these strategies, is that they ensure operational compatibility of the current power systems, which are mainly based on SGs [26]. Compared to SGs, synchronverters have some extra benefits, such as the possibility of adjusting parameters and the ability to generate practically instantaneous frequency and voltage droop [3].

However, as a result of the reduced system inertia, power systems world-wide are more frequently operating closer to their security margins [9]. Hence, the provision of stability and robustness guarantees beyond merely local properties, *i.e.*, ensuring transient stability, Input-to-State Stability (ISS) and integral ISS (iISS), is swiftly gaining in relevance [9].

Unfortunately, as with SGs in classical power systems, the standard synchronverter algorithm only provides limited robustness properties of the closed-loop system with respect to perturbations [25]. By recognizing this fact, an additional control termed *virtual friction* is considered in [25], which has been proven to be effective to (locally) improve the system behavior and, in particular, its robustness [3], [4]. The present paper is dedicated to addressing the same challenge by designing an additional control for the synchronverter, which confers the closed-loop system with the ISS property.

To achieve this objective, we use the concept of Leonov functions [5], [6], [12], [19]. This is motivated by the fact that the dynamics of AC power systems are periodic with respect to a part of the state vector. This periodicity of the dynamics implies that the power system exhibits multiple equilibria. Therefore, the construction of Lyapunov functions becomes very complicated. For instance, an application of existing approaches [7] in a state periodic system would require to construct a Lyapunov function, which respects the system periodicity. These strong requirements can often not be met in power systems applications and, consequently, the majority of stability results are merely local [13]. By using a Leonov function instead, we explicitly exploit the periodicity of the system dynamics to relax the (definiteness and periodicity) requirements compared to standard Lyapunov functions. Recently, the Leonov function framework has been extended to a universal formula for ISS and iISS stabilization via the concepts of an ISS- and iISS-Control Leonov Function (ISS- and iISS-CLeF, respectively) [14].

Using the Leonov function method, global stability and robustness properties are derived for a SG connected to a infinite bus in [1], [21]. Likewise, first extensions to provide almost global stability properties of power systems and microgrids are given in [22], [23].

While the above results use the Leonov function framework concern and aiming on the global analysis of system properties, this work is focused on control *design*. More precisely, our main contribution is to apply the ISS-CLeF concept from [14] to confer the standard synchronverter algorithm from [26] with additional robustness properties in the ISS sense. The control is derived and the *global* ISS properties are rigorously established for the case of a microgrid consisting of two synchronverters feeding a common load. Compared to this, the analysis of stability and robustness is only locally proven in [18], [25]. The efficacy of the proposed control is illustrated in a simulation example.

This paper is structured as follows. Some preliminaries on practical ISS- and iISS-CLeFs are presented in Section II. The description of the system and its respective model is developed in Section III. The main result is provided in

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Section IV. A simulation example is shown in Section V. Finally, conclusions and future work are given in Section VI.

II. Preliminaries

A. Notation

We denote the sets of natural, real and integer numbers by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{Z}$, respectively. Let $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$ and $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$. The variable $v \in \mathbb{R}^k$ ($v \in \mathbb{Z}^k$) denotes a vector of real (integer) numbers with $k \in \mathbb{N}$ components, whose infinity norm is represented by $\|v\|_\infty = \max(|v_1|, ..., |v_k|)$ and its Euclidean norm is denoted by $\|v\|_2 = \sqrt{v_1^2 + \cdots + v_n^2}$. The distance from a point $p \in \mathbb{R}^n$ to the set $\mathcal{S} \subset \mathbb{R}^n$ is defined as $\|p\|_{\mathcal{S}} = \inf_{a \in \mathcal{S}} \|p - a\|_2$. The identity matrix is represented by $\mathbf{I}_k \in \mathbb{R}^{k \times k}$. Let $i \in \mathbb{Z}$, then the notations $\mathbf{i}_n$ and $\mathbf{i}_{n \times m}$, $n \in \mathbb{N}$, $m \in \mathbb{N}$, represent a column vector of n elements and a matrix with n rows and m columns, respectively, with all its elements being identical to i. The variable $v \in \mathcal{L}_\infty^k$ with $k \in \mathbb{N}$ denotes a function whose components belong to the set of Lebesgue ∞-space.

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if, for each fixed $t \in \mathbb{R}_+$, $\beta(\cdot, t) \in \mathcal{K}_\infty$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is non-increasing and it tends to zero for $t \to \infty$.

B. Practical ISS- and iISS-Control Leonov Functions

Consider a system of the form

$$\dot{\bar{\theta}} = f_1(\bar{x}, d), \quad \dot{\bar{z}} = f_2(\bar{x}, d) + g(\bar{x})u, \tag{1}$$

where $\bar{x} \in \mathbb{R}^n$ is the state vector, $\bar{x} = (\bar{z}, \bar{\theta}) \in \mathbb{R}^n$ with $\bar{z} \in \mathbb{R}^k$ and $\bar{\theta} \in \mathbb{R}^q$, $n = k + q$ for $k, q > 0$, $u \in \mathcal{L}_\infty^p$ is an admissible control input, and $d \in \mathcal{L}_\infty^m$ is a vector of perturbation signals. Moreover, $f_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^q$, $f_2 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ and $g : \mathbb{R}^n \to \mathbb{R}^{k \times p}$ are assumed to be locally Lipschitz continuous vector functions. Furthermore, define the vector $\xi_j = [\mathbf{0}_k^\top \ 2\pi j^\top]^\top \in \mathbb{R}^n$, $j \in \mathbb{Z}^q$. Let us introduce the following assumptions for the system (1).

Assumption 1: The vector fields f_1, f_2 and g in (1) are 2π-periodic with respect to $\bar{\theta}$.

Assumption 2: Let the set $\bar{\mathcal{W}} \subset \mathbb{R}^n$ contain all α- and ω-limit sets of (1) for $d = \mathbf{0}_m$ and $u = \mathbf{0}_p$. Its projection $\mathcal{W}$, on $M = \mathbb{R}^k \times [0, 2\pi)^q$, is compact and decomposable (in the sense of [7]).

Then, we introduce the following definition.

Definition 1 ([14]): A $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be a practical ISS-CLeF for the system (1) if $\frac{\partial V(\bar{x})}{\partial \bar{z}}$ is 2π-periodic in $\bar{\theta}$, and there exist functions $\underline{\vartheta}, \overline{\vartheta}, \underline{\sigma}, \overline{\sigma}, \varpi, \upsilon \in \mathcal{K}_\infty$ and scalars $\underline{g}, \bar{g}, \zeta \geq 0$, such that for all $\bar{x} = (\bar{z}, \bar{\theta}) \in \mathbb{R}^n$:

$$\underline{\vartheta}(\|\bar{x}\|_{\bar{\mathcal{W}}'}) - \underline{\sigma}(\|\bar{\theta}\|) - \underline{g} \leq V(\bar{x}) \leq \overline{\vartheta}(\|\bar{x}\|_{\bar{\mathcal{W}}'} + \bar{g}) - \overline{\sigma}(\|\bar{\theta}\|), \tag{2}$$

and for all $\bar{x} \in \{\bar{x} \in \mathbb{R}^n \,|\, V(\bar{x}) \geq 0\}$ and all $d \in \mathbb{R}^m$:

$$\inf_{u \in \mathbb{R}^p} \{a(\bar{x}, d) + \beta(\bar{x})u\} \leq -\varpi(V(\bar{x})) + \upsilon(\|d\|) + \zeta, \tag{3}$$

where $a(\bar{x}, d) = \frac{\partial V(\bar{x})}{\partial \theta} f_1(\bar{x}, d) + \frac{\partial V(\bar{x})}{\partial \bar{z}} f_2(\bar{x}, d)$ and $\beta(\bar{x}) = \frac{\partial V(\bar{x})}{\partial \bar{z}} g(\bar{x})$, and $\mathcal{W}'$ (the projection of $\bar{\mathcal{W}}'$ on M) is formed by atoms $\mathcal{W}_i$ of the decomposition of $\mathcal{W}$, such that $(\mathcal{W}_i + \{\xi_j\}) \subset \Omega_1 \cup \Omega_2$ for some $j \in \mathbb{Z}^q$, where $\Omega_1 = \{\bar{x} \in \mathbb{R}^n \,|\, V(\bar{x}) < 0\}$, $\Omega_2 = \{\bar{x} \in \mathbb{R}^n \,|\, \beta(\bar{x}) = 0\}$. Additionally, if (3) holds for $\zeta = 0$, then V is said to be an ISS-CLeF. Moreover, if (3) holds for $\zeta = 0$ and a non-negative function $\varpi : \mathbb{R}_+ \to \mathbb{R}_+$, then V is said to be an iISS-CLeF.

Now, following [14], we introduce the functions

$$\bar{a}(\bar{x}) = \sup_{d \in \mathbb{R}^m} \left(a(\bar{x}, d) - \overline{\upsilon}(\|d\|) - \overline{\zeta}\right), \tag{4}$$

with $\overline{\upsilon}(s) \geq \upsilon(s)$ for all $s \geq 0$ and $\overline{\zeta} \geq \zeta$, and

$$\phi(\|\bar{x}\|_{\bar{\mathcal{W}}}) = \sup_{\overline{\sigma}(\|\bar{\theta}\|) \leq \overline{\vartheta}(\|\bar{x}\|_{\bar{\mathcal{W}}'} + \bar{g})} \left(\bar{a}(\bar{x}) + \frac{1}{2}\varpi(V(\bar{x}))\right). \tag{5}$$

Likewise, in [14] the following variant of the small control property is introduced: for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\sup_{0 < \|\bar{x}\|_\Omega \leq \delta} \frac{\phi(\|\bar{x}\|_{\bar{\mathcal{W}}'}, \bar{z})}{\|\beta(\bar{x})\|^2} < \varepsilon, \tag{6}$$

where $\Omega = \{\bar{x} \in \mathbb{R}^n \,|\, V(\bar{x}) = 0; \beta(\bar{x}) = 0\}$. We write the well-known *Sontag's universal formula* [24] as follows

$$K(\phi(\cdot), \beta(\cdot)) = \begin{cases} \frac{\phi + \sqrt{\phi^2 + \|\beta\|^4}}{\|\beta\|^2} & if \quad \beta \neq 0, \\ 1 & if \quad \beta = 0. \end{cases} \tag{7}$$

Finally, we define the set

$$\mathcal{W}^- = \cup_{i=1}^{l^-} \mathcal{W}_i, \tag{8}$$

such that $(\mathcal{W}_i + \{\xi_j\}) \subset \Omega_1 \setminus \Omega_2$ for some $j \in \mathbb{Z}^q$, $\mathcal{W}_i$ are atoms of the decomposition of $\mathcal{W}$ and $\epsilon \in (0, \bar{\epsilon}]$, where $\bar{\epsilon} > 0$ is the distance between the sets $\{\bar{x} \in \mathbb{R}^n \,|\, V(\bar{x}) = 0\}$ and $\bar{\mathcal{W}}^-$. Then, the following theorem can be stated.

Theorem 1 ([14]): Let the system (1) satisfy Assumptions 1 and 2, and the set $\bar{\mathcal{W}}_{cl}$, which contains all invariant solutions of the closed-loop system, be decomposable. If there exists a (practical) ISS-CLeF (iISS-CLeF) for the system (1) satisfying the inequality (6), then the controller

$$u(\bar{x}) = -\min\left(1, \frac{\|\bar{x}\|_{\bar{\mathcal{W}}^-}}{\epsilon}\right) \bar{K}(\bar{x}) \beta^\top(\bar{x}), \tag{9}$$

with $\bar{K}(\bar{x}) \geq K(\phi(\|\bar{x}\|_{\bar{\mathcal{W}}}), \beta(\bar{x}))$, which is continuous for all $\bar{x} \in \mathbb{R}^n \setminus \Omega_1$, provides the (practical) ISS (iISS) property to the system (1) with respect to $\mathcal{W}_{cl}$.

III. Considered System and Problem Statement

A. Model of two synchronverters coupled via dynamic power lines in abc-coordinates

We consider a small power system composed of two inverters connected in parallel to a load. We assume that both inverters are operated with the standard synchronverter algorithm [26]. Similarly to [25], we assume that the load is purely resistive. So, we obtain the scheme of Figure 1, where for $k = 1, 2$; the parameters $R_k > 0$ and $L_k > 0$ are the line resistance and inductance, respectively, while the variables $\omega_k(t) \in \mathbb{R}$, $i_{k,abc}(t) \in \mathbb{R}^3$, $e_{k,abc}(t) \in \mathbb{R}^3$ represent frequency, currents and voltage, respectively. Likewise,

$R_L > 0$ represents the load resistance and $i_{L_{abc}}(t) \in \mathbb{R}^3$ is the load current and

$$V_{L,abc}(t) = R_L \sum_{k=1}^{2} i_{k,abc}(t) \in \mathbb{R}^3 \tag{10}$$

is the load voltage[1].

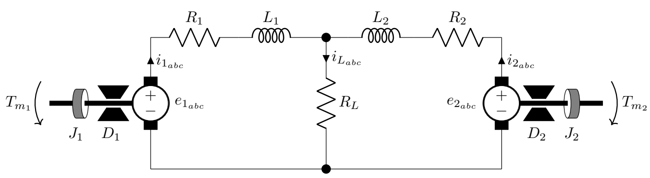


Fig. 1: Two parallel synchronverters feeding a resistive load.

Following [26], the electromotive forces are given by

$$e_{k,abc} = \sqrt{\tfrac{2}{3}} b_k \omega_k \begin{bmatrix} \sin(\delta_k) \\ \sin\left(\delta_k - \frac{2}{3}\pi\right) \\ \sin\left(\delta_k + \frac{2}{3}\pi\right) \end{bmatrix}, \quad b_k = \sqrt{\tfrac{3}{2}} M_{f_k} i_{f_k},$$

for $k = 1, 2$, where δ_k are the phase angles, M_{f_k} is the mutual inductance and i_{f_k} denotes the rotor current of the k-th synchronverter. Then, the dynamics of the k-th synchronverter are given by [26]

$$\begin{aligned} \dot{\delta}_k =& \omega_k, \\ J_k \dot{\omega}_k =& - D_k \omega_k + T_{m_k} - T_{e_k} + \rho_{m_k}, \\ L_k \frac{di_{k,abc}}{dt} =& - R_k i_{k,abc} + e_{k,abc} - V_{L,abc} + \rho_{e_k}, \end{aligned} \tag{11}$$

where the electrical torque T_{e_k} can be expressed as

$$T_{e_k} = \omega_k^{-1} i_{k,abc}^{\top} e_{k,abc}, \tag{12}$$

which is well-defined for $\omega_k = 0$, since the term ω_k^{-1} cancels out with ω_k in $e_{k,abc}$. The positive parameters J_k as well as D_k are the virtual inertia and damping coefficients, respectively. The signal $T_{m_k}(t) \in \mathbb{R}$ represents a virtual torque, which is usually taken as control input for active power and frequency control [26]. The signal $\rho_{m_k}(t) \in \mathbb{R}$ represents perturbations arising in the implementation of the synchronverter algorithm, such as measurement noise or oscillations pertaining to harmonics. Finally, $\rho_{e_k}(t) \in \mathbb{R}^3$ is an external perturbations vector, which can represent, e.g., additional, unmodeled load variations.

We aim at designing a control law that confers the synchronverter dynamics (11) with improved robustness properties. Similarly to [25], we write the control input T_{m_k} as

$$T_{m_k} = \bar{T}_{m_k} + u_k, \tag{13}$$

where $\bar{T}_{m_k}$ is a positive constant representing the nominal torque, and u_k is a free control input for $k = 1, 2$.

Remark 1: The current i_f could be used as an additional control input for reactive power and voltage control [26]. In the present work, we follow [25] and assume that i_f is constant, see also [26].

[1] In the sequel, the time-dependency of signals is omitted, whenever it is clear from the context.

B. Model of two synchronverters coupled via dynamic power lines in dq-coordinates

For the subsequent derivations, it is convenient to represent the system (11) in dq-coordinates [20]. Thus, let $\phi(t) \in \mathbb{R}$ and consider the dq-transformation matrix defined as follows

$$T_{dq}(\phi) = \sqrt{\tfrac{2}{3}} \begin{bmatrix} \cos(\phi) & \cos\left(\phi - \frac{2}{3}\pi\right) & \cos\left(\phi + \frac{2}{3}\pi\right) \\ \sin(\phi) & \sin\left(\phi - \frac{2}{3}\pi\right) & \sin\left(\phi + \frac{2}{3}\pi\right) \end{bmatrix}, \tag{14}$$

as well as the rotation matrix

$$T_r(\phi_1 - \phi_2) = \begin{bmatrix} \cos(\phi_1 - \phi_2) & -\sin(\phi_1 - \phi_2) \\ \sin(\phi_1 - \phi_2) & \cos(\phi_1 - \phi_2) \end{bmatrix}, \tag{15}$$

which allows to transform dq-variables from a dq-frame with $\phi = \phi_1$ to another dq-frame with $\phi = \phi_2$, see [20].

By applying (14) with $\phi = \delta_k$ to the k-th synchronverter, the model (11) can be written in its local dq-coordinates as

$$\begin{aligned} \dot{\delta}_k =& \omega_k, \\ J_k \dot{\omega}_k =& - D_k \omega_k + \bar{T}_{m_k} + u_k - T_{e_k} + \rho_{m_k}, \\ L_k \frac{di_{k,dq}}{dt} =& \left[L_k \mathcal{J}(\omega_k) - R_k \mathbf{I}_2\right] i_{k,dq} + e_{k,dq} - \\ & T_{dq}(\delta_k) V_{L,abc} + \mu_{e_k}, \end{aligned} \tag{16}$$

where $\mu_{e_k} \in \mathbb{R}^2$ is the transformation of ρ_{e_k} and

$$\mathcal{J}(\omega_k) = \begin{bmatrix} 0 & -\omega_k \\ \omega_k & 0 \end{bmatrix}, \quad e_{k,dq} = b_k \omega_k \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

So, electrical torque and load voltage result in $T_{e_k} = b_k i_{k,q}$ and $T_{dq}(\delta_k) V_{L,abc} = R_L \sum_{j=1}^{2} T_r(\delta_k - \delta_j)\, i_{j,dq}$.

To combine both synchronverter models (16) in a single system, we define the difference between angles $\theta = \delta_1 - \delta_2$, and the state vectors $z_k = \begin{bmatrix} \omega_k & i_{k,d} & i_{k,q} \end{bmatrix}^{\top}$, for $k = 1, 2$. Therefore, the model for the whole system of Figure 1 can be written compactly as

$$\begin{aligned} \Gamma_1 \dot{z}_1 =& A_1(\omega_1) z_1 - B(\theta) z_2 + \hat{e}_1 T_{m_1} + \mu_1, \\ \Gamma_2 \dot{z}_2 =& A_2(\omega_2) z_2 - B(\theta)^{\top} z_1 + \hat{e}_1 T_{m_2} + \mu_2, \\ \dot{\theta} =& \omega_1 - \omega_2, \end{aligned} \tag{17}$$

where $\hat{e}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^{\top}$, $\mu_k = \begin{bmatrix} \rho_{m_k} & \mu_{e_k} \end{bmatrix}^{\top}$, and

$$A_k(\omega_k) = \begin{bmatrix} -D_k & 0 & -b_k \\ 0 & -(R_k + R_L) & -L_k \omega_k \\ b_k & L_k \omega_k & -(R_k + R_L) \end{bmatrix},$$

$$B(\theta) = R_L \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}, \Gamma_k = \begin{bmatrix} J_k & 0 & 0 \\ 0 & L_k & 0 \\ 0 & 0 & L_k \end{bmatrix}.$$

Following [25], we assume that the two synchronverters have identical parameters, such that $\Gamma_k = \Gamma$. Likewise, recall that $T_{m_k} = \bar{T}_{m_k} + u_k$. Hence, we obtain

$$\begin{aligned} \Gamma \dot{z}_1 =& \hat{A}(\omega_1) z_1 - B(\theta) z_2 + \hat{e}_1 \bar{T}_m + \hat{e}_1 u_1 + \mu_1 \\ \Gamma \dot{z}_2 =& \hat{A}(\omega_2) z_2 - B(\theta)^{\top} z_1 + \hat{e}_1 \bar{T}_m + \hat{e}_1 u_2 + \mu_2 \\ \dot{\theta} =& \omega_1 - \omega_2, \end{aligned} \tag{18}$$

where

$$\hat{A}(\omega_k) = \begin{bmatrix} -D & 0 & -b \\ 0 & -(R + R_L) & -L\omega_k \\ b & L\omega_k & -(R + R_L) \end{bmatrix}, \quad k = 1, 2.$$

C. Equilibrium set

Now, the equilibria of the unforced nominal system (18) (i.e., with $u_1 = u_2 = 0$ and $\mu_1 = \mu_2 = \mathbf{0}_2$) are given by

$$\begin{aligned} \omega_1 - \omega_2 = 0, \quad \hat{A}(\omega_1) z_1 - B(\theta) z_2 + \hat{e}_1 \bar{T}_m = \mathbf{0}_3, \\ \hat{A}(\omega_2) z_2 - B(\theta)^\top z_1 + \hat{e}_1 \bar{T}_m = \mathbf{0}_3. \end{aligned} \tag{19}$$

Based on [8], [25], we consider the following assumption.

Assumption 3: All solutions of (19) belong to the set

$$S_E = S_{E^*} \cup S_{E^{**}}, \tag{20}$$

where

$$\begin{aligned} S_{E^*} &= \left\{[z_1, z_2, \theta] \in \mathbb{R}^7 \,|\, z_1 = z_2 = z^*, \, \theta = 2j\pi \right\}, \\ S_{E^{**}} &= \left\{[z_1, z_2, \theta] \in \mathbb{R}^7 \,|\, z_1 = z_2 = z^{**}, \, \theta = (2j-1)\pi \right\}, \end{aligned}$$

for $j \in \mathbb{Z}$. Moreover, S_{E^*} corresponds to the locally stable equilibrium points and $S_{E^{**}}$, contains the unstable ones.

Remark 2: Assumption 3 is reasonable, because we are considering identical parameters as [25]. In general, the system may also possess other equilibria.

Under Assumption 3, we denote $(z^*, 0)$ as the reference equilibrium and the shifted state vector is defined as follows

$$\tilde{x} = \begin{bmatrix} \tilde{z}_1 & \tilde{z}_2 & \tilde{\theta} \end{bmatrix}^\top = \begin{bmatrix} z_1 - z^* & z_2 - z^* & \theta \end{bmatrix}^\top. \tag{21}$$

Then, the shifted system can be written as

$$\begin{aligned} \Gamma \dot{\tilde{z}}_1 &= \tilde{A}(\tilde{z}_{1,1}) \tilde{z}_1 - B(\tilde{\theta}) \tilde{z}_2 - \tilde{B}(\tilde{\theta}) z^* + \hat{e}_1 u_1 + \mu_1, \\ \Gamma \dot{\tilde{z}}_2 &= \tilde{A}(\tilde{z}_{2,1}) \tilde{z}_2 - B(\tilde{\theta})^\top \tilde{z}_1 - \tilde{B}(\tilde{\theta})^\top z^* + \hat{e}_1 u_2 + \mu_2, \\ \dot{\tilde{\theta}} &= \hat{e}_1^\top (\tilde{z}_1 - \tilde{z}_2), \end{aligned} \tag{22}$$

where $\tilde{B}(\tilde{\theta}) = B(\tilde{\theta}) - B(0)$ and

$$\tilde{A}(\tilde{z}_{k,1}) = \begin{bmatrix} -D & 0 & -\bar{b} \\ -Li_q^* & -(R+R_L) & -L(\tilde{z}_{k,1} + \omega^*) \\ \bar{b} + Li_d^* & L(\tilde{z}_{k,1} + \omega^*) & -(R+R_L) \end{bmatrix}.$$

Hence, all invariant solutions of the system (22) are contained in the set

$$\begin{aligned} \tilde{\mathcal{W}} &= \tilde{\mathcal{W}}_1 \cup \tilde{\mathcal{W}}_2, \\ \tilde{\mathcal{W}}_2 &= \{\tilde{x} \in \mathbb{R}^7 \,|\, \tilde{z}_1 = \tilde{z}_2 = z^{**} - z^*, \, \tilde{\theta} = (2j+1)\pi\}, \\ \tilde{\mathcal{W}}_1 &= \{\tilde{x} \in \mathbb{R}^7 \,|\, \tilde{z}_1 = \tilde{z}_2 = \mathbf{0}_3, \, \tilde{\theta} = 2j\pi\}, \quad j \in \mathbb{Z}. \end{aligned} \tag{23}$$

Now, we formulate the problem addressed in this work.

Problem 1: Consider the system (22). Design a feedback control law for u_k, $k = 1, 2$, such that the closed-loop set $\tilde{\mathcal{W}}_{cl} \supset \tilde{\mathcal{W}}_1$ is *globally* ISS with respect to μ_1 and μ_2.

IV. ISS-Control Leonov Function Design

In this section, we present a control law, which solves Problem 1. Since the system dynamics (22) are periodic with respect to the variable $\tilde{\theta}$, Problem 1 is hard to solve using ISS methods based on conventional Lyapunov theory, see also the discussion in [5], [6], [19]. Thus, to design the controller, we follow our recent work [14].

For the sake of a compact presentation of the subsequent derivations, we introduce the following variables

$$\begin{aligned} \sigma_1 &= (2(R + R_L) - \lambda) p_2 - R_L \| (z_2^*, z_3^*) \|^2, \\ \sigma_c &= 2p_2 R_L \cos(\tilde{\theta}), \quad \sigma_s = 2p_2 R_L \sin(\tilde{\theta}), \\ \sigma_1^* &= \sigma_1 + 0.5\lambda p_2, \quad \sigma_{k,2}^* = p_{k,1} b - p_2 (L i_d^* + b), \\ \sigma_k^{**} &= (2D - 0.5\lambda) p_{k,1} - 8\epsilon\lambda^{-1}, \end{aligned} \tag{24}$$

for $p_{1,1} > 0$, $p_{2,1} > 0$, $p_2 > 0$, $\lambda > 0$, $\epsilon > 0$ and $k = 1, 2$.

A. Preliminary lemmata

In order to verify the required properties of a ISS-CLeF for the system (22), we introduce the matrices

$$\tilde{M}_S = \begin{bmatrix} M_{S,1} & \mathbf{0}_{4\times1} \\ \mathbf{0}_{1\times4} & \lambda\epsilon - 4p_2^2 R_L \end{bmatrix}, \; \tilde{M}_{S,1} = \begin{bmatrix} \sigma_1 \mathbf{I}_2 & \tilde{M}_\sigma \\ \tilde{M}_\sigma^\top & \sigma_1 \mathbf{I}_2 \end{bmatrix}, \tag{25}$$

$$\tilde{M}_{S,1}^* = \begin{bmatrix} \sigma_1^* \mathbf{I}_2 & \tilde{M}_\sigma \\ \tilde{M}_\sigma^\top & \sigma_1^* \mathbf{I}_2 \end{bmatrix}, \quad \tilde{M}_\sigma = \begin{bmatrix} \sigma_c & -\sigma_s \\ \sigma_s & \sigma_c \end{bmatrix}, \tag{26}$$

and

$$\hat{M}^* = \begin{bmatrix} \left[\frac{8\epsilon}{\lambda}\hat{e}_1 \; \mathbf{0}_{3\times2}\right] & \mathbf{0}_{3\times3} & \epsilon\hat{e}_1 \\ \mathbf{0}_{3\times3} & \left[\frac{8\epsilon}{\lambda}\hat{e}_1 \; \mathbf{0}_{3\times2}\right] & -\epsilon\hat{e}_1 \\ \epsilon\hat{e}_1^\top & -\epsilon\hat{e}_1^\top & \frac{\lambda\epsilon}{2} - 4p_2^2 R_L \end{bmatrix}, \tag{27}$$

to present the following three lemmata.

Lemma 1: The matrix $\tilde{M}_S$ is positive definite for any $\tilde{\theta} \in \mathbb{S}$ if and only if

$$\epsilon > 4\lambda^{-1} p_2^2 R_L, \; \lambda > p_2^{-1} \left(2p_2 R - R_L \| (z_2^*, z_3^*) \|^2\right). \tag{28}$$

Lemma 2: The matrix $\hat{M}^*$ is positive semidefinite for any $\tilde{\theta} \in \mathbb{S}$ if and only if

$$\epsilon \geq 16\lambda^{-1} p_2^2 R_L. \tag{29}$$

Lemma 3: The matrix $\tilde{M}_{S,1}^*$ is positive definite for any $\tilde{\theta} \in \mathbb{S}$ if (28) is satisfied.

Due to space limitations, the previous proofs are omitted. However, they are based on Schur's complement.

B. Main Result

To present the main result, we define

$$\bar{K}(\bar{\Phi}, \beta) = \begin{cases} \frac{\bar{\Phi} + \sqrt{\bar{\Phi}^2 + \|\beta\|^4}}{\|\beta\|^2} & \text{if} \quad \|\beta\| \neq 0 \\ 1 & \text{if} \quad \|\beta\| = 0 \end{cases}, \tag{30}$$

where

$$\begin{aligned} \beta(\tilde{z}_1, \tilde{z}_2) &= 2 \begin{bmatrix} p_{1,1} \tilde{z}_{1,1} & p_{2,1} \tilde{z}_{2,1} \end{bmatrix}, \\ \bar{\Phi}(\tilde{z}_1, \tilde{z}_2) &= - \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix}^\top \tilde{M}^* \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix}, \end{aligned} \tag{31}$$

with

$$\tilde{M}^* = \begin{bmatrix} \sigma_1^{**} & Lp_2 i_q^* & \sigma_{1,2}^* & 0 & 0 & 0 \\ Lp_2 i_q^* & \sigma_1^* & 0 & 0 & \sigma_c & -\sigma_s \\ \sigma_{1,2}^* & 0 & \sigma_1^* & 0 & \sigma_s & \sigma_c \\ 0 & 0 & 0 & \sigma_2^{**} & Lp_2 i_q^* & \sigma_{2,2}^* \\ 0 & \sigma_c & \sigma_s & Lp_2 i_q^* & \sigma_1^* & 0 \\ 0 & -\sigma_s & \sigma_c & \sigma_{2,2}^* & 0 & \sigma_1^* \end{bmatrix}. \tag{32}$$

In line with Problem 1, we are interested in robustifying the set of *stable* equilibria, i.e., $\tilde{\mathcal{W}}_1$. This is a particular application of the approach presented in [14], which leads to our main result of this work. We stress that the control may lead to new, additional equilibria of the closed-loop system and, thus, $\tilde{\mathcal{W}}_{cl} \supset \tilde{\mathcal{W}}_1$.

Theorem 2: Choose $p_{1,1} > 0$, $p_{2,1} > 0$, $\epsilon > 0$, $\lambda > 0$ and $p_2 > 0$ in (24), such that (28) and (29) are satisfied. Assume that the set $\mathcal{W}_{cl}$ is decomposable. Then, the control law

$$u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^\top = -\bar{K}(\tilde{z}_1, \tilde{z}_2)\, \beta^\top(\tilde{z}_1, \tilde{z}_2). \tag{33}$$

ensures that the system (22) is ISS with respect to the closed-loop set $\tilde{\mathcal{W}}_{cl} \supset \tilde{\mathcal{W}}_1$ and to the external inputs μ_1 and μ_2.

Proof: We propose the following ISS-CLeF candidate

$$V(\tilde{x}) = \tilde{z}_1^\top P_1 \Gamma \tilde{z}_1 + \tilde{z}_2^\top P_2 \Gamma \tilde{z}_2 - \epsilon \tilde{\theta}^2, \quad (34)$$

where the matrices $P_1 = \operatorname{diag}(p_{1,1}, p_{1,2}, p_{1,3}) > 0$ and $P_2 = \operatorname{diag}(p_{2,1}, p_{2,2}, p_{2,3}) > 0$ and $\epsilon \in \mathbb{R}_+$ are design parameters. Since all its parameters are positive, we can readily see that the function (34) satisfies condition (2) with, see (23),

$$\bar{\mathcal{W}}' = \tilde{\mathcal{W}}_2 = \{\tilde{x} \in \mathbb{R}^7 \mid \tilde{z}_1 = \tilde{z}_2 = \mathbf{0}_3,\ \tilde{\theta} = 2j\pi\}. \quad (35)$$

Now, defining $\tilde{\mu}^\top = [\mu_1^\top, \mu_2^\top]$, the derivative of (34) along the trajectories of the system (22) is given by

$$\dot{V} = a(\tilde{x}, \tilde{\mu}) + \beta(\tilde{z}_1, \tilde{z}_2)\, u, \quad (36)$$

where

$$\begin{aligned} a(\cdot,\cdot) = &- \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix}^\top \begin{bmatrix} -\tilde{A}_{P_{1_\text{sym}}}(\tilde{z}_{1,1}) & B_P(\tilde{\theta}) \\ B_P(\tilde{\theta})^\top & -\tilde{A}_{P_{2_\text{sym}}}(\tilde{z}_{2,1}) \end{bmatrix} \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} - \\ & 2\Big(\tilde{z}_1^\top P_1 \tilde{B}(\tilde{\theta}) + \tilde{z}_2^\top P_2 \tilde{B}(\tilde{\theta})^\top\Big) z^* + \\ & 2\epsilon\left(\tilde{z}_2^\top - \tilde{z}_1^\top\right) \hat{e}_1 \tilde{\theta} + 2\begin{bmatrix} \tilde{z}_1 & \tilde{z}_2 \end{bmatrix} \beta_\mu \tilde{\mu}, \\ \beta(\cdot,\cdot) = & 2\begin{bmatrix} p_{1,1}\tilde{z}_{1,1} & p_{2,1}\tilde{z}_{2,1} \end{bmatrix}, \quad \beta_\mu = \frac{1}{2}\begin{bmatrix} \Gamma P_1 & \mathbf{0}_{3\times 2} \\ \mathbf{0}_{3\times 2} & \Gamma P_2 \end{bmatrix}, \end{aligned}$$

with $B_P(\tilde{\theta}) = P_1 B(\tilde{\theta}) + B(\tilde{\theta}) P_2$ and $\tilde{A}_{P_{k_\text{sym}}}(\tilde{z}_{k,1}) = P_k \tilde{A}(\tilde{z}_{k,1}) + \tilde{A}(\tilde{z}_{k,1})^\top P_k$, for $k = 1, 2$.

Analyzing the terms in (36) that do not depend on u, for any positive constant $\upsilon > 0$, we have

$$\begin{aligned} a(\tilde{x}, \tilde{\mu}) + \lambda V(\tilde{x}) - \upsilon\|\tilde{\mu}\|^2 = 2\begin{bmatrix} \tilde{z}_1 & \tilde{z}_2 \end{bmatrix} \beta_\mu \tilde{\mu} - \upsilon \tilde{\mu}^\top \mathbf{I}_6 \tilde{\mu} \\ - \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix}^\top \begin{bmatrix} -\tilde{A}_{P_{1_\text{sym}}}(\tilde{z}_{1,1}) & B_P(\tilde{\theta}) \\ B_P(\tilde{\theta})^\top & -\tilde{A}_{P_{2_\text{sym}}}(\tilde{z}_{2,1}) \end{bmatrix} \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} - \\ 2\left(\tilde{z}_1^\top P_1 \tilde{B}(\tilde{\theta}) + \tilde{z}_2^\top P_2 \tilde{B}(\tilde{\theta})^\top\right) z^* + 2\epsilon\left(\tilde{z}_2^\top - \tilde{z}_1^\top\right) \hat{e}_1 \tilde{\theta} \\ + \lambda\left(\tilde{z}_1^\top P_1 \tilde{z}_1 + \tilde{z}_2^\top P_2 \tilde{z}_2 - \epsilon\tilde{\theta}^2\right). \end{aligned}$$

For the next step, it is important to stress that, in general, $\tilde{A}_{P_{k_\text{sym}}}(\tilde{z}_{k,1})$ depends on $\tilde{z}_{k,1}$, $k = 1, 2$. This dependency generates cross terms of order 3, which cannot be compensated by the quadratic terms. To avoid this issue and taking advantage of the structure of the system (22), we choose $p_{1,3} = p_{1,2}$ and $p_{1,3} = p_{1,2}$, so that all cross terms of order 3 are canceled. Likewise, we fix $p_{1,2} = p_{2,2} = p_2$ to reduce some terms and using trigonometric properties, we obtain

$$\begin{aligned} \tilde{B}(\tilde{\theta}) z^* &= R_L \|(z_2^*, z_3^*)\| \begin{bmatrix} 0 & \Delta_c(\tilde{\theta}, \phi_i) & \Delta_s(\tilde{\theta}, \phi_i) \end{bmatrix}^\top, \\ \tilde{B}(\tilde{\theta})^\top z^* &= R_L \|(z_2^*, z_3^*)\| \begin{bmatrix} 0 & \Delta_c(\tilde{\theta}, -\phi_i) & \Delta_s(\tilde{\theta}, -\phi_i) \end{bmatrix}^\top, \end{aligned}$$

where $\Delta_c(\tilde{\theta}, \phi_i) = \cos(\tilde{\theta} + \phi_i) - \cos(\phi_i)$ and $\Delta_s(\tilde{\theta}, \phi_i) = \sin(\tilde{\theta} + \phi_i) - \sin(\phi_i)$ with $\phi_i = \arctan\left(\frac{z_3^*}{z_2^*}\right)$. Since

$$\left|\Delta_c(\tilde{\theta}, \pm\phi_i)\right| \le \min(|\tilde{\theta}|, 1), \ \left|\Delta_s(\tilde{\theta}, \pm\phi_i)\right| \le \min(|\tilde{\theta}|, 1),$$

we have

$$\tilde{z}_k^\top P \tilde{B}(\tilde{\theta}) z^* \le p_2 R_L \|(z_2^*, z_3^*)\|\, |\tilde{z}_{k,2} + \tilde{z}_{k,3}|\, |\tilde{\theta}|, \quad k = 1, 2.$$

Using Young's inequality and the triangle inequality, we can bound $a(\tilde{x}, \tilde{\mu})$ in (36) as follows

$$a(\tilde{x}, \tilde{\mu}) \le -\lambda V(\tilde{x}) + \upsilon\|\tilde{\mu}\|^2 - \begin{bmatrix} \tilde{x} \\ \tilde{\mu} \end{bmatrix}^\top \begin{bmatrix} \tilde{M} & \tilde{\beta}_\mu \\ \tilde{\beta}_\mu^\top & \upsilon \mathbf{I}_6 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{\mu} \end{bmatrix}, \quad (37)$$

where $\tilde{\beta}_\mu = \left[\beta_\mu^\top, \mathbf{0}_{6\times 1}\right]^\top$ and

$$\tilde{M} = \begin{bmatrix} D_{\lambda_1} & Lp_2 i_q^* & \sigma_{1,2}^* & 0 & 0 & 0 & \epsilon \\ Lp_2 i_q^* & \sigma_1 & 0 & 0 & \sigma_c & -\sigma_s & 0 \\ \sigma_{1,2}^* & 0 & \sigma_1 & 0 & \sigma_s & \sigma_c & 0 \\ 0 & 0 & 0 & D_{\lambda_2} & Lp_2 i_q^* & \sigma_{2,2}^* & -\epsilon \\ 0 & \sigma_c & \sigma_s & Lp_2 i_q^* & \sigma_1 & 0 & 0 \\ 0 & -\sigma_s & \sigma_c & \sigma_{2,2}^* & 0 & \sigma_1 & 0 \\ \epsilon & 0 & 0 & -\epsilon & 0 & 0 & \epsilon_\lambda \end{bmatrix},$$

with $D_{\lambda_1} = (2D - \lambda)\, p_{1,1}$, $D_{\lambda_2} = (2D - \lambda)\, p_{2,1}$, $\epsilon_\lambda = \lambda\epsilon - 4p_2^2 R_L$, and the auxiliary variables defined in (24).

To show that condition (3) of Definition 1 is verified, we consider the set, where β in (37) is zero, *i.e.*, the set

$$S = \left\{\tilde{x} \in \mathbb{R}^7 \mid \|\beta(\tilde{z}_1, \tilde{z}_2)\| = 0\right\}. \quad (38)$$

Then, we have from (37) that

$$\begin{aligned} a(\tilde{x}, \tilde{\mu})|_{\tilde{x}\in S} \le -\lambda\, V(\tilde{x})|_{\tilde{x}\in S} + \upsilon\|\tilde{\mu}\|^2 - \upsilon\tilde{\mu}_m^\top \tilde{\mu}_m - \\ \begin{bmatrix} \tilde{x}_S \\ \tilde{\mu}_e \end{bmatrix}^\top \begin{bmatrix} \tilde{M}_S & \tilde{\beta}_S \\ \tilde{\beta}_S^\top & \upsilon \mathbf{I}_4 \end{bmatrix} \begin{bmatrix} \tilde{x}_S \\ \tilde{\mu}_e \end{bmatrix}, \quad (39) \end{aligned}$$

where $\tilde{x}_S = \left[\tilde{z}_{1,2}, \tilde{z}_{1,3}, \tilde{z}_{2,2}, \tilde{z}_{2,3}, \tilde{\theta}\right]^\top$, $\tilde{M}_S$ is defined in (25), $\tilde{\mu}_m = \left[\rho_{m_1}, \rho_{m_2}\right]^\top$, $\tilde{\mu}_e = \left[\mu_{e_1}^\top, \mu_{e_2}^\top\right]^\top$ and $\tilde{\beta}_S = \frac{1}{2}Lp_2\left[\mathbf{I}_4 \quad \mathbf{0}_{1\times 4}\right]^\top$. By invoking Lemma 1, under the required conditions by the theorem, the matrix $\tilde{M}_S$ is positive definite. Thus, using the Schur complement for $\upsilon \ge \frac{L^2 p_2^2}{4\lambda_{\min}(\tilde{M}_s)}$, we conclude that condition (3) of Definition 1 is satisfied. Hence, (34) is an ISS-CLeF for the system (22).

The last step of the proof is to show that the controller (33) provides the ISS property in closed-loop with the system (22) with respect to $\tilde{\mathcal{W}}_1$. For this, we use Theorem 1 and the fact that $\bar{\mathcal{W}}' = \tilde{\mathcal{W}}_1$. It, thus, remains to verify condition (6).

Consider the following function

$$\Phi(\tilde{z}_1, \tilde{z}_2) = \sup_{V\ge 0}\left(\bar{a}(\tilde{x}) + \frac{\lambda}{2}V(\tilde{x})\right) \le \sup_{V\ge 0}\left(-\tilde{x}^\top M^* \tilde{x}\right),$$

where $\bar{a} = \sup_{\tilde{\mu}\in\mathbb{R}^6}\left(a(\tilde{x}, \tilde{\mu}) - \upsilon\|\tilde{\mu}\|^2\right)$, and $M^* = \hat{M}^* + \tilde{M}^*$, with $\hat{M}^*$ and $\tilde{M}^*$ defined in (27) and (32), respectively. By Lemma 2, the matrix $\hat{M}^*$ is positive semidefinite if (29) is satisfied. Therefore, the previous inequality is reduced to

$$\Phi(\tilde{z}_1, \tilde{z}_2) \le \bar{\Phi}(\tilde{z}_1, \tilde{z}_2). \quad (40)$$

The functions $\bar{\Phi}(\tilde{z}_1, \tilde{z}_2)$ and $\beta(\tilde{z}_1, \tilde{z}_2)$ given in (31) are substituted in the formula (30) to define the controller (33). Clearly the 2π-periodicity follows trivially. So, we only have to check the continuity of the formula of (30) for all $\tilde{x} \in \mathbb{R}^7$.

We have that $\bar{\Phi}(\tilde{z}_1, \tilde{z}_2)$ and $\beta(\tilde{z}_1, \tilde{z}_2)$ are continuous by construction. Hence, (30) is also continuous for $\|\beta(\tilde{z}_1, \tilde{z}_2)\| \ne 0$. To analyze the case $\|\beta(\tilde{z}_1, \tilde{z}_2)\| = 0$, recall the property $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ for any $a, b \ge 0$. Hence,

$$\bar{K}(\bar{\Phi}, \beta) \le \left(\bar{\Phi} + |\bar{\Phi}|\right)\|\beta\|^{-2} + 1. \quad (41)$$

Recall the set S in (38). On S, we have from (31) that

$$\bar{\Phi}(\tilde{z}_1, \tilde{z}_2)\big|_{\tilde{x} \in S} \leq -\zeta \tilde{M}_S^* \zeta, \quad \zeta = \begin{bmatrix} \tilde{z}_{1,2} & \tilde{z}_{1,3} & \tilde{z}_{2,2} & \tilde{z}_{2,3} \end{bmatrix}^\top,$$

where $\tilde{M}_S^*$ is presented in (26). By using Lemma 3, under the conditions of the theorem, $\tilde{M}_S^*$ is positive definite. So, $\|\beta(\tilde{z}_1, \tilde{z}_2)\| = 0 \implies \bar{\Phi}(\tilde{z}_1, \tilde{z}_2) \leq 0$. Moreover, $\|\beta(\tilde{z}_1, \tilde{z}_2)\|$ and $\bar{\Phi}(\tilde{z}_1, \tilde{z}_2)$ are zero simultaneously only at the origin and $\bar{\Phi}(\tilde{z}_1, \tilde{z}_2)$ and $\|\beta(\tilde{z}_1, \tilde{z}_2)\|^2$ are quadratic polynomial-like functions, so (6) is satisfied for some $\varepsilon > 0$ and for all $\tilde{x} \in \mathbb{R}^7$. From (41), we can see that $\bar{K}(\bar{\phi}, \beta)\big|_S \leq 1 + \varepsilon$ for all $\tilde{x} \in \mathbb{R}^7$. So, function (30) is continuous for all $\tilde{x} \in \mathbb{R}^7$.

Finally, since $\tilde{\mathcal{W}}_1 = \tilde{\mathcal{W}}' \subset \tilde{\mathcal{W}}_{cl}$, the controller (33) vanishes at every element of the set $\tilde{\mathcal{W}}_1$. Thus, the set $\tilde{\mathcal{W}}_1$ is kept and the function (34) is an ISS-Leonov function for the closed-loop system (22), (33) with respect to $\tilde{\mathcal{W}}_{cl}$. ■

V. Simulation Results

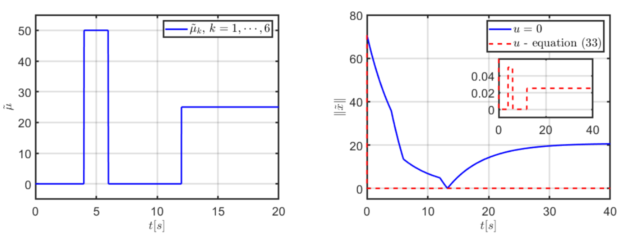
For the system in Figure 1, we consider identical piecewise constant perturbations, see Figure 2a. We compare the behavior of the system with and without the controller (33) under initial condition as $\mathbf{0}_7$.

Based on [16], we consider two identical synchronverters (18), whose parameters are $J = 10[kg \cdot m^2/rad]$, $D = 1.7[N \cdot m/(rad/s)]$, $T_m = 85.04[N \cdot m]$, $b = \sqrt{3/2}[V \cdot s]$, $R = 0.75[\Omega]$, $L = 2.2[mH]$ and $R_L = 1000[\Omega]$.

By numerical computation, we can determine that the equilibria of the unforced system (18) are given by

$$z_1^* = z_2^* = \begin{bmatrix} 50, & -1.68 \times 10^{-5}, & 3.06 \times 10^{-2} \end{bmatrix}^\top, \quad \theta^* = 0;$$
$$z_1^{**} = z_2^{**} = [23.04, \ -2.53, \ 37.45]^\top, \quad \theta^{**} = \pi.$$

By using a mathematical software, we verify that the Jacobian A^* is Hurwitz and A^{**} has one eigenvalue with positive real part. Hence, Assumption 3 is satisfied. Likewise, we calculate the control law (33) with $\tilde{x}$. By fixing $p_2 = 1$, the inequalities (28) and (29) are satisfied for $\lambda = 0.5629$ and $\epsilon = 28423$. So, we fix $p_{2,1} = p_{1,1}$, such that $\sigma_k^{**} = 1$.



(a) Behavior of perturbations. (b) Norm of states $\tilde{x}$.

In Figure 2b, the norm of the state $\tilde{x}$ is shown. We see that in both cases, the trajectories converge to zero but the trajectories for the system with controller (33) converge much faster. Likewise, we can verify that in presence of perturbations, the final norm of the state is by far smaller, when the controller (33) is used.

VI. Conclusions and Future Work

We designed an ISS controller, which provides robustness to two identical synchronverters connected in parallel with a resistive load. The closed-loop system is ISS against external perturbations in the frequency and current dynamics.

The control was designed by using the novel theory of ISS-CLeFs. In comparison to the classic control Lyapunov function, this new methodology relaxes the restrictions to design the control function by exploiting the periodicity of the system dynamics. This allows us to obtain the controller by analyzing a reduced system instead of the whole system.

Future work is geared towards extending the presented control design approach to other settings in low-inertia power systems, such as multiple, non-identical synchronverters, different types of loads and other type of inverter controls. Also, we plan to implement and validate our proposed control in our power-hardware-in-the-loop laboratory [10].

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