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Model Reduction by Matching Zero-Order Moments for 2-D Discrete Systems

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Abstract— **In this paper, the problem of model reduction for two-dimensional (2-D) systems in the Fornasini-Marchesini local state-space form is addressed by matching zero-order moments. Two characterizations of zero-order moments are proposed: the first based on the notion of interpolation of complex points and the second based on the concept of steady state. A parameterized family of reduced-order models that achieves moment matching while preserving the 2-D structure of the original system is presented. The developed theory is illustrated by means of a 2-D low-pass filter reduction problem.**

I. INTRODUCTION

Two-dimensional systems have attracted research interest over the past few decades owing to their natural ability in modelling the dynamics of signals evolving along two independent variables, which are widely seen in a broad spectrum of applications, such as digital imaging processing [1], seismic data processing [2] and grid sensor networks [3]. Some recent work [4] has also noted the advantages in modelling Convolutional Neural Networks (CNNs) as a 2-D dynamical system, revealing the prospective use of 2-D system theory in complex neural network models. Unlike one-dimensional (1-D) systems that propagate information only along the *time* dimension, 2-D systems possess two independent directions that enable information flow. Several forms of state-space models for representing 2-D systems, such as the Roesser model [5], Attasi model [6], Fornasini-Marchesini first (FM-I) model [7] and Fornasini-Marchesini local state-space (FMLSS) model[1] [8] have been proposed. Note that all the modelling frameworks listed above can be embedded in an FMLSS form with no model order increase [9]. Given this generality, this paper focuses on the FMLSS representation.

The problem of *model order reduction* consists in finding a simplified description (*e.g.* a lower-order model) to approximate a given system in some sense (*e.g.* $\mathcal{H}_\infty$ and $\mathcal{H}_2$) while some structures (*e.g.* port-Hamiltonian) or properties (*e.g.* stability) are also preserved. As such, model reduction techniques provide a powerful solution to the complexity issues arising in many modern engineering problems. Compared with 1-D systems, it has been noted in [1] that 2-D systems suffer an enhanced *curse of dimensionality* due to the fact that one more dimension is added to the field of data, making the number of computations increase as the square of the order of the system. This scenario commonly arises, for instance, in the field of 2-D digital filtering, where finding a lower-order filter subject to the design specifications is of primary significance [1], thus motivating the study of model reduction methods for 2-D systems.

In the context of 1-D systems, model reduction techniques have been extensively developed from diverse perspectives, *e.g.*, optimal Hankel norm approximation methods, balanced truncation methods, and Krylov projection method [10]. The Krylov projection methods are also known as *moment matching* methods since the resulting reduced-order model interpolates the so-called "moments" at prescribed frequencies (also referred to as interpolation points). An alternative notion of moment matching, based on the concept of steady state, has been extended to a broad range of systems, for instance, nonlinear [11], time-delay [12], hybrid [13] and stochastic systems [14].

Some model reduction frameworks have been proposed for 2-D systems, *e.g.*, balanced truncation [15], [16], weighted balanced truncation [17], the singular perturbation methods [18], [19] and LMI-based methods [20], [21]. However, to the best knowledge of the authors, moment-matching methods, which generally prove to be computationally faster than other reduction methods and allow extensions to nonlinear systems, have not been proposed for 2-D systems. To address this gap, this paper proposes a novel result that extends the moment-matching framework to the FMLSS model. To this end, we propose a definition of moment for 2-D systems, we relate these moments to the solution of a Sylvester-like equation and to the steady-state response of a particular system interconnection, and we develop reduced-order models. We restrict this work to zero-order moments. In fact, model reduction of 2-D systems is not a straightforward extension of 1-D techniques as 2-D systems are infinite-dimensional, see [22]. As such, we will show that both interpolation-based moments and steady-state-based moments explode in dimensionality and mathematical complexity in comparison to the 1-D version. Our findings on zero-order moments lay the foundations for future research in moment-matching methods for 2-D linear and nonlinear systems and general n-D systems.

The remainder of this paper is structured as follows. In Section II, after briefly recalling the theory of moment matching for 1-D linear systems we introduce the formal system description of the FMLSS model. In Section III-A we define the notion of moment for 2-D FMLSS models, and characterize 0-moments using the solution of a Sylvester-like equation. A steady-state characterization is presented in

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[1]The FMLSS model is also referred to as Fornasini-Marchesini second (FM-II) model in some literature.

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III-B. Section IV presents a family of reduced-order models achieving moment matching and an approach to enforce an additional property. In Section V the developed theory is illustrated by a 2-D filter reduction problem. The paper ends with some concluding remarks.

Notation: Throughout this paper, we use standard notation. $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. The set of non-negative integers is denoted by $\mathbb{Z}_{\geq 0}$. The $n \times n$ identity matrix is denoted by the symbol I_n, and $\sigma(A)$ denotes the spectrum of a square matrix A. The operator $\mathrm{vec}(A)$ indicates the vectorization of a matrix $A \in \mathbb{R}^{n\times m}$, which is the $nm \times 1$ vector obtained by stacking the columns of the matrix A one on top of the other. For a vector a, $\mathrm{diag}(a)$ denotes a square diagonal matrix with the elements of a on the main diagonal. For a set X, $||X||$ denotes the supremum norm of the set. ι denotes the imaginary unit.

II. PRELIMINARIES

In this section we recall the theory of model reduction by moment matching for 1-D linear systems and introduce a state-space representation of 2-D systems described by the FMLSS model, together with its notion of asymptotic stability.

A. Model Reduction by Moment Matching for 1-D Systems

Consider a linear, discrete-time 1-D system described by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k), \end{aligned} \tag{1}$$

with state $x(k) \in \mathbb{R}^n$, input $u(k) \in \mathbb{R}$, output $y(k) \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$. Let

$$W(z) = C(zI - A)^{-1}B$$

be the associated transfer function. Assume system (1) is minimal, *i.e.*, reachable and observable. The definition of the moments for such system is given as follows.

Definition 1. The 0-moment of system (1) at $z_i \in \mathbb{C} \setminus \sigma(A)$ is the complex number $\eta_0(z_i) = W(z_i)$. The k-moment of system (1) at z_i is the complex number $\eta_k(z_i) = \frac{(-1)^k}{k!}\left[\frac{d^k}{dz^k}W(z)\right]_{z=z_i}$, where $k \geq 1$ is an integer.

It has been shown in [23] that the moments can be linked to the solution of a particular Sylvester equation that involves only real-valued matrices. We recall this result as formulated in [11], adapting it straightforwardly to the discrete-time case.

Theorem 1. Consider system (1) and let $z_i \in \mathbb{C} \setminus \sigma(A)$, for $i = 1, \ldots, \rho$. Assume that the matrix $S \in \mathbb{R}^{\nu\times\nu}$ is non-derogatory[2] with characteristic polynomial $p(\lambda) = \prod_{i=1}^{\rho}(\lambda - z_i)^{k_i}$, and $\nu = \sum_{i=1}^{\rho} k_i$. Let $L \in \mathbb{R}^{1\times\nu}$ be such that (S, L) is observable. Then, the moments of system (1), namely $\eta_0(z_1), \ldots, \eta_{k_1-1}(z_1), \ldots, \eta_0(z_\rho), \ldots, \eta_{k_\rho-1}(z_\rho)$, are in a one-to-one relation with the matrix $C\Pi$, in which $\Pi \in \mathbb{R}^{n\times\nu}$ is the unique solution of the Sylvester equation

$$A\Pi + BL = \Pi S. \tag{2}$$

A family of models that achieve moment matching at S, *i.e.*, system (1) and the reduced-order model have the same moments at the frequencies $z_i \in \sigma(S)$, is described by equations of the form

$$\xi(k+1) = (S - GL)\xi(k) + Gu(k), \tag{3a}$$
$$\psi(k) = C\Pi\xi(k), \tag{3b}$$

for any $G \in \mathbb{R}^{\nu\times 1}$ such that $\sigma(S - GL) \cap \sigma(S) = \emptyset$. This parameterized family contains all the ν-order models that achieve moment matching at $\sigma(S)$.

B. FMLSS Description of 2-D Linear Systems

Throughout this paper, we consider a linear, strictly causal[3], discrete-time 2-D system described by an FMLSS model of the form

$$\begin{aligned} x(i+1, j+1) &= A_1x(i, j+1) + A_2x(i+1, j) \\ &\quad + B_1u(i, j+1) + B_2u(i+1, j), \\ y(i, j) &= Cx(i, j), \end{aligned} \tag{4}$$

where $i \in \mathbb{Z}_{\geq 0}$ and $j \in \mathbb{Z}_{\geq 0}$ are coordinates of the horizontal and vertical directions[4], respectively, $x(i, j) \in \mathbb{R}^n$ is the *local* state vector, $u(i, j) \in \mathbb{R}$ is the input, $y(i, j) \in \mathbb{R}$ is the output and $A_1 \in \mathbb{R}^{n\times n}$, $A_2 \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times 1}$, $B_2 \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$ are known matrices. The boundary conditions are given by $x(i, 0)$ and $x(0, j)$, for $i, j \in \mathbb{Z}_{\geq 0}$. By applying the 2-D z-transform [1], the transfer function of system (4) is obtained as

$$W(z_1, z_2) = C(z_1z_2I - z_2A_1 - z_1A_2)^{-1}(z_2B_1 + z_1B_2), \tag{5}$$

in which z_1 and z_2 are the z-transform shift operators in the horizontal and vertical directions, respectively.

2-D systems fundamentally differ from 1-D systems in the fact that there exist two[5] independent dimensions/directions along which the information is propagated, hence the *local* state $x(i, j)$ provides the minimal size of recursion rather than a sufficient summary of the past. To summarize all past information of the system, we introduce the *global* state defined as the set $\mathcal{X}(\kappa) = \{x(i, j) : i + j = \kappa\}$, indicating that 2-D system (4) is an infinite-dimensional system.

We conclude this section, by recalling the notion of asymptotic stability for 2-D systems and a sufficient and necessary condition that ensures such a property.

Definition 2 (see [8])**.** System (4) is *asymptotically stable*, if for $u(i, j) \equiv 0$ and any finite $||\mathcal{X}(0)||$, $\lim_{\kappa\to+\infty} ||\mathcal{X}(\kappa)|| = 0$.

Theorem 2 (see [8])**.** System (4) is asymptotic stable if and only if the polynomial $\rho(z_1, z_2) = \det(z_1z_2I - z_2A_1 - z_1A_2) \neq 0$ in $\mathcal{D} = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : |z_1| \geq 1, |z_2| \geq 1\}$.

[2]A matrix is non-derogatory if its characteristic and minimal polynomial coincide.

[3]An FMLSS model is said to be strictly causal if it possesses no feedforward term.

[4]Neither of these two directions is necessarily associated with time.

[5]2-D systems and the FMLSS model (4) can be generalized to multi-dimensional (n-D) systems, see, *e.g.*, [9].

III. Moments for 2-D Systems

In this section we first define the notion of moment for 2-D FMLSS models, based on which two characterizations of moment are derived by means of a Sylvester-like equation and the steady-state response of a particular system interconnection.

A. Definition of Moments

Based on the notion of 2-D transfer function, a natural definition of the moments for system (4) at spatial frequency pairs (z_1, z_2) is proposed as follows.

Definition 3. Let $z_1^*, z_2^* \in \mathbb{C}$ with $\rho(z_1^*, z_2^*) \neq 0$. The 0-*moment* of system (4) at (z_1^*, z_2^*) is the complex number

$$\eta_0(z_1^*, z_2^*) = W(z_1^*, z_2^*).$$

The 0-moments (provided they exist) can be characterized as the (unique) solution of certain Sylvester-like equations, as shown in the following results.

Lemma 3. Consider system (4) and $(z_1^*, z_2^*) \in \mathbb{C} \times \mathbb{C}$ with $\rho(z_1^*, z_2^*) \neq 0$. Then, the moment $\eta_0(z_1^*, z_2^*) = C\Pi$, with $\Pi \in \mathbb{C}^{n\times 1}$ the unique solution of the Sylvester-like equation

$$A_1\Pi z_2^* + A_2\Pi z_1^* + B_1 z_2^* + B_2 z_1^* = \Pi z_1^* z_2^*. \qquad (6)$$

Lemma 4. Consider system (4) and let the set $(z_1^{(1)}, z_2^{(1)}), (z_1^{(2)}, z_2^{(2)}), \ldots, (z_1^{(\nu)}, z_2^{(\nu)}) \subset \mathbb{C}\times\mathbb{C}$ be such that $\rho(z_1^{(k)}, z_2^{(k)}) \neq 0$ for $k = 1, \ldots, \nu$. Then

$$\begin{bmatrix}\eta_0(z_1^{(1)}, z_2^{(1)}) & \cdots & \eta_0(z_1^{(\nu)}, z_2^{(\nu)})\end{bmatrix} = C\Pi,$$

in which $\Pi \in \mathbb{C}^{n\times\nu}$ is the unique solution of the Sylvester-like equation

$$A_1\Pi\Sigma_2 + A_2\Pi\Sigma_1 + B_1\tilde{L}\Sigma_2 + B_2\tilde{L}\Sigma_1 = \Pi\Sigma_1\Sigma_2, \qquad (7)$$

with $\tilde{L} = [1, 1, \ldots, 1] \in \mathbb{R}^{1\times\nu}$, $\Sigma_1 = \mathrm{diag}([z_1^{(1)}, \ldots, z_1^{(\nu)}])$ and $\Sigma_2 = \mathrm{diag}([z_2^{(1)}, \ldots, z_2^{(\nu)}])$.

The above result characterizes the moments with a linear equation of matrices with complex values or special structure. In what follows we relax this restriction by linking the moments to the solution of a real-valued Sylvester-like equation with matrices S_1, S_2 and L of greater structural flexibility.

Theorem 5. Consider system (4) and let the set $(z_1^{(1)}, z_2^{(1)}), (z_1^{(2)}, z_2^{(2)}), \ldots, (z_1^{(\nu)}, z_2^{(\nu)}) \subset \mathbb{C}\times\mathbb{C}$ be such that $\rho(z_1^{(k)}, z_2^{(k)}) \neq 0$ for $k = 1, \ldots, \nu$. Let $S_1 \in \mathbb{R}^{\nu\times\nu}$ and $S_2 \in \mathbb{R}^{\nu\times\nu}$ be any simultaneously diagonalizable[6] matrices with diagonal form Σ_1 and Σ_2, respectively, where $\Sigma_1 = \mathrm{diag}([z_1^{(1)}, \ldots, z_1^{(\nu)}])$ and $\Sigma_2 = \mathrm{diag}([z_2^{(1)}, \ldots, z_2^{(\nu)}])$. Let $L \in \mathbb{R}^{1\times\nu}$ be such that (S_1, L) and (S_2, L) are both observable. Then, there exists a one-to-one relation between the moments $\eta_0(z_1^{(1)}, z_2^{(1)}), \eta_0(z_1^{(2)}, z_2^{(2)}), \ldots, \eta_0(z_1^{(\nu)}, z_2^{(\nu)})$ and the matrix $C\Pi$, with $\Pi \in \mathbb{R}^{n\times\nu}$ the unique solution of the Sylvester-like equation

$$A_1\Pi S_2 + A_2\Pi S_1 + B_1 L S_2 + B_2 L S_1 = \Pi S_1 S_2. \qquad (8)$$

Remark 1. The condition that the real matrices S_1 and S_2 are simultaneously diagonalizable is without loss of generality and can be guaranteed with the use of Jordan blocks. To show this, convert the complex diagonal matrices Σ_1, Σ_2 into the real matrices $\bar{\Sigma}_1, \bar{\Sigma}_2$ with 2-by-2 Jordan blocks with multiplicity 1 (each associated with a conjugate pair of complex eigenvalues) in the main diagonal. Under the choice $S_1 = \bar{\Sigma}_1$ and $S_2 = \bar{\Sigma}_2$, S_1 and S_2 are diagonalizable in $\mathbb{C}$ and they commute, hence they are simultaneously diagonalizable. In addition, selecting $L = \bar{L} := [1\ 0\ 1\ 0\ \ldots\ 1\ 0]$ with each entry of 1 associated with a Jordan block, ensures the observability for both pairs (S_1, L) and (S_2, L). Note further that for any non-singular matrix $M \in \mathbb{R}^{\nu\times\nu}$, the matrices $S_1 = M^{-1}\bar{\Sigma}_1 M$, $S_2 = M^{-1}\bar{\Sigma}_2 M$ and $L = \bar{L}M$ are such that the conditions on simultaneous diagonalizability and observability hold.

Remark 2. Each interpolation pair is determined by an exact pairing of specific eigenvalues of S_1 and S_2. As such, the order in which the eigenvalues of S_1 and S_2 appear in their simultaneous diagonalized forms plays a role in characterizing the interpolation pairs. This is in contrast with the 1-D case in Theorem 1, where any complex diagonal form of S is equivalent as long as the characteristic polynomial remains the same.

Remark 3. It can be observed that if $A_1 = 0$ and $B_1 = 0$, or $A_2 = 0$ and $B_2 = 0$ in system (4), the information is now only propagated in a single direction. In other words, system (4) reduces to a 1-D system in the form of system (1). Note that in this case, the Sylvester-like equation (8) also reduces to the Sylvester equation (2) accordingly.

As the Sylvester-like equation (8) is a linear matrix equation in Π, it can be solved, for instance by exploiting the property[7] of the vectorization operator, yielding

$$\begin{aligned}(S_1^\top S_2^\top \otimes I_n - S_2^\top \otimes A_1 - S_1^\top \otimes A_2)\,\mathrm{vec}(\Pi) \\ = (S_2^\top \otimes B_1 + S_1^\top \otimes B_2)L^\top.\end{aligned}$$

The above equation has a unique solution if and only if the matrix $(S_1^\top S_2^\top \otimes I_n - S_2^\top \otimes A_1 - S_1^\top \otimes A_2)$ is non-singular, which holds if and only if $\rho(z_1^{(k)}, z_2^{(k)}) \neq 0$ for $k = 1, \ldots, \nu$.

B. Steady-State Characterization of Moments

We now link the moments with the steady-state response of a particular interconnection between a signal generator and the system to be reduced.

Theorem 6. Let $S_1 \in \mathbb{R}^{\nu\times\nu}$ and $S_2 \in \mathbb{R}^{\nu\times\nu}$ be any simultaneously diagonalizable matrices with diagonal form Σ_1 and Σ_2, respectively, where $\Sigma_1 = \mathrm{diag}([z_1^{(1)}, \ldots, z_1^{(\nu)}])$ and $\Sigma_2 = \mathrm{diag}([z_2^{(1)}, \ldots, z_2^{(\nu)}])$. Consider system (4) and assume

[6] Square matrices S_1 and S_2 are simultaneously diagonalizable if there is a single non-singular matrix T such that TS_1T^{-1} and TS_2T^{-1} are both diagonal. An equivalent statement is that S_1 and S_2 are diagonalizable and $S_1S_2 = S_2S_1$. See [24, Section 1.3] for more details.

[7] $\mathrm{vec}(\mathcal{ABC}) = (\mathcal{C}^\top \otimes \mathcal{A})\,\mathrm{vec}(\mathcal{B})$, with $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ of compatible dimensions.

$\rho(z_1^{(k)}, z_2^{(k)}) \neq 0$ with $k = 1, \dots, \nu$. Assume that system (4) is asymptotically stable with finite $||\mathcal{X}(0)||$. Consider the interconnection between system (4) and the 2-D signal generator

$$\begin{aligned} \omega(i+1, j) &= S_1 \omega(i, j), \\ \omega(i, j+1) &= S_2 \omega(i, j), \\ u(i, j) &= L\omega(i, j), \end{aligned} \tag{9}$$

with L such that (S_1, L) and (S_2, L) are both observable, and $\omega(0,0)$ such that $(S_1, \omega(0,0))$ and $(S_2, \omega(0,0))$ are both excitable[8]. Then, there exists a one-to-one relation between the moments $\eta_0(z_1^{(1)}, z_2^{(1)}), \eta_0(z_1^{(2)}, z_2^{(2)}), \dots, \eta_0(z_1^{(\nu)}, z_2^{(\nu)})$ and the steady-state response of the output of such interconnected system.

Theorem 6 provides an alternative characterization of moments for 2-D systems based on the steady-state response of the interconnected system. This implies that a reduced-order model by moment matching would exhibit the same steady-state output response of the original system when excited by the same class of "matched" input signals (those generated by (9)). A major advantage of this steady-state-based characterization comes from the fact that it opens the possibility of extending the notion of moment to systems that do not admit a transfer function, *e.g.*, 2-D nonlinear systems (see [22]). Furthermore, from a practical perspective, the theorem also allows developing algorithms to asymptotically estimate the matrix $C\Pi$ from input-output *space-domain* samples, in line with the ideas in [26].

Remark 4. The signal generator (9) is in a special Roesser form, not in the FMLSS form. If the generator (9) is replaced with a general FMLSS model then two observations are in order. First, the resulting steady-state becomes much more complex. In fact, it can be shown that the steady-state dynamics become infinite-dimensional and it is not possible to characterize this with a finite-dimensional Sylvester-like equation. This leads to the second observation: the steady-state response of such a system would not describe the moments formalised in Definition 3, but much richer objects. Currently, it is unclear what those objects would be or if they are of any use for model reduction.

IV. Model Reduction by Moment Matching

Based on the result in Theorem 5, in this section we present a parameterized family of models which interpolate the 0-moments at frequency pairs characterized by S_1 and S_2, and discuss how to exploit some free parameters to impose an additional property.

Theorem 7. Consider system (4) and let the set $(z_1^{(1)}, z_2^{(1)}), (z_1^{(2)}, z_2^{(2)}), \dots, (z_1^{(\nu)}, z_2^{(\nu)}) \subset \mathbb{C} \times \mathbb{C}$ be such that $\rho(z_1^{(k)}, z_2^{(k)}) \neq 0$ for $k = 1, \dots, \nu$. Let $S_1 \in \mathbb{R}^{\nu \times \nu}$ and $S_2 \in \mathbb{R}^{\nu \times \nu}$ be any simultaneously diagonalizable matrices with diagonal form Σ_1 and Σ_2, respectively, where $\Sigma_1 = \text{diag}([z_1^{(1)}, \dots, z_1^{(\nu)}])$ and $\Sigma_2 = \text{diag}([z_2^{(1)}, \dots, z_2^{(\nu)}])$. Let $L \in \mathbb{R}^{1 \times \nu}$ be such that (S_1, L) and (S_2, L) are both observable. Then, the system described by the equations

$$\begin{aligned} \xi(i+1, j+1) &= F_1 \xi(i, j+1) + F_2 \xi(i+1, j) \\ &\quad + G_1 u(i, j+1) + G_2 u(i+1, j), \\ \phi(i, j) &= H \xi(i, j), \end{aligned} \tag{10}$$

with $\xi(i,j) \in \mathbb{R}^\nu$, $\phi(i,j) \in \mathbb{R}$, $F_1 \in \mathbb{R}^{\nu \times \nu}$, $F_2 \in \mathbb{R}^{\nu \times \nu}$, $B_1 \in \mathbb{R}^{\nu \times 1}$, $B_2 \in \mathbb{R}^{\nu \times 1}$ and $H \in \mathbb{R}^{1 \times \nu}$, is a model of system (4) at (S_1, S_2), if

$$\det(z_1^{(k)} z_2^{(k)} I - z_2^{(k)} F_1 - z_1^{(k)} F_2) \neq 0, \tag{11}$$

for all $k = 1, \dots, \nu$, and there exists a unique solution $P \in \mathbb{R}^{\nu \times \nu}$ to the Sylvester-like equation

$$F_1 P S_2 + F_2 P S_1 + G_1 L S_2 + G_2 L S_1 = P S_1 S_2, \tag{12}$$

such that

$$C\Pi = HP, \tag{13}$$

where Π is the unique solution of (8).

Note that condition (11) ensures that (12) has a unique solution, condition (12) ensures well-defined moments for system (10), namely HP, and condition (13) enforces a moment matching condition. F_1, F_2, G_1, G_2, H and P are variables to be used to satisfy the conditions of the theorem.

System (10) matches the moments of system (4) at all interpolation pairs $(z_1^{(k)}, z_2^{(k)})$ characterized by the pair (S_1, S_2). In particular, it is called a reduced-order model of system (4) if $\nu < n$. Without loss of generality, we assume $\nu < n$ throughout the remainder of the paper.

Based on Theorem 7, by simply selecting $P = I$ (which is a choice that can be always performed without loss of generality), a family of reduced-order models that achieves moment matching at (S_1, S_2) is obtained with the parameterization

$$\begin{gathered} F_1 = \Gamma_1, \qquad F_2 = \Gamma_2, \qquad G_1 = \Delta_1, \\ G_2 = \Delta_2, \qquad H = C\Pi, \end{gathered} \tag{14}$$

with Δ_1, Δ_2, Γ_1 and Γ_2 any matrices such that

$$\Gamma_1 S_2 + \Gamma_2 S_1 = S_1 S_2 - \Delta_1 L S_2 - \Delta_2 L S_1 \tag{15}$$

and condition (11) holds. In particular, when S_2 (S_1, respectively) is non-singular, *i.e.*, $z_2^{(k)}$ ($z_1^{(k)}$, respectively) $\neq 0$ with $k = 1, \dots, \nu$, Γ_1 (Γ_2, respectively) can be uniquely determined as $\Gamma_1 = S_1 - \Delta_1 L - \Delta_2 L S_1 S_2^{-1} - \Gamma_2 S_1 S_2^{-1}$ ($\Gamma_2 = S_2 - \Delta_2 L - \Delta_1 L S_2 S_1^{-1} - \Gamma_1 S_2 S_1^{-1}$, respectively) for any Δ_1, Δ_2 and Γ_2 (Γ_1, respectively) such that condition (11) holds. In the case that S_1 and S_2 are both singular, the solution can be determined by solving a linear programme with the equality constraint (15) for any Δ_1, Δ_2 such that condition (11) holds.

With a change of coordinates one can show that system (14) contains all models of order ν that achieve moment matching at (S_1, S_2).

The above shows that when we match a set of ν moments, we have a great level of design freedom to select parameters

[8]See [25] for the definition of excitable pair.

Δ_1, Δ_2, Γ_1 and Γ_2. A special selection of parameters is given by $\Gamma_2 = 0$ and $\Delta_2 = 0$, resulting in a 1-D reduced-order model. This selection leads to reducing an infinite-dimensional system with a finite-dimensional system. This may not be desirable for some applications since it would alter the structure of the system and may result in loss of essential behaviours and characteristics. Thus, instead of this trivial selection which "wastes" the design freedom, in the next subsection we show how to exploit these free parameters to enforce one desired property onto the reduced-order model.

A. Interpolating with Prescribed Eigenvalues

Consider the parameterization (14) and the problem of determining Δ_1 and Δ_2 such that the eigenvalues of the reduced-order model are prescribed, *i.e.*, $\sigma(F_1) = \{\lambda_1, \dots, \lambda_\nu\}$ and $\sigma(F_2) = \{\lambda_{\nu+1}, \dots, \lambda_{2\nu}\}$, for some λ's such that condition (11) holds. To this end, from (15) we are able to select $\Gamma_1 S_2 = \alpha S_1 S_2 - \Delta_1 L S_2$ and $\Gamma_2 S_1 = (1-\alpha) S_1 S_2 - \Delta_2 L S_1$, for any $0 < \alpha < 1$. Then we notice that the goal can be achieved by selecting Δ_1 such that

$$\sigma(\alpha S_1 - \Delta_1 L) = \{\lambda_1, \dots, \lambda_\nu\}, \tag{16}$$

and selecting Δ_2 such that

$$\sigma((1-\alpha) S_2 - \Delta_2 L) = \{\lambda_{\nu+1}, \dots, \lambda_{2\nu}\}. \tag{17}$$

By the observability of (S_1, L) and (S_2, L), there exist unique values of Δ_1 amd Δ_2 such that conditions (16) and (17) hold, and then F_1 and F_2 can be obtained as $\Gamma_1 = \alpha S_1 - \Delta_1 L$ and $\Gamma_1 = (1-\alpha) S_2 - \Delta_2 L$. The constant α can be adjusted to control the magnitude of the obtained Δ_1 and Δ_2.

V. EXAMPLE

To demonstrate the developed 2-D moment matching theory, we consider the problem of reducing a low-pass filter used in [16]. The system is originally described by a Roesser model, which we embed into an FMLSS model of order $n = 12$ by performing the transformation in [9] (see equation (1.23) therein).

We consider the set of interpolation pairs as $(e^{\frac{\pi}{3}\iota}, e^{\frac{\pi}{4}\iota}), (e^{-\frac{\pi}{3}\iota}, e^{-\frac{\pi}{4}\iota}), (e^{\frac{\pi}{7}\iota}, e^{\frac{\pi}{5}\iota}), (e^{-\frac{\pi}{7}\iota}, e^{-\frac{\pi}{5}\iota})$, resulting in a reduced order of $\nu = 4$. The signal generator matrices S_1 and S_2 are constructed using the technique presented in Remark 1. We look for a stable reduced-order model. To this end, we use the parameterization (14) and determine the free parameters by optimizing over $\Gamma_1, \Gamma_2, \Delta_1$ and Δ_2 to minimize the sum of the $\mathcal{L}_2$ norms of Γ_1 and Γ_2 subject to constraint (15). Driven by the inputs $u(i,j)$ generated by the signal generator characterized by S_1 and S_2, the space-domain responses of the original and the reduced-order model are recorded in Fig. 1 and Fig. 2, respectively. It can be observed that although the (transient) responses are significantly different at the boundary, the steady-state responses exhibit a similar pattern. This observation is also verified by computing the absolute errors between $y(i,j)$ and $\phi(i,j)$, which decay rapidly to 0 as $i + j$ increases, as shown in Fig. 3. These plots demonstrate the fact that the reduced-order model interpolates the moments of the original system at our selected pairs.

VI. CONCLUSIONS

This paper presented a solution to the problem of model reduction by matching the zero-order moments of 2-D systems in the FMLSS framework. A parameterized family of reduced-order models that achieve moment matching was proposed. To showcase the practical use of the developed theory, a 2-D filter reduction problem was presented. Various potential research paths have been highlighted for future interest. These include the matching of high-order moments and the extension of the moment matching framework to 2-D nonlinear systems and general n-D systems.

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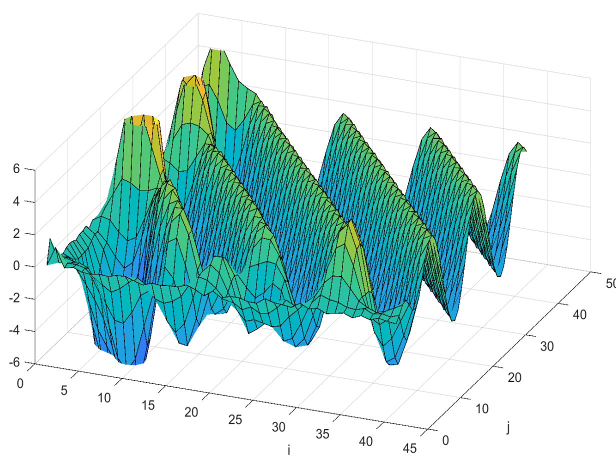

Fig. 1. Space-domain response of the *original* model as a function of coordinates i and j.

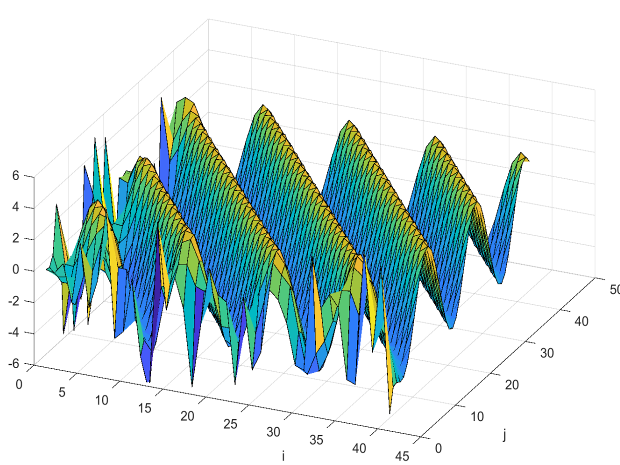

Fig. 2. Space-domain response of the *reduced-order* model as a function of coordinates i and j.

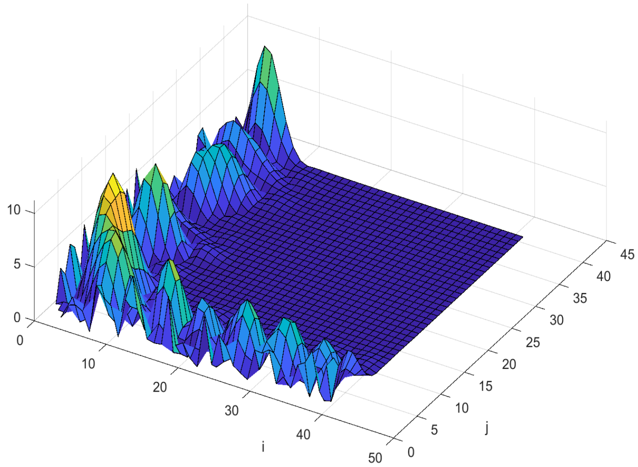

Fig. 3. Absolute errors between the outputs of the original model and the obtained reduced-order model.

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