2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Distributed Algorithms for *Edge*-Agreements: More than Consensus

Ayush Rai and Shaoshuai Mou

Abstract— **In this paper, we propose distributed algorithms for multi-agent systems to achieve edge-agreements. Different from consensus, where all agents' states converge to be the same value, the edge agreement is characterized by linear constraints defined for edges, i.e. one linear constraint involving two neighboring agents' states for each edge. Such agreement allows more general coordination among agents, with consensus on a special case. Given the underlying graph of the multi-agent system is undirected (not necessarily to be connected), we propose two discrete-time distributed algorithms that enable all agents' states to converge to constants satisfying edge agreements. Besides theoretical proofs, effectiveness of the proposed algorithms is also shown by simulations on a four-agent multi-agent system.**

I. INTRODUCTION

Distributed algorithms to coordinate multi-agent systems (MAS) have recently attracted a significant amount of research attention, which aim to accomplish global objectives only through local coordination [1]. In order to guarantee all agents in MAS work as a cohesive whole, the concept of *consensus*, which drives all agents in the MAS to reach an agreement regarding a certain quantity [2]–[4], has naturally arisen and started to serve as a basis for developing distributed algorithms for MAS. Based on the idea of consensus, a lot of distributed algorithms for MAS have been developed in the past decade such as distributed algorithms for achieving a global average [5], [6], solving linear algebraic equations [7]–[11], multi-agent optimization [12]–[17], multi-agent formation control [18], [19], and multi-agent reinforcement learning [20]–[23] . With so many distributed algorithms developed based on consensus, they are designed only for scenarios when all agents need to reach the same value regarding a specific quantity. Recognition of this has motivated us the goal of this paper to investigate coordination among agents beyond consensus.

Different from consensus, which enforces a global constraint to all agents in the whole MAS, we propose to consider a group of *edge agreements*, i.e. linear constraints defined for nearby neighbor agents, with each pair of neighboring agents corresponding to one such constraint. Note right away that such edge agreements are defined locally, providing a higher degree of flexibility compared to global consensus, and encompassing global consensus as a particular instance. They can deal with cases when only local coordination is needed as shown in later-on simulations of using edge agreements to define multi-agent formations and other local coordination. Second, the global consensus could be implemented as a special case as edge agreements, as shown later in the problem formulation. In the following, we will develop two discrete-time and distributed algorithms to achieve edge agreements followed by analytical proofs and numerical simulations.

Notations: The transpose for vectors and matrices is denoted by $(\cdot)'$, while $\ker(\cdot)$ and $\mathrm{image}\ (\cdot)$ denote the kernel and image of a matrix, respectively. A column stack of vectors $x_i, i = 1, 2, \ldots, r$ is denoted as $\mathrm{col}\,\{x_1, \ldots, x_r\}$, and $\mathrm{diag}\,\{A_1, \ldots, A_r\}$ represents a block diagonal matrix with $A_i, i = 1, 2, \ldots, r$ as the ith block diagonal entry. The notation I_r represents the identity matrix in $\mathbb{R}^{r\times r}$, and $(\cdot)^\dagger$ denotes the Moore–Penrose inverse, applicable to matrices. We use $\|\cdot\|$ to represent the 2-norm of a vector.

II. THE PROBLEM

In a multi-agent systems (MAS) consisting of m agents with labels $\mathbf{m}\ = \{1, 2, ..., m\}$, suppose each agent is able to communicate bi-directionally with certain other agents called *neighbors*. The set of agent i's neighbors is denoted by $\mathcal{N}_i$, where $i \notin \mathcal{N}_i$. Such neighbor relations can be characterized by a fixed graph $\mathbb{N} = \{\mathcal{V}, \mathcal{E}\}$ with $\mathcal{V} = \mathbf{m}$ and $|\mathcal{E}| = \bar{m}$ such that there is an undirected edge $(i, j) \in \mathcal{E}$ if any only if agent i and agent j are neighbors.

Suppose each agent i, $i \in \mathbf{m}$ controls a state $x_i(t) \in \mathbb{R}^n$ at time $t = 0, 1, 2, ...$ The problem of interest in this paper is to develop an iterative update rule for each agent i to update its state based on information from its neighbors at time t such that for any initialization $x_i(0)$, each $x_i(t)$ converges exponentially fast to a constant vector $x_i^* \in \mathbb{R}^n$, $i \in \mathbf{m}$, satisfying the following ***edge-agreement*** in $\mathbb{N}$, i.e.

$$A_{ij}(x_i^* - x_j^*) = b_{ij}, \quad \forall (i, j) \in \mathcal{E} \tag{1}$$

where $A_{ij} \in \mathbb{R}^{a_{ij}\times n},\ b_{ij} \in \mathbb{R}^{a_{ij}}$ are constant matrices.

Remark 1: Note that when each $b_{ij} = 0$, the consensus $x_1^* = x_2^* = \cdots = x_m^*$ is one solution to the edge-agreement in (1), although edge agreement does not necessarily imply consensus but includes it as a special case. The matrices $A_{ij}.b_{ij}$ in the edge-agreement (1) can enable more flexible coordination among multi-agent systems by enforcing a linear constraint to each pair of neighboring agents, rather than a global requirement of consensus in the whole MAS. Such characterization of edge agreements can be employed to define a multi-agent formation or other local coordination in general.

The research is supported in part by a grant from the NASA University Leadership Initiative (80NSSC20M0161) and a gift funding from Northrop Grumman Corporation.

Ayush Rai and Shaoshuai Mou are with the School of Aeronautics and Astronautics, Purdue University, West Lafayette, IN 47906 USA. (e-mail: rai29@purdue.edu, mous@purdue.edu). Corresponding Author: Shaoshuai Mou.

For the problem formulation to be complete, certain assumptions regarding the consistency and existence of the solution are required. Without losing generality, we adopt the following assumptions in achieving the edge-agreement in this paper.

Assumption 1: (**Consistency**) The linear constraints in (1) are consistent with each other for each pair of neighbor agents $(i, j) \in \mathcal{E}$, i.e.

$$A_{ij} = A_{ji}, \quad b_{ij} = -b_{ji}$$

Remark 2: Assumption 1 aims to avoid the inconsistent situation, for example, agent 1 thinks the sum of its scalar state x_1 and its nearby neighbor agent 2's state x_2 should be 2, while agent 2 thinks $x_1 + x_2 = 3$.

Assumption 2: (**Existence**) There exist constant vectors $x_i^* \in \mathbb{R}^n$, $i \in \mathbf{m}$ that satisfy the edge-agreement in (1).

Remark 3: Assumption 2 guarantee that there at least exist a set of constants x_i^*, $i = 1, 2, ..., m$, to satisfy the edge-agreement in (1). Otherwise, the whole problem of achieving edge agreements does not make sense.

III. Algorithms and Main Results

In this section, we will develop two discrete-time, distributed algorithms to achieve all the edge agreements defined in (1).

A. A Distributed Algorithm for Edge Agreement

Suppose each agent i can receive information from all its neighbors, we propose the following discrete-time update for each agent i, $i \in \mathbf{m}$, at time t:

$$x_i(t+1) = x_i(t) - \frac{1}{d_i + 1} \sum_{j \in \mathcal{N}_i} P_{ij}(x_i - x_j - \bar{b}_{ij}) \quad (2)$$

where $P_{ij} \in \mathbb{R}^{n \times n}$ denote a projection matrix and $\bar{b}_{ij} \in \mathbb{R}^n$ as follows[1]

$$P_{ij} = A'_{ij}(A_{ij}A'_{ij})^{\dagger}A_{ij}, \quad \bar{b}_{ij} = A'_{ij}(A_{ij}A'_{ij})^{\dagger}b_{ij}, \quad (3)$$

and $d_i = |\mathcal{N}_i|$ serves as a scaling factor for later-on convergence analysis. From the definition of P_{ij} and $\bar{b}_{ij}$ in (3) and noting $P_{ij}\bar{b}_{ij} = \bar{b}_{ij}$, one has the linear constraints in (1) for edge agreement is equivalent to

$$P_{ij}(x_i^* - x_j^* - \bar{b}_{ij}) = 0, \quad \forall (i,j) \in \mathcal{E}. \quad (4)$$

To facilitate our analysis, we introduce

$$\bar{P} = \text{diag}\,\{P_{i_1j_1}, P_{i_2j_2}, ..., P_{i_{\bar{m}}j_{\bar{m}}}\} \in \mathbb{R}^{\bar{m}n \times \bar{m}n} \quad (5)$$

and

$$\bar{b} = \text{col}\,\{b_{i_1j_1}, b_{i_2j_2}, ..., b_{i_{\bar{m}}j_{\bar{m}}}\} \quad (6)$$

where (i_k, j_k) denote its kth edge of $\mathbb{N}$, $k = 1, 2, ..., \bar{m}$.

For the m-node-$\bar{m}$-edge undirected graph $\mathbb{N} = \{\mathcal{V}, \mathcal{E}\}$, by assigning each undirected edge in $\mathbb{N}$ a direction, one then defines the *incidence matrix* of $\mathbb{N}$ denoted by $H \in \mathbb{R}^{\bar{m} \times m}$ such that its kjth element is 1 if edge k is an incoming edge to node j;-1 if edge k is an outgoing edge to node j; and 0, elsewhere. From (4), the definitions of H, $\bar{P}$, and

$$\bar{H} = H \otimes I_n \quad (7)$$

one has the following lemma:

Lemma 1: There exist constant vectors $x_i^* \in \mathbb{R}^n$, $i \in \mathbf{m}$ to satisfy the edge-agreement in (1) if and only if there exists a constant vector $x^* = \text{col}\,\{x_1^*, x_2^*, ..., x_m^*\}$ with $x_i^* \in \mathbb{R}^n$ such that

$$\bar{P}(\bar{H}x^* - \bar{b}) = 0$$

where $\bar{P}$, $\bar{b}$ and $\bar{H}$ are as defined in (5), (6) and (7), respectively.

Let $x(t) = \text{col}\,\{x_1(t), x_2(t), ..., x_m(t)\}$. Since $P_{ij} = P_{ji}$ and $b_{ij} = -b_{ji}$ as in Assumption 1, then the distributed update in (2) can be written in the following compact form

$$x(t+1) = x(t) - \bar{D}^{-1}\bar{H}'\bar{P}(\bar{H}x(t) - \bar{b}), \quad i \in \mathbf{m} \quad (8)$$

where $\bar{P}$, $\bar{b}$, and $\bar{H}$ are as defined in (5), (6), and (7), respectively, and

$$\bar{D} = (I_m + D) \otimes I_n \quad (9)$$

with $D = \text{diag}\,\{d_1, d_2, ..., d_m\}$.

In order to further prove the convergence of (8), , and drawing inspiration from Lemma 1, we introduce an error term

$$e(t) = \bar{P}\Big(\bar{H}x(t) - \bar{b}\Big) \quad (10)$$

which together with $\bar{P}^2 = \bar{P}$ and (8) leads to

$$e(t+1) = (I_{\bar{m}n} - Q)e(t) \quad (11)$$

where

$$Q = \bar{P}\bar{H}\bar{D}^{-1}\bar{H}'\bar{P}. \quad (12)$$

Lemma 2: For Q defined in (12), we have

$$0 \leq Q < 2I_{\bar{m}n}. \quad (13)$$

Proof: The proof is provided in the appendix. ■

From Lemma 2 and (11), one has $e(t)$ converges to a constant in the eigenspace of $I - Q$ corresponding to its eigenvalue 1, which is the kernel of Q. Thus $Qe(t) \to 0$ exponentially fast. Note that $Q = (\bar{D}^{-\frac{1}{2}}\bar{H}'\bar{P})'(\bar{D}^{-\frac{1}{2}}\bar{H}'\bar{P})$. It follows that $\bar{H}'\bar{P}e(t) \to 0$ exponentially fast. From this, the definition of $e(t)$ in (10) and $\bar{P}^2 = \bar{P}$ that $\bar{H}'\bar{P}(\bar{H}x(t) - \bar{b}) \to 0$ exponentially fast. From this and the linear time-invariant system (8), one has there exist a constant vector $x^* = \{x_1^*, x_2^*, ..., x_m^*\}$ with $x_i^* \in \mathbb{R}^n$ such that $x(t) \to x^*$ exponentially fast and $\bar{H}'\bar{P}(\bar{H}x^* - \bar{b}) = 0$. To perform further analysis, we find it necessary to introduce the following assumption.

Assumption 3: Assume the graph $\mathbb{G}$ is *well-configured for edge-agreement*, i.e.

$$\ker \bar{H}' \cap \text{image}\,\bar{P} = 0. \quad (14)$$

This along with the fact $\bar{H}'\bar{P}(\bar{H}x^* - \bar{b}) = 0$ imply $\bar{P}(\bar{H}x^* - \bar{b}) = 0$. From this, Assumption 2, and Lemma

[1]Note that $A_{ij}A'_{ij}$ is invertible only if A_{ij} has linearly independent columns.

1, one has each $x_i(t) \to x_i^*$, $i \in \mathbf{m}$, exponentially fast, which reach the edge-agreement in (1). In summary, one has the following theorem:

Theorem 1: Under Assumption 1, Assumption 2 and Assumption 3, the distributed update in discrete-time (2) drives each $x_i(t)$ to converge exponentially fast to a constant vector $x_i^* \in \mathbb{R}^n$, $i \in \mathbf{m}$, which reach the edge-agreement in $\mathbb{N}$ as defined in (1).

B. A Gossiping-based Distributed Algorithm for Edge-agreement

Edge agreements in (1) by imposing a constraint to each edge in the graph (namely, each pair of neighboring agents) naturally aligns with the concept of gossiping, in which each pair of neighbor agents communicate and update. In this subsection, we develop a gossiping-based distributed algorithm in which the update rule can be applied independently to each edge while still achieving convergence to a constant vector that satisfies edge agreement.

In a gossiping process, each agent engages in gossip with a maximum of one neighboring agent at any time step. The sequence of gossip pairs that occur during the gossiping process might be determined either probabilistically [24] or deterministically [25]. Similar to gossiping algorithms, we propose that at a given time t only one pair of neighbor agents i and j corresponding to the kth edge of $\mathbb{N}$, communicate and perform the following updates:

$$\begin{aligned} x_i(t+1) &= x_i(t) - \frac{1}{2}P_{ij}(x_i(t) - x_j(t) - \bar{b}_{ij}), \\ x_j(t+1) &= x_j(t) - \frac{1}{2}P_{ji}(x_j(t) - x_i(t) - \bar{b}_{ji}), \end{aligned} \tag{15}$$

and $x_l(t+1) = x_l(t)$ $\forall l \neq i, j$, where P_{ij} and $\bar{b}_{ij}$ are as defined in (3).

With (i_k, j_k) representing the kth edge of $\mathbb{N}$, $k = 1, 2, ..., \bar{m}$, and using Assumption 1, (15) can be compactly written as

$$x(t+1) = M_{i_k,j_k}(t)x(t) + b_{i_k,j_k}(t), \tag{16}$$

where $M_{i_k,j_k}(t) \in \mathbb{R}^{mn \times mn}$ is the gossip matrix corresponding to kth edge defined as

$$M_{i_k,j_k} = \begin{bmatrix} I_n & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & I_n - \frac{1}{2}P_{ij} & \cdots & \frac{1}{2}P_{ij} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \frac{1}{2}P_{ij} & \cdots & I_n - \frac{1}{2}P_{ij} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & I_n \end{bmatrix}$$

and $b_{i_k,j_k} = \text{col}\,\{0, \cdots, \bar{b}_{ij}, \cdots, \bar{b}_{ji}, \cdots, 0\} \in \mathbb{R}^{mn}$.

Before embarking on the analysis of this update law, we first review nonlinear maps which are *paracontractions* and their basic properties. A continuous nonlinear map $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ is a paracontraction with respect to a given norm $\|\cdot\|$ on $\mathbb{R}^n$, if for any fixed point $z \in \mathbb{R}^n, z = \mathcal{T}(z)$ and for any $y \in \mathbb{R}^n$, either $\|\mathcal{T}(y) - z\| < \|y - z\|$ or $y = \mathcal{T}(y)$ [26]. We observe the following important result for paracontraction.

Lemma 3: [26] Suppose $\mathcal{P}$ is a finite set of paracontractions with respect to some given norm on $\mathbb{R}^n$. Additionally, suppose that all paracontractions within this set share at least one common fixed point. If a sequence $\mathcal{P}_1, \mathcal{P}_2, \dots$ composed of paracontractions drawn from $\mathcal{P}$, then the state $x(t)$ of the iteration

$$x(t+1) = \mathcal{P}_t(x(t)),$$

converges to a common fixed point for those paracontractions appearing in the sequence an infinite number of times.

We also note the following result for paracontraction.

Lemma 4: Consider a affine linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ defined as $L(\bar{x}) = P\bar{x} + b$, where $P \in \mathbb{R}^{n \times n}$ is projection matrix. Then L is a paracontraction with respect to the 2-norm on $\mathbb{R}^n$.

Proof of Lemma 4: Let z be a fixed point of the linear map, $L(z) = z$. For $y \in \mathbb{R}^n$, we have $L(y) - z = P(y - z)$. The inequality $y \neq L(y)$ is equivalent to $(y - z) \notin \text{image } P$. Consequently, we have $\|P(y - z)\|_2 < \|y - z\|_2$, which demonstrates that L is a paracontraction. ■

Thus, to establish the convergence of the gossip algorithm (15), employing Lemma 3 and Lemma 4, our sole requirement is to demonstrate that each update rule (15) can be characterized as an affine linear map featuring a projection matrix, as we shall confirm with the following result.

Lemma 5: The linear map defined by (16) is a paracontraction.

Proof of Lemma 5: Consider a block matrix

$$\bar{M} = \begin{bmatrix} M_{ij} & 0 \\ 0 & I_{n(m-2)} \end{bmatrix},$$

where $M_{ij} = \begin{bmatrix} I_n - \frac{1}{2}P_{ij} & \frac{1}{2}P_{ij} \\ \frac{1}{2}P_{ij} & I_n - \frac{1}{2}P_{ij} \end{bmatrix}$. We observe that $\bar{M}^2 = \begin{bmatrix} M_{ij}^2 & 0 \\ 0 & I_{n(m-2)} \end{bmatrix}$, and we have

$$\begin{aligned} M_{ij}^2 &= \begin{bmatrix} (I_n - \frac{1}{2}P_{ij})^2 + \frac{1}{4}P_{ij}^2 & P_{ij}(I_n - \frac{1}{2}P_{ij}) \\ P_{ij}(I_n - \frac{1}{2}P_{ij}) & (I_n - \frac{1}{2}P_{ij})^2 + \frac{1}{4}P_{ij}^2 \end{bmatrix}, \\ &= \begin{bmatrix} I_n - \frac{1}{2}P_{ij} & \frac{1}{2}P_{ij} \\ \frac{1}{2}P_{ij} & I_n - \frac{1}{2}P_{ij} \end{bmatrix} = M_{ij}, \end{aligned}$$

where the last equality is obtained using the fact that P_{ij} is a projection matrix as defined in (3). Therefore, we have $\bar{M}^2 = \bar{M}$, establishing that $\bar{M}$ is indeed a projection matrix. Since any linear map $M(e_{ij})$ in update rule (16) can be transformed from matrix $\bar{M}$ by row and column swaps, the projection property remains unchanged. Hence, by applying Lemma 4, the linear map defined by (16) is a paracontraction. ■

From Lemma 5, we establish that the gossip rule (16) for the kth edge of $\mathbb{N}$, where $k = 1, 2, ..., \bar{m}$, qualifies as a paracontraction. Before presenting the final results, let's review some definitions and conditions related to when an

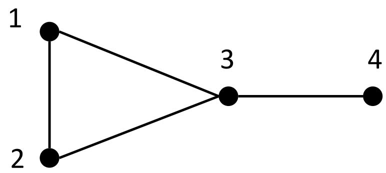

Fig. 1. A Multi-Agent System of Four Agents

infinite sequence is considered repetitively complete [25]. A gossiping sequence is defined as the sequence of gossip pairs that occur during the gossiping process. A finite sequence of gossip pairs is deemed complete if the corresponding set of edges in $\mathbb{N}$ forms a connected spanning subgraph. An infinite sequence of gossips is classified as repetitively complete if there exists a time period T for which each successive subsequence of gossips with a length of T in the sequence is complete. Note that these conditions for an infinite sequence to be repetitively complete also apply to multi-gossiping. Multi-gossiping refers to the scenario in which multiple pairs of agents engage in gossip simultaneously, with no agent participating in more than one pair. For instance, in Fig. 1, we have four edges (1,2), (1,3), (2,3), and (3,4). In the gossiping sequence, edges (1,2) and (3,4) can engage in gossip simultaneously.

A repetitively complete infinite sequence ensures that each gossip pair repeats itself infinitely in the gossiping process. This readily allows us to deduce the convergence of the update rule to a common fixed point for all paracontractions, corresponding to a constant vector satisfying all edge agreements, directly from Lemma 3 and Lemma 5. In summary, we now present the following theorem.

Theorem 2: Under Assumption 1, Assumption 2, if the gossiping sequence is infinite and repetitively complete, i.e each gossip pair appears in the gossiping sequence an infinite number of times, then distributed update using the gossip algorithm (15) drives each $x_i(t)$ to a constant vector $x_i^* \in \mathbb{R}^n$, $i \in \mathbf{m}$, which reach the edge-agreement in $\mathbb{N}$ as defined in (1).

Remark 4: The convergence guarantee in Theorem 2 is independent of the specific sequence employed in the gossiping process, as long as the gossiping sequence is infinite and repetitively complete.

IV. Simulations

In this section, we will perform simulations on the multi-agent systems of four agents as in Fig. 1 with the edge set

$$\mathcal{E} = \{(1,2),(2,3),(3,1),(3,4)\}. \tag{17}$$

Two examples will be introduced to show that the proposed distributed algorithms (2) and (15) are able to drive each agent i's state x_i to converge to x_i^* satisfying a group of edge agreements in (1), i.e.

$$A_{ij}(x_i^* - x_j^*) = b_{ij}, \quad \forall (i,j) \in \mathcal{E} \tag{18}$$

with $A_{ij} \in \mathbb{R}^{a_{ij} \times n}$, $b_{ij} \in \mathbb{R}^n$ given constant matrices and $\mathcal{E}$ in (17). To measure the closeness of all agents' states to the desired states satisfying edge agreements, we introduce the following index

$$V(t) = \frac{1}{2} \sum_{(i,j)\in\mathcal{E}} \|A_{ij}(x_i(t) - x_j(t)) - b_{ij}\|^2. \tag{19}$$

Note that $V(t) \geq 0$ with equality holds if and only if all the edge agreements in (18) are reached.

A. Heterogeneous and Local Coordination using discrete-time algorithm

We first consider applying the proposed discrete-time algorithm in (2) to achieve a group of heterogeneous and local coordination that can be characterized by edge agreements. Suppose $x_i \in \mathbb{R}^4$, $i = 1,2,3,4$ and the local coordination can be described by the edge agreement in (18) with the second element of x_1 and x_2 are equal,i.e.

$$A_{12} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \quad b_{12} = 0,$$

the first element of x_1 is maintained away from x_2 by a constant 3 while the third element of x_1 is away from the third element of x_3 by -2, i.e.

$$A_{23} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad b_{23} = \begin{bmatrix} 3 \\ -2 \end{bmatrix},$$

no local coordination needed between agent 2 and 3,i.e.

$$A_{31} = 0, \quad b_{31} = 0$$

and the sum of differences between the first three elements in x_3 and x_4 is equal to 10, i.e.

$$A_{34} = \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix}, \quad b_{34} = 10.$$

One employs the discrete-time distributed control in (2) with initial values randomly chosen for $x_i(0) \in \mathbb{R}^4$. The Fig. 2 studying the evolution $V(t)$ with time shows $V(t) \to 0$ exponentially fast, and thus all edge agreements described above are reached.

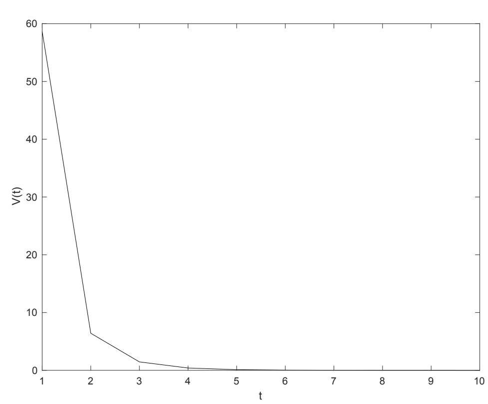

Fig. 2. Evolution of $V(t)$ for Heterogeneous and Local Coordination using discrete-time algorithm.

B. Heterogeneous and Local Coordination using gossip algorithm

We consider the problem as presented in Section IV-A but approach the problem using the proposed gossip algorithm in (15) to achieve a group of heterogeneous and local coordination which can be characterized by edge agreements.

We employ the gossip algorithm (15) using two different gossiping sequences: a deterministic periodic gossiping sequence and a repetitively complete infinite random gossiping sequence. Both implementations utilize the same update rule (15), but they differ in the method used to select edges. In the periodic gossiping, as shown in Fig. 3, the evolution of $V(k)$ is depicted, with k denoting an iteration (gossip subsequence) that consists of one complete round of gossiping. In contrast, Fig. 4 illustrates the evolution of $V(t)$ for a repetitively complete infinite random gossiping sequence where the length of the complete subsequence is randomly selected between [4,10]. In both cases, the plots demonstrate $V(t) \to 0$ as $t \to \infty$, indicating the achievement of all edge agreements described above.

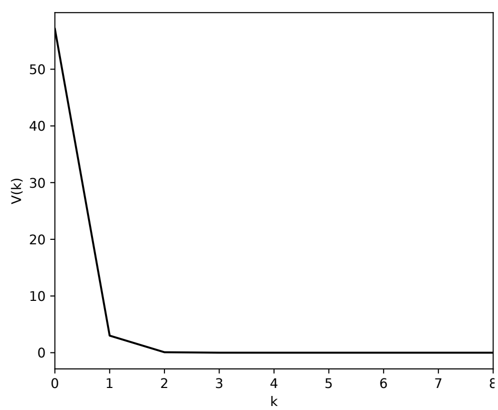

Fig. 3. Evolution of $V(t)$ for Heterogeneous and Local Coordination using a periodic gossiping sequence. Here on x-axis, each iteration represents one cycle of periodic gossiping.

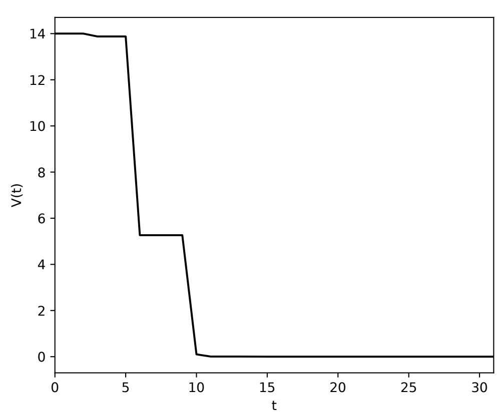

Fig. 4. Evolution of $V(t)$ for Heterogeneous and Local Coordination using a repetitively complete infinite random gossiping sequence. Here on x-axis, each time-step represents one gossip update.

V. Conclusion

This paper has addressed the problem of edge agreement, which allows for more flexible coordination among agents and encompasses consensus as a special case. Two discrete-time distributed algorithms have been developed to achieve edge agreements. It has been demonstrated that, under undirected graphs, both algorithms lead to the convergence of all agent states towards edge agreements as $t \to \infty$, with the discrete-time algorithm exhibiting exponential convergence. Simulations have confirmed the effectiveness of these algorithms. Future work includes extending these algorithms for time-varying directed graphs and investigating the convergence rates of the gossip algorithm.

Appendix

We first introduce the following lemma which will be used in proving Lemma 2.

Lemma 6: For an m-node-$\bar{m}$-edge undirected graph $\mathbb{N} = \{\mathcal{V}, \mathcal{E}\}$, let

$$L = (I_m + D)^{-\frac{1}{2}} H' H (I_m + D)^{-\frac{1}{2}} \tag{20}$$

where H denote the incidence matrix of $\mathbb{N}$ and D is the degree matrix, defines as

$$D = \mathrm{diag}\,\{d_1, d_2, ..., d_m\} \tag{21}$$

with d_i the number of node i's neighbors. Then

$$0 \leq L < 2I_m. \tag{22}$$

Proof of Lemma 6: Note that L is positive semi-definite. Then all eigenvalues of L are non-negative. Moreover, by Rayleigh-Ritz Theorem one has the largest eigenvalue of L is such that

$$\lambda_{\max}(L) = \max_{q \neq 0, q \in \mathbb{R}^n} \frac{q' L q}{q' q}$$

Let $p = (I_m + D)^{-\frac{1}{2}} q$, where the ith element of p is denoted by p_i. Then

$$\frac{q' L q}{q' q} = \frac{\sum_{(i,j) \in \mathcal{E}} (p_i - p_j)^2}{\sum_{i=1}^{n} (d_i + 1) p_i^2}$$

which reaches its largest value when

$$p_i = -p_j, \quad \forall (i,j) \in \mathcal{E}. \tag{23}$$

Suppose the graph $\mathbb{N}$ consist of a number of c connected components with $\mathcal{V}_k$ and $\bar{\mathcal{E}}_k$ denoting vertex set and edge set of the kth connected component, $k = 1, 2, 3, ..., c$. Specially an isolated vertex without any adjacency edges is also looked at a connected component, which is with one vertex and empty edge set. From (23), one has all p_i^2 are equal within the same connected component, i.e. for all $i \in \mathcal{V}_k$. Suppose they are equal to $\bar{p}_k^2$ in the kth connected component. Then

$$\frac{\sum_{(i,j) \in \mathcal{E}} (p_i - p_j)^2}{\sum_{i=1}^{n} (d_i + 1) p_i^2} \leq \frac{\sum_{k=1}^{c} \sum_{(i,j) \in \mathcal{E}_k} 4 \bar{p}_k^2}{\sum_{k=1}^{c} \sum_{i \in \mathcal{V}_k} (d_i + 1) \bar{p}_k^2}$$

By the Handshaking Theorem in graph theory, one has

$$\sum_{i \in \mathcal{V}_k} (d_i + 1) = 2\bar{r}_k + r_k$$

where $r_k = |\mathcal{V}_k|$ and $\bar{r}_k = |\bar{\mathcal{E}}_k|$. It follows that

$$\frac{q'Lq}{q'q} \leq \frac{\sum_{k=1}^{c} 4\bar{r}_k \bar{p}_k^2}{\sum_{k=1}^{c}(2\bar{r}_k + r_k)\bar{p}_k^2} < 2$$

for $q \neq 0$. Then

$$\lambda_{\max}(L) < 2$$

Note that L is positive-semidefinite, then (22) holds. We complete the proof. ∎

Remark 5: It is known that the normalized Laplacian $L_{\mathbb{N}} = D^{-\frac{1}{2}}H'HD^{-\frac{1}{2}}$ for an undirected graph $\mathbb{N}$ is such that

$$0 \leq L_{\mathbb{N}} \leq 2I_m \tag{24}$$

Here, we introduce a little bit different normalization to eliminate the equality at the right of (24) and achieve (22), which to be shown later is very useful for us to develop a distributed algorithm for edge agreement.

Proof of Lemma 2: By Lemma 6 and the fact that the matrix product AB shares the same non-zero eigenvalues with BA, one has

$$0 \leq H(I_m + D)^{-1}H' < 2I_{\bar{m}}$$

which and $\bar{H} = H \otimes I_n$ and $\bar{D} = (I_m + D) \otimes I_n$ imply

$$0 \leq \bar{H}\bar{D}^{-1}\bar{H}' < 2I_{\bar{m}}.$$

This, the fact that $\bar{P}$ is the projection matrix, and $Q = \bar{P}\bar{H}\bar{D}^{-1}\bar{H}'\bar{P}$ imply $0 \leq Q < 2I_{\bar{m}n}$. We complete the proof. ∎

REFERENCES

[1] F. Bullo, J. Cortes, and S. Martinez. *Distributed Control of Robotic Networks*. Princeton University Press, 2009.

[2] W. Ren and R. W. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, 50(5):655–661, May 2005.

[3] M. Cao, A. S. Morse, and B. D. O. Anderson. Agree asychronously. *IEEE Transactions on Automatic Control*, 53(8):1826–1838, 2008.

[4] M. Cao, A. S. Morse, and B. D. O. Anderson. Reaching a consensus in a dynamically changing enviornment: a graphical approach. *SIAM Jounal on Control and Optimization*, 47(2):575–600, 2008.

[5] X. Chen, M. A. Belabbas, and T. Basar. Distributed averaging with linear objective maps. *Automatica*, 70(3):179–188, 2016.

[6] S. Mou and M. Cao. Distributed averaging using compensation. *IEEE Communication Letters*, 17(8):1672–1675, 2013.

[7] S. Mou, J. Liu, and A. S. Morse. A distributed algorithm for solving a linear algebraic equation. *IEEE Transactions on Automatic Control*, 60(11):2863–2878, 2015.

[8] S. Mou, Z. Lin, L. Wang, D. Fullmer, and A. S. Morse. A distributed algorithm for efficiently solving linear equations and its applications (special issue jcw). *Systems & Control Letters*, 91:21–27, 2016.

[9] G. Shi, B. D. O. Anderson, and U. Helmke. Network flows that solve linear equations. *IEEE Transactions on Automatic Control*, 62(6):2659–2674, June 2017.

[10] X. Wang, J. Zhou, S. Mou, and M. J. Corless. A distributed algorithm for least square solutions. *IEEE Transactions on Automatic Control*, 64(10):4217–4222, 2019.

[11] X.Wang, S. Mou, and B.D.O.Anderson. Scalable, distributed algorithms for solving linear equations. *IEEE Transactions on Automatic Control*, 65(3):1132–1143, 2020.

[12] A. Nedic, A. Ozdaglar, and P. A. Parrilo. Constrained consensus and optimization in multi-agent networks. *IEEE Transactions on Automatic Control*, 55(4):922–938, 2010.

[13] J. C. Duchi, A. Agarwal, and M. J. Wainwright. Dual averaging for distributed optimization: Convergence analysis and network scaling. *IEEE Transactions on Automatic Control*, 57(3):592–606, 2012.

[14] T. Chang, A. Nedic, and A. Scaglione. Distributed constrained optimization by consensus-based primal-dual perturbation method. *IEEE Transactions on Automatic Control*, 59(6):1524–1538, 2014.

[15] A. Nedic and A. Olshevsky. Distributed optimization over time-varying directed graphs. *IEEE Transactions on Automatic Control*, 60(3):601–615, 2015.

[16] X. Zeng, P. Yi, and Y. Hong. Distributed continuous-time algorithm for constrained convex optimizations via nonsmooth analysis approach. *IEEE Transactions on Automatic Control*, 62(10):5227–5233, 2017.

[17] Q. Liu, S. Yang, and Y. Hong. Constrained consensus algorithms with fixed step size for distributed convex optimizations over multiagent networks. *IEEE Transactions on Automatic Control*, 62(8):4259–4265, 2017.

[18] X. Chen, M. A. Belabbas, and T. Basar. Controllability of formations over directed time-varying graphs. *IEEE Transactions on Control of Network Systems*, pages 4362–4367, 2015.

[19] H. Marina, B. Jayawardhana, and M. Cao. Distributed rotational and translational maneuvering of rigid formations and their applications. *IEEE Transactions on Robotics*, 32(3):1552–3098, 2016.

[20] K. Zhang, Z. Yang, H. Liu, T. Zhang, and T. Başar. Fully decentralized multi-agent reinforcement learning with networked agents. *Proceedings of the 35th International Conference on Machine Learning*, 80:5872–5881, 2018.

[21] K. Zhang, Z. Yang, and T. Başar. Multi-agent reinforcement learning: A selective overview of theories and algorithms. *Handbook of Reinforcement Learning and Control*, 325, 2021.

[22] G. Qu, A. Wierman, and N. Li. Scalable reinforcement learning of localized policies for multi-agent networked systems. *Proceedings of the 2nd Conference on Learning for Dynamics and Control*, 120(256-266), 2020.

[23] G. Qu, Y. Lin, A. Wierman, and N. Li. Scalable multi-agent reinforcement learning for networked systems with average reward. *Proceedings of the 34th Conference on Neural Information Processing Systems (NeurIPS),*, 2020.

[24] Stephen Boyd, Arpita Ghosh, Balaji Prabhakar, and Devavrat Shah. Randomized gossip algorithms. *IEEE transactions on information theory*, 52(6):2508–2530, 2006.

[25] Ji Liu, Shaoshuai Mou, A Stephen Morse, Brian DO Anderson, and Changbin Yu. Deterministic gossiping. *Proceedings of the IEEE*, 99(9):1505–1524, 2011.

[26] Ludwig Elsner, Israel Koltracht, and Michael Neumann. Convergence of sequential and asynchronous nonlinear paracontractions. *Numerische mathematik*, 62(1):305–319, 1992.