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Optimal active sensing control for two-frame systems

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Abstract— **This paper provides a complete characterization of the trajectories that maximize the information collected by a moving vehicle, through sensors' measurements, for the recently introduced class of nonlinear "two-frame systems". The information is quantified in terms of the trace of the observability Gramian (OG) along a trajectory. In general, this quantity nontrivially depends on the control inputs and the state trajectory, resulting in a difficult optimal control problem. Herein, we leverage the property of invariant filtering that Jacobians are state-trajectory independent, that is, only depend on the control inputs, which enables us to mathematically derive optimal trajectories in closed form. We illustrate the results numerically on problems from robotics such as 3D robot localization, and 2D simultaneous localization and mapping.**

Index Terms— **localization, Lie groups, active estimation, invariant Kalman filtering.**

I. Introduction

The notion of observability of linear systems does not depend on the system's trajectory. By contrast, when turning to nonlinear systems, some trajectories may make the state unobservable, whereas others make it observable. As an extreme example, think of a wheeled robot whose position is measured: if it stays still, its orientation is unobservble, whereas if it is moving in straight line, its orientation is readily recovered. Beyond observability, which is a binary criterion, some trajectories are more informative than others in terms of state estimation of nonlinear systems. In the context of navigation of vehicles, amongst others, it may be useful to know which trajectory brings the most information about the state, to safely move in a constrained environment.

This has prompted previous authors to develop methods for trajectory generation that facilitate state estimation [1]–[3], or online parameter estimation [4]. This pertains to the theory of optimal experiment design [5] as these methods seek to find a control sequence to be applied in order to maximize the information acquired along the trajectory.

As an information measure, it is customary to use the observability Gramian, e.g., [6]. Some other criteria were used, though, such as Fisher's information, or the covariance matrix associated to the Riccati equation of an extended Kalman filter [7], or measures specifically tailored to the problem, e.g., [8] which targets trajectories leading to easier-to-exploit images from the onboard camera. More generally, *active* simultaneous localization and mapping (SLAM) improves state estimation through control, see e.g., [9].

The most widespread criterion to measure the extent of observability brought by a trajectory is the local OG, introduced by Krener and Ide [10]. In some simple cases, the optimal trajectories might be computed analytically, see [6], but this is generally not the case, and one needs to resort to numerical methods for trajectory planning [11] or trajectory modification for sensors calibration [1]. Despite the drawbacks of Gramian maximization based planning methods [7], the local OG or the closely related Constructibility Gramian [3] remains a relevant tool to establish the most informative trajectories, especially when the system's model is well-known, i.e., the process noise is low.

In this paper, we consider the recently introduced theory of two-frame systems [12], which introduces a large class of navigating vehicles having remarkable properties in terms of state estimation. For those systems, we benefit from the properties of invariant filtering that provide state-trajectory independent Jacobians [13]. This makes the OG independent of the trajectory being followed, and only dependent on the controls, and hence simplifies the associated optimal control problem. Leveraging this property enables us to derive optimal trajectories in closed form that maximize the trace of the OG for two-frames systems.

The main contributions of this paper are:

- The use of the invariant errors to make the observability Gramian independent of the system's state;
- A general analytical computation of the optimal control that maximizes the observability Gramian for a broad class of systems that fall into the theory of two-frames navigating systems;
- Application to three problems: a vehicle that navigates in 3D and estimates its state from position measurements performed by a GNSS, a 2D vehicle equipped with a GNSS mounted on an arm with unknown position in the vehicle (lever arm), and the problem of SLAM for a 2D wheeled robot.

Section II formulates the perception-aware control problem, Section III summarizes the theory of two-frame systems, Section IV calculates the optimal control to apply when facing world frame measurements, Section V addresses the problem when facing vehicle frame measurements, and Section VI illustrates the results numerically.

II. Optimal perception-aware trajectories

In this section we recall the problem of active sensing. It is a path planning problem where the objective function is an information criterion. Therefore, it will be easier to

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estimate the system state along the optimized trajectory. Let us consider a general discrete-time dynamical system:

$$x_{n+1} = f(x_n, u_n),\ x_0 = x^0 \tag{1}$$

$$y_n = h(x_n) \tag{2}$$

where $x_n \in \mathbb{R}^p$ represents the state of the system, $u_n \in \mathbb{R}^m$ is the control input, $y_n \in \mathbb{R}^q$ is the sensor output at time n. One approach to measure the extent of observability associated to a given trajectory is based on the observability Gramian (OG). The discrete-time OG is defined by:

$$W_K = \sum_{n=1}^{K} \Phi_n^T H_n^T H_n \Phi_n \tag{3}$$

where $H_n = \frac{\partial h(x_n)}{\partial x_n}$ and $\Phi_n = F_n \Phi_{n-1}$, $\Phi_{-1} = I_d$ with $F_n = \frac{\partial f(x_n, u_n)}{\partial x_n}$ and K the estimation horizon. It should be noted that in the case of linear systems, F_n neither depends on the state of the system nor the controls. This implies that the OG does not depend on the trajectory: all trajectories are equally informative. However, in the nonlinear case, observability depends on the trajectory, see [6], [10]. The optimal inference problem consists then in finding the trajectories along which the OG is maximum. Thus, the trajectory planning problem writes:

$$\boxed{\begin{array}{ll} & u^* = \underset{u_0,\dots,u_{K-1}}{\arg\max}\ Tr(W_K) \\ \text{subject to} & \forall n\ x_{n+1} = f(x_n, u_n) \\ & \forall n\ u_n \in C_n \end{array}} \tag{4}$$

$u^* \in \mathbb{R}^{pK}$ refers to the sequence of vectors of controls to apply. At each time instant n, the control vector is required to belong to a bounded set $C_n \subset \mathbb{R}^p$.

The interest of the above criterion is its simple interpretation in terms of the (linearized) error variables denoted by $\delta x_n := x_n - \bar{x}_n$ and $\delta y_n := y_n - \bar{y}_n$. Up to the first order, they evolve according to the linearized Jacobians, that is, $\delta x_{n+1} = F_n \delta x_n$ and $\delta y_n = H_n x_n$. As the initial error is unknown, we may assume it be random and isotropically distributed, for instance $\delta x_0 \sim \mathcal{N}(0, I_d)$. We have then.

Lemma 1: Assume $\delta x_0 \sim \mathcal{N}(0, I_d)$ denotes the discrepancy between two initial states. We have, neglecting second order terms, the equality

$$E\left(\sum_{n=1}^{K} ||\delta y_n(\delta x_0)||_2^2\right) = Tr\left(W_K\right). \tag{5}$$

This is proved by writing

$$E\left(\sum_{n=1}^{K} ||\delta y_n(\delta x_0)||_2^2\right) = E\left(\sum_{n=1}^{K} \delta x_0^T \Phi_n^T H_n^T H_n \Phi_n \delta x_0\right)$$
$$= \sum_{n=1}^{K} E\left(Tr\left(\delta x_0^T \Phi_n^T H_n^T H_n \Phi_n \delta x_0\right)\right)$$
$$= Tr\left(\sum_{n=1}^{K} \Phi_n^T H_n^T H_n \Phi_n E(\delta x_0 \delta x_0^T)\right) = Tr\left(W_K\right)$$

where we used the commutation properties of the trace and that $E(\delta x_0 \delta x_0^T) = I_d$.

This provides the following simple interpretation: the optimal sequence of control inputs makes *on average* the (first-order) discrepancy between the predicted and measured outputs as large as possible, for a random error about which we do not have any information. The trace of the OG is thus a way of measuring the extent of observability gained from the generated trajectory.

In general, the optimisation problem is difficult to solve because of the numerous constraints. To overcome this difficulty, several authors [3], [1] restrict the study to flat systems [14]. In this case one may perform parametric (suboptimal) optimization, where the parameters are a number of way-points for the flat output. In the following, we consider the recently introduced class of two-frames navigating systems, and leverage their properties to show problem (4) is amenable to a more simple optimal control problem, that may be solved analytically.

III. Reminder: the theory of two-frame systems

The theory of two-frame natural systems [12] is general and encompasses numerous examples from robotics and inertial navigation. The idea is to consider systems consisting of a body that navigates, and whose state is defined by a set of vectors, as well as a variable encoding a change of frame from the world frame to the frame attached to the body. In the present paper, we will focus on the following two-frames systems. The state writes

$$\begin{pmatrix} R \\ x \\ \mathrm{X} \end{pmatrix} \in SO(d) \times \mathbb{R}^{dN_x} \times \mathbb{R}^{dN_\mathrm{X}},\ x = \begin{pmatrix} x^1 \\ \vdots \\ x^{N_x} \end{pmatrix},\ \mathrm{X} = \begin{pmatrix} \mathrm{X}^1 \\ \vdots \\ \mathrm{X}^{N_\mathrm{X}} \end{pmatrix}.$$

$R \in G = SO(d)$ with $d = 2$ or $d = 3$ is a rotation matrix that encodes the orientation of the frame attached to the navigating body with respect to the world-fixed frame, x is a set of N_x vectors of $\mathbb{R}^d$, that is, the d-dimensional world in which the body is navigating, which encode physical quantities expressed in the fixed frame, such as the position and velocity of a craft, and X is a set of vectors expressed in the body frame, such as a sensor bias, where the sensor is attached to the body, see Figure 1.

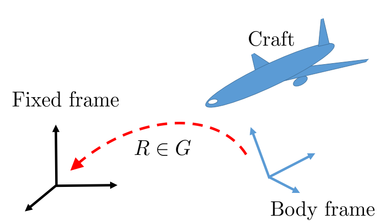


Fig. 1. The state of a navigating two-frames system, to be estimated, consists of a rotation matrix R, and vectors, e.g., the position, velocity, or sensor biases, stacked in x (when expressed in the fixed frame) and X (when expressed in the body frame). R transforms vectors of the body frame denoted by uppercase letters to vectors of the world-fixed frame denoted by lowercase letters, and as such encodes the orientation of the craft.

To fix ideas, let us start by giving two examples of interest, before we turn to the unifying theory of two-frame systems.

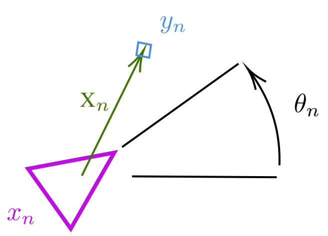
A. Examples

The first example is in 2D and the second one in 3D.

1) First example, from [12]: Consider the classical 2D model of a non-holonomic car, see e.g., [13], or equivalently a wheeled robot or a unicyle. The position of the car in 2D is described by the middle point of the rear wheels axle $x_n \in \mathbb{R}^2$, and its orientation (heading) denoted by $\theta_n \in \mathbb{R}$ and parameterized by the planar rotation matrix R_n of angle θ_n that is $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$. The car is equipped with a GNSS located at unknown position $\mathrm{x}_n \in \mathbb{R}^2$ in the car frame with respect to point x_n (a lever arm to be estimated), which provides the world (fixed) frame position measurements

$$y_n = x_n + R_n \mathrm{x}_n \in \mathbb{R}^2. \tag{6}$$

In the schematic diagram below, the triangle is the car and the square is the position measured by the GNSS.



The discrete-time dynamics of a mobile wheeled robot write:

$$R_{n+1} = R_n\Omega_n, \quad x_{n+1} = x_n + R_n \mathrm{u}_n, \quad \mathrm{x}_{n+1} = \mathrm{x}_n, \tag{7}$$

The control inputs are the angular velocity Ω_n, which can be controlled through steering or via the difference between wheel speeds, and u_n that is the average of wheel speeds.

2) Second example: We consider a body that navigates in $3D$, and which is equipped with a GNSS. We consider the acceleration in the body frame as well as the angular velocity to be (possibly high-level) control inputs, and discretize the continuous-time equations, yielding dynamics of the form

$$\begin{cases} R_{n+1} = R_n\Omega_n \\ p_{n+1} = p_n + \Delta t \; v_n \\ v_{n+1} = v_n + R_n \mathrm{u}_n \end{cases}, \tag{8}$$

with observation $Y_n = p_n$.

Let us now turn to the unifying framework of two-frame systems, that encompasses the two latter examples.

B. Two-frames general theory

In this paper, we consider the following subclass of discrete-time dynamics amongst the two-frame dynamics of [12]

$$\boxed{\begin{pmatrix} R_{n+1} \\ x_{n+1} \\ \mathrm{x}_{n+1}\end{pmatrix} = \begin{pmatrix} R_n\Omega_n \\ \boldsymbol{F} x_n + \mathrm{d}_n + R_n * \mathrm{u}_n \\ \mathrm{x}_n \end{pmatrix},} \tag{9}$$

where $*$ defines the following operation, called an action

$$R * x = R * \begin{pmatrix} x^1 \\ \vdots \\ x^{N_x}\end{pmatrix} := \begin{pmatrix} Rx^1 \\ \vdots \\ Rx^{N_x}\end{pmatrix}, \quad R * \mathrm{x} := \begin{pmatrix} R\mathrm{x}^1 \\ \vdots \\ R\mathrm{x}^{N_\mathrm{x}}\end{pmatrix}, \tag{10}$$

and where the matrix $\boldsymbol{F} : \mathbb{R}^{dN_x} \mapsto \mathbb{R}^{dN_x}$ is of the form

$$\boldsymbol{F} = \begin{pmatrix} \alpha_{11}\boldsymbol{I}_d & \dots & \alpha_{1N_x}\boldsymbol{I}_d \\ \vdots & & \vdots \\ \alpha_{N_x1}\boldsymbol{I}_d & \dots & \alpha_{N_xN_x}\boldsymbol{I}_d \end{pmatrix} \tag{11}$$

with $\alpha_{ij} \geq 0$'s real nonnegative numbers, and $\mathrm{d}_n, \mathrm{u}_n \in \mathbb{R}^{dN_x}$. It can be readily noticed that such a matrix commutes with the action, that is $\boldsymbol{F}(R * x) = R * (\boldsymbol{F}x)$.

The rotation R maps vectors of the body frame to vectors of the world-fixed frame. It is reasonable to assume that the control inputs are vectors expressed in the body frame, as actuators are attached to the craft. We will therefore consider d_n to be a known input, and Ω_n, u_n to consist of the (possibly high-level) control inputs we will use for trajectory generation. The fact that $\mathrm{u}_n \in \mathbb{R}^{dN_x}$, does not preclude the system from being underactuated, since as many components of u_n as necessary may be set to 0 (through constraints). The reason for the equation $\mathrm{x}_{n+1} = \mathrm{x}_n$ in (9) is that the variables of the body frame that we seek to estimate are considered as parameters, as is the case of lever arms (see the first example above) and accelerometer and gyrometer biases in the general theory of two-frames.

Regarding state estimation, the two-frames systems are endowed with sensors providing the vehicle with measurements. In the theory of two-frames systems of [12], they are of two types. Indeed we use either an output map h representing measurements in the fixed frame, or H representing measurements in the frame of the vehicle, defined as

$$\boxed{\text{fixed-frame: } h(R, x, \mathrm{x}) = \boldsymbol{H}^x x + R * [\boldsymbol{H}^\mathrm{x} \mathrm{x}]} \tag{12}$$

$$\boxed{\text{body-frame: } \mathrm{H}(R, x, \mathrm{x}) = R^{-1} * [-\boldsymbol{H}^x x] - \boldsymbol{H}^\mathrm{x} \mathrm{x}} \tag{13}$$

with $\boldsymbol{H}^x : \mathbb{R}^{dN_x} \mapsto \mathcal{Y}$ and $\boldsymbol{H}^\mathrm{x} : \mathbb{R}^{dN_\mathrm{x}} \mapsto \mathcal{Y}$ two linear maps that commute with the action of G, that is, are of the form

$$\boldsymbol{H}^x = \begin{pmatrix} \beta_{11}\boldsymbol{I}_d & \dots & \beta_{1N_x}\boldsymbol{I}_d \\ \vdots & & \vdots \\ \beta_{N_y1}I_d & \dots & \beta_{N_yN_x}I_d \end{pmatrix} \tag{14}$$

and similarly for $\boldsymbol{H}^\mathrm{x}$, and where $\mathcal{Y} = \mathbb{R}^{dN_y}$ denotes the output space. In the theory, R also acts on $\mathcal{Y} = \mathbb{R}^{dN_y}$ via term-by-term action similarly to (10).

1) Link to the first example: We see the first example fits the present framework of two-frames systems indeed, as (7) correspond to (9), with $d = 2, N_x = 1, N_\mathrm{x} = 1$, hence action $*$ is trivially the matrix-vector product, and $\boldsymbol{F} = I_2$. The measured output corresponds to fixed-frame measurements (12), with $\boldsymbol{H}^x = \boldsymbol{H}^\mathrm{x} = I_2$.

2) Link to the second example: We see the second example above fits the present framework of two-frames systems indeed, as (7) correspond to (9), with $d=3, N_x=2, N_{\mathrm{X}}=0$, $\mathrm{d}_n=0$, action $*$ is $R*(p,v)=(RP,Rv)$, and

$$\boldsymbol{F}=\begin{pmatrix} I_3 & \Delta t I_3 \\ 0_3 & I_3 \end{pmatrix},$$

which is of the form (11). The measured output corresponds to fixed-frame measurement (12), with $\boldsymbol{H}^x=I_2$.

C. State-trajectory independent error evolution

The theory of invariant filtering [15], having its roots in the theory of invariant observers [16], [17], and which underlies the recent theory of two-frame systems, uses alternative error variables to recover some properties of linear systems, especially the fact the errors evolve independently from the state trajectory. This notably allows for state-trajectory independent Jacobians, when linearizing the system.

When confronted with observations in the fixed frame (12), we use as an alternative state error, the left-invariant error between two states defined as

$$\text{Obs. (12)} \Rightarrow \boxed{\mathrm{E}=\begin{pmatrix} \bar{R}^{-1}R \\ \bar{R}^{-1}*(x_n-\bar{x}) \\ \mathrm{X}-(R^{-1}\bar{R})*\bar{\mathrm{X}} \end{pmatrix}} := \begin{pmatrix} \mathrm{E}^R \\ \mathrm{E}^x \\ \mathrm{E}^{\mathrm{X}} \end{pmatrix}. \quad (15)$$

If observations are performed in the body frame instead (13) one should use the right-invariant error defined as

$$\text{Obs. (13)} \Rightarrow \boxed{e=\begin{pmatrix} R\bar{R}^{-1} \\ x-(R\bar{R}^{-1})*\bar{x} \\ \bar{R}*(\mathrm{X}-\bar{\mathrm{X}}) \end{pmatrix}} := \begin{pmatrix} e^R \\ e^x \\ e^{\mathrm{X}} \end{pmatrix}. \quad (16)$$

If we evolve two state variables through (9), the corresponding *left-invariant* error evolves as:

$$\begin{aligned} \mathrm{E}^R_{n+1} &= \Omega_n^{-1}\mathrm{E}^R_n\Omega_n \\ \mathrm{E}^x_{n+1} &= \Omega_n^{-1}*[\boldsymbol{F}\mathrm{E}^x_n+\mathrm{E}^R_n*\mathrm{U}_n-\mathrm{U}_n] \\ \mathrm{E}^{\mathrm{X}}_{n+1} &= \mathrm{X}-\Omega_n^{-1}\mathrm{E}^R_n\Omega_n*\bar{\mathrm{X}}. \end{aligned} \quad (17)$$

We see the evolution of the error is independent of the state variable (R,x,X) but for the last line. However, if $d=2$, rotations commute and the last line becomes $\mathrm{E}^{\mathrm{X}}_{n+1}=\mathrm{E}^{\mathrm{X}}_n$ and the error evolution is wholly state-trajectory independent. This is Theorem 5 of [12].

If we evolve two state variables through (9), the corresponding *right-invariant error* evolves as:

$$\begin{aligned} e^R_{n+1} &= e^R_n \\ e^x_{n+1} &= \boldsymbol{F}_n e^x_n+\mathrm{d}_n-e^R_n\mathrm{d}_n \\ e^{\mathrm{X}}_{n+1} &= \bar{R}\Omega_n*(\mathrm{X}-\bar{\mathrm{X}}). \end{aligned} \quad (18)$$

We see the evolution of the error is independent of the state variable (R,x,X) but for the last line. However, if $d=2$, rotations commute and the last line becomes $e^{\mathrm{X}}_{n+1}=\Omega_n*e^{\mathrm{X}}_n$ and the error evolution is wholly state-trajectory independent. This is Theorem 5 of [12].

D. State-trajectory independent prediction error

Instead of using the usual prediction error $y-\bar{y}=h(R,x,\mathrm{X})-h(\bar{R},\bar{x},\bar{\mathrm{X}})$ of state estimation, called innovation in the literature devoted to Kalman filtering, the theory of invariant filtering advocates the use of alternative innovation variables, that also reflect the prediction error. The innovation variable Z associated to fixed-frame output y, i.e., Obs. (12), writes:

$$\boxed{\mathrm{Z}=\bar{R}^{-1}*(y-\boldsymbol{H}^x\bar{x})-\boldsymbol{H}^{\mathrm{X}}\bar{\mathrm{X}}} \quad (19)$$

while the innovation variable z associated to body-frame output Y, i.e., Obs. (13), writes:

$$\boxed{z=\bar{R}*(\mathrm{Y}+\boldsymbol{H}^{\mathrm{X}}\bar{\mathrm{X}})+\boldsymbol{H}^x\bar{x}} \quad (20)$$

The interest of those alternative innovation variables, is the strong link they bear with the alternative errors we have defined. Indeed,

$$\text{Error (15)} \Rightarrow \mathrm{Z}_n=\boldsymbol{H}^x\mathrm{E}^x_n+\boldsymbol{H}^{\mathrm{X}}\mathrm{E}^R_n*\mathrm{E}^{\mathrm{X}}_n, \quad (21)$$

$$\text{Error (16)} \Rightarrow z_n=-\boldsymbol{H}^{\mathrm{X}}e^{\mathrm{X}}_n-\boldsymbol{H}^x\left(e^R_n\right)^{-1}*e^x_n. \quad (22)$$

This result of [12] is easily proved. It will allow for state-independent Jacobian of the output map h, that we are about to leverage to simplify our perception-aware control problem.

IV. Optimal solutions to active sensing control with world frame measurements

This section addresses the problem of optimal perception-aware trajectory generation for the class of two-frames systems we have considered. Our main result is to prove that they can be computed explicitly for this general enough class of systems modeling vehicles that navigate.

Our goal is to maximize the global information acquired over a given future horizon, i.e., K-steps trajectory. This is measured as the trace of the observability Gramian matrix, see problem (4). To get this matrix using our alternative errors, we need the Jacobians resulting from linearizing the error system. In Subsection IV-C, we will relate this to the original control objective based on standard linear errors in the original variables, as classically used by the extended Kalman filter.

In this section we consider the case of measurements performed in the world frame. In the next, we will consider body-frame measurements.

A. Linearized error system and observability Gramian

We would like, for (R,x,X) close to $(\bar{R},\bar{x},\bar{\mathrm{X}})$, a first-order (i.e., linearized) approximation to the left-invariant error (15). Recall that we may relate a rotation matrix with a vector through the matrix exponential $\bar{R}^{-1}R=\exp((\xi^R)_\times)$ where

$$d=3 \Rightarrow \xi^R\in\mathbb{R}^3,\ (\xi^R)_\times=\begin{pmatrix} 0 & -\xi^R_3 & \xi^R_2 \\ \xi^R_3 & 0 & -\xi^R_1 \\ -\xi^R_2 & \xi^R_1 & 0 \end{pmatrix}$$

and $d=2 \Rightarrow \xi^R\in\mathbb{R},\ (\xi^R)_\times=\xi^R J$ with $J=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Using a first order expansion of the matrix exponential we have $\mathrm{E}^R = \bar{R}^{-1}R \approx I_d + (\xi^R)_\times$ and the linearized error is then defined as $(\xi^R, \xi^x, \xi^{\mathrm{X}})$, where we let $\xi^x = x - \bar{x}$ and $\xi^{\mathrm{X}} = \mathrm{X} - \bar{\mathrm{X}}$. Substituting those approximations in (17) we find in the 2D case $d = 2$

$$\begin{aligned} \xi^R_{n+1} &= \xi^R_n \\ \xi^x_{n+1} &= \Omega_n^{-1} * [\boldsymbol{F}\xi^x_n + J\mathrm{U}_n\xi^R_n] \\ \xi^{\mathrm{X}}_{n+1} &= \xi^{\mathrm{X}}_n, \end{aligned} \tag{23}$$

and in the 3D case $d = 3$ by not including X in the state

$$\begin{aligned} \xi^R_{n+1} &= \Omega_n^{-1}\xi^R_n \\ \xi^x_{n+1} &= \Omega_n^{-1} * [\boldsymbol{F}\xi^x_n - \mathrm{U}_n \times \xi^R_n] \end{aligned} \tag{24}$$

where we also used that $\Omega_n^{-1}(\xi^R_n)_\times\Omega_n = (\Omega_n^{-1}\xi^R_n)_\times$. This yields the linearized system (hence the Jacobian) associated to the dynamics, when written in the invariant variables. Regarding the Jacobian of map h, it is related to the linearization of the innovation, which yields from (21) neglecting second order terms

$$\delta \mathrm{Z}_n = \boldsymbol{H}^x\xi^x_n + \boldsymbol{H}^{\mathrm{X}}\xi^{\mathrm{X}}_n. \tag{25}$$

Our goal is to find the control inputs Ω_n, U_n's that maximize the trace of the observability Gramian associated to the linearized systems (23), (25) in 2D and (24)-(25) in 3D. Recalling the derivation (5), this is equivalent to maximizing $\sum_{k=1}^K E\left(||\delta \mathrm{Z}_k||^2\right)$, for a random isotropically distributed initial error.

B. Optimal trajectory generation

Recalling that $\mathrm{U} \in \mathbb{R}^{dN_x}$ is a multi-vector, we take as constraints that at all time i, each component U_i^k of the input must remain in a predefined bounded set $C_{ik} \subset \mathbb{R}^{N_x}$.

Our first main theoretical result states that, in the case of world-frame measurements, optimal solutions are obtained by maintaining a fixed orientation R_n and taking the control inputs collinear and of maximum authorized norm.

Theorem 1: Consider a two-frames system with dynamics (9) and observations in the fixed frame given by (12), such that the β_{ij}'s in (14) related to $\boldsymbol{H}^x$ (only) are either all nonnegative or all nonpositive. Moreover, suppose that:

1) Either $d = 2$, that is, the vehicle evolves in 2D space,
2) or $d = 3$ and there is no variable X in the state, that is, the state consists of variables (R, x).

Then, assume we may take $\Omega_k^* = I_d$ and U_k^* such that the scalar product across components and times satisfies $\mathrm{U}_i^k.\mathrm{U}_j^l = \max_{C_{ik}}||\mathrm{U}_i^k||\max_{C_{jl}}||\mathrm{U}_j^l||$ for $0 \leq i, j \leq K-1, 1 \leq k, l \leq N_x$. This defines an optimal trajectory.

Proof: We begin noticing from (24) the variable ξ^{X} is fixed when $d = 2$, and when $d = 3$ we supposed it was not part of the state. As the components of the error are assumed initially independent, this yields $E\left(||\delta \mathrm{Z}_k||^2\right) = E\left(||\boldsymbol{H}^x\xi^x_k||^2\right) + E\left(||\boldsymbol{H}^{\mathrm{X}}\xi^{\mathrm{X}}_k||^2\right)$. As a result all we need is to maximize $\sum_{k=1}^K E\left(||\boldsymbol{H}^x\xi^x_k||^2\right)$. Without loss of generality let us focus on the 3D case (24). By recursion we may prove

$$\begin{aligned} \xi^R_k &= \Omega_{k-1}^{-1}\dots\Omega_0^{-1}\xi^R_0 = (\Pi_0^{k-1}\Omega_j)^{-1}\xi^R_0 \\ \xi^x_k &= (\Pi_0^{k-1}\Omega_l)^{-1} * \boldsymbol{F}^k\xi^x_0 \\ &\quad - \sum_{j=1}^k (\Pi_{l=j-1}^{k-1}\Omega_l)^{-1} * \boldsymbol{F}^{k-j}\mathrm{U}_{j-1} \times \xi^R_{j-1} \end{aligned} \tag{26}$$

where we have used that $\Omega*$ and $\boldsymbol{F}$ commute. We see the goal boils down to finding the control sequence that maximizes $E||\boldsymbol{H}^x\sum_{j=1}^k(\Pi_{l=j-1}^{k-1}\Omega_l)^{-1} * \boldsymbol{F}^{k-j}\mathrm{U}_{j-1} \times (\Pi_0^{k-1}\Omega_j)^{-1}\xi^R_0||^2$.

Lemma 2: Let $u_1, u_2 \in \mathbb{R}^d$ with $d = 2, 3$, let $\Omega \in SO(d)$ and let $\xi \sim \mathcal{N}(0, I_d)$. We have $E(u_1 \times \xi \ . \ u_2 \times \Omega\xi) \leq (d-1)||u_1||\ ||u_2||$. Equality is obtained for $\Omega = I_d$ and $u_2 = \gamma u_1, \gamma > 0$. For $d = 3$ it is unique. For $d = 2$ the maximum is attained for all combinations $\Omega \in SO(2), u_2 = \gamma\Omega u_1, \gamma > 0$.

Proof: Using the properties of mixed product followed by triple product expansion we find $u_1 \times \xi \ . \ u_2 \times \Omega\xi = (u_1.u_2)(\xi.\Omega\xi) - (u_1.\xi)(u_2.\Omega\xi) = (u_1.u_2)Tr(\xi^T\Omega\xi) - Tr(u_1^T\xi\xi^T\Omega^Tu_2)$, whose expectation equals $(u_1.u_2)Tr(\Omega E\xi\xi^T)) - Tr(u_1^TE(\xi\xi^T)\Omega^Tu_2) = Tr(\Omega)(u_1.u_2) - u_1^T\Omega^Tu_2 := g(u_1, u_2, \Omega)$. We write that

$$\max_{||u_1||=||u_2||=1, \Omega\in SO(d)} g = \max_{||u_1||=1}\ \max_{\Omega\in SO(d)}\ \max_{||u_2||=1} g$$

As $g(u_1, u_2, \Omega) = (Tr(\Omega)u_1 - \Omega u_1).u_2$ the method of Lagrange multipliers yields that $\max_{||u_2||=1} g$ is attained for $u_2^* = [Tr(\Omega)u_1 - \Omega u_1]/||Tr(\Omega)u_1 - \Omega u_1||$, and $g(u_1, u_2^*, \Omega) = ||Tr(\Omega)u_1 - \Omega u_1|| = \sqrt{1 + Tr(\Omega)^2 - 2Tr(\Omega)u_1^T\Omega u_1}$. To conclude we use two facts. First, letting θ be the rotation angle associated to Ω, we have $Tr(\Omega) = 1 + 2\cos\theta$ for $d = 3$. Then, we have $\cos\theta \leq \Omega u_1.u_1 \leq 1$, for $||u_1|| = 1$. This stems from using the Rodrigues formula and the triple product expansion that proves $\Omega u.u = 1 + (1-\cos(\theta))((v.u)^2 - 1) \geq 1 + (1-\cos(\theta))(-1) = \cos\theta$, with v the unit vector encoding the rotation axis. Thus $g^2 \leq 1 + Tr(\Omega)^2 - 2Tr(\Omega)\cos\theta = 2 + 2\cos\theta$. This proves $g \leq 2$ with equality attained if and only if $\cos\theta = 0$, that is, $\Omega = I_3$.

If $d = 2$, only the trace is modified and we find $g = 1$, that is, the maximum is non uniquely attained for arbitrary Ω, letting $u_2 = u_2^*$. In particular it is attained for $\Omega = I_2$ and $u_2^* = u_2$. ■

Let us go back to our control objective of maximizing $E||\boldsymbol{H}^x\sum_{j=1}^k(\Pi_{l=j-1}^{k-1}\Omega_l)^{-1} * \boldsymbol{F}^{k-j}\mathrm{U}_{j-1} \times (\Pi_0^{k-1}\Omega_j)^{-1}\xi^R_0||^2$. We need to recall we are dealing with multi-vectors, and that $\boldsymbol{H}^x$ is of the form (14), where the β_{ij}'s have the same sign. We may expand the squared norm, and we recover a linear combination with positive coefficients of scalar products of the form $E(\tilde{u}_1 \times \xi \ . \ \tilde{u}_2 \times \Omega\xi)$, using our assumption on $\boldsymbol{F}$ and using that for rotation matrices A, B we have $(A\tilde{u}_1)\times(A\xi) \ . \ (B\tilde{u}_2)\times(B\xi) = \tilde{u}_1\times\xi \ . \ (A^TB\tilde{u}_2)\times(A^TB\xi)$. We may bound those terms using the lemma, and to get an equality we take all Ω_k to be I_d. If the condition of the theorem $\mathrm{U}_i^k.\mathrm{U}_j^l = \max_{C_{ik}}||\mathrm{U}_i^k||\max_{C_{jl}}||\mathrm{U}_j^l||$ is met, it implies from Cauchy-Schwarz inequality the collinearity of the components and we see from the lemma the sum reaches its maximum. ■

For instance, if we consider Example 2 with dynamics (8) and constraints $||\mathrm{U}_n|| \leq \gamma$, we see the optimal trajectory is obtained by fixing the orientation, and taking full acceleration. Note that, there is no requirement regarding R_n apart from it being fixed: the maximizing acceleration could be lateral if constraints permit. Besides, we note that such a system is not flat, as the vehicle's orientation is not related to the trajectory, contrary to wheeled robots.

Remark 1: The optimal trajectory of the theorem is not necessarily unique mathematically. However, when $d = 3$, the proof shows the condition $\Omega_k^* = I_d$ is inevitable. Moreover, when $d = 2$, the other maximizing trajectories are in practice not feasible (they violate the constraints) and in any case more complicated to achieve while not improving the objective.

C. Connection to the original variables

We now relate this to the original Gramian maximization problem. Indeed, the fact that the trajectory be optimal for given modified error and innovation, does not a priori mean it is optimal in the problem's original error variables, or more prosaically speaking using the conventionally defined Jacobians in (4). It turns out, though, that in the present case both are equivalent.

Proposition 1: The optimal trajectories of Theorem 1 are optimal too for the problem (4) using the original variables of the problem.

Proof: The proof is based on the use of the equivalent formulation (5). Indeed, we see that if we take a two-frame system above, we have $\mathrm{Z} = \bar{R}^{-1} * (y - \bar{y})$, where $\bar{y} = \boldsymbol{H}^x \bar{x} - \bar{R} * [\boldsymbol{H}^{\mathrm{X}} \bar{\mathrm{X}}]$ denotes the predicted output. As a result $||\mathrm{Z}|| = ||y - \bar{y}||$ so that both objectives coincide. ■

V. Optimal solutions to active sensing control with body frame measurements

In this section, we first provide the counterpart of Theorem 1 for body-frame measurements. Then, we step back and discuss the trace criterion.

A. Optimal trajectory generation

When facing body-frame measurements, one should use right-invariant errors, leading to the error system (18). Along the lines of the theory of two-frames, we see the two following cases lead to a state-independent error:

1) Either $d = 2$, that is, the vehicle evolves in 2D space,
2) or $d = 3$ and there is no variable X in the state, that is, the state consists of variables (R, x),

which are the hypotheses we used in Theorem 1. In the first case, the rotations commute, leading to autonomy of the error evolution. In the case of body-frame measurements, it turns out that all trajectories bring equal information, exactly as is the case for linear systems. This is due to the linearized error system being virtually independent of the controls.

Theorem 2: Consider a two-frame system with dynamics (9) and observations in the body frame given by (13). Then, under any of the two assumptions above, all trajectories yield an equal trace of the observability Gramian, in other words all trajectories are optimal for problem (4).

Proof: The proof is in two steps. First we show the result using the Jacobians related to the right-invariant error, as a natural consequence of the fact the error evolution is (almost) independent of the controls. Then, we relate the result to the Jacobians expressed using the original variables.

As before, we may linearize the innovation (22) which yields $\delta z_k = -\boldsymbol{H}^{\mathrm{X}} \xi_k^{\mathrm{X}} - \boldsymbol{H}^x \xi_k^x$. Both ξ^x and ξ^{X} evolve independently, so that we have $E\left(||\delta z_k||^2\right) = E\left(||\boldsymbol{H}^{\mathrm{X}} \xi_k^{\mathrm{X}}||^2\right) + E\left(||\boldsymbol{H}^x \xi_k^x||^2\right)$. The latter term is independent of the controls, see (18). Regarding ξ^{X}, we see from (18) that in the case where it is present in the state, that is, when $d = 2$, it evolves as $\xi_{n+1}^{\mathrm{X}} = \Omega_n * \xi_n^{\mathrm{X}}$. Recalling we are dealing with multivectors and the form of the matrix (14), and that rotations are isometries, we see $E\left(||\boldsymbol{H}^{\mathrm{X}} \xi_k^{\mathrm{X}}||^2\right)$ is also independent of the controls. This proves all trajectories yield the same trace of the Gramian, when using the Jacobians of invariant filtering, that is, those related to the right-invariant error.

If we turn to the original variables of the problem, we may exploit the link between problem (4) and relation (5). Besides, we see that $z = \bar{R} * (\mathrm{Y} - \bar{\mathrm{Y}})$, where $\bar{\mathrm{Y}} = \bar{R}^{-1} * b - \bar{R}^{-1} * \boldsymbol{H}^x \bar{x} - \boldsymbol{H}^{\mathrm{X}} \bar{\mathrm{X}}$, denotes the predicted output. As a result $||z|| = ||\mathrm{Y} - \bar{\mathrm{Y}}||$ so that both objectives coincide. ■

B. Discussion and further results

Whether in the case of fixed-frame or body-frame observations, we were able to derive closed-form expressions for the trajectories that maximize the information for a large class of problems, in the sense of the trace of the OG criterion, for which we provided a probabilistic interpretation in terms of averaged observed linearized error. In this subsection, we would like to step back and reflect on the limitations of this widespread criterion.

Albeit useful, and allowing for closed-form expressions, the trace criterion might prove insufficient for a number of applications revolving around navigation of 2-frames systems. Indeed, as shown in (5), the trace is related to an average over random (unknown) initial error. A trajectory that optimizes such a criterion does not preclude the risk of falling into an unobservable configuration for a particular initial error, though. As a result, the following objective shall also be interesting for a variety of applications

$$\max_{\mathrm{U}_0, \ldots \mathrm{U}_{K-1}} \min_{||\xi_0||=1} \sum_{k=1}^{K} ||\delta \mathrm{Z}_k(\xi_0)||_2^2, \tag{27}$$

which coincides with $\max_{\mathrm{U}_0, \ldots \mathrm{U}_{K-1}} \min_i \lambda_i(W_K)$, i.e., maximizing the smallest eigenvalue of the Gramian as seen from (5), and advocated in [3]. We see that in 3D a uniformly accelerated motion fails to allow for estimation of the vehicle's angle around the acceleration axis, and as such it may be optimal for the Gramian's trace criterion but not for the latter. Deriving the optimal trajectories in closed form - as we just did - for this max-min eigenvalue may prove involved, and is left for future research.

There is some overlap between the objectives, though.

Corollary 1: a) Assume that $d = 2$ and there is no body-frame-expressed variables X to be estimated in the state. Then the optimal trajectory of Theorem 1 solves the minimium eigenvalue optimization problem (27) in the case of fixed-frame measurements. *b)* In the case of body-frame measurements, under the assumption of Theorem 2 all trajectories are optimal in the sense of (27).

Proof: In the 2D case, the last expression of (26) becomes $\left(\sum_{j=1}^{k}(\Pi_{l=j-1}^{k-1}\Omega_l)^{-1} * \boldsymbol{F}^{k-j} J \mathrm{U}_{j-1}\right)\xi_0^R$. Maximizing the expectation of its squared norm w.r.t. $\xi_0^R \sim \mathcal{N}(0,1)$ or its squared norm with $(\xi_0^R)^2 = 1$ yields identical problems. For body-frame measurements we have independence w.r.t. the controls. ∎

VI. Numerical illustrations

We now illustrate the theoretical results for the two motivating examples of Section III-A, plus a 2D SLAM example.

A. Navigation with GNSS measurement in 3D

In the case of 3D navigation as presented in section III-A.2, the position of the system is directly observed. However, the attitude of the vehicle must be estimated. The quality of the estimation depends on the chosen trajectory. To illustrate the result of Theorem 1 of Section IV-B, we compare the evolution of the OG's trace over time for the optimal trajectory and an arbitrary trajectory with same (maximum) energy U. The results in Figure 2 confirm the theorem.

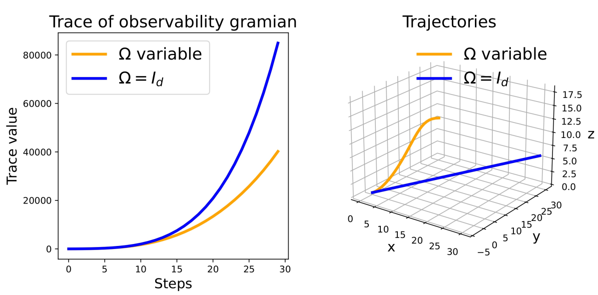


Fig. 2. On the left is shown the evolution of the OG trace for the optimal trajectory in blue and an arbitrary trajectory in orange. On the right, the two trajectories in 3D.

B. Navigation with online GNSS lever arm calibration in 2D

One of the advantages of the two-frame theory is the possibility to apply invariant filtering to state variables expressed in the body frame. The modelling of a wheeled robot with unknown lever arm between the GNSS and the middle point of the rear axle can be found in Section III-A.1. In Theorem 1 of Section IV-B, optimal trajectories were computed. To illustrate the result, we compute the trace of the OG of the optimal trajectory and compare it with another trajectory with identical energy. The results in Figure 3 confirm the theorem.

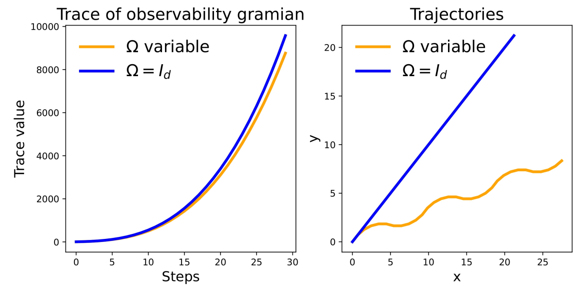


Fig. 3. On the left is shown the evolution of the OG trace for the optimal trajectory in blue and an arbitrary trajectory in orange. On the right, the two trajectories in the 2D plane are plotted.

C. 2D Simultaneous localization and mapping

Consider the 2D SLAM problem with two unknown landmarks and observations of the relative distance between the vehicle and these points [18]. The state vector is:

$$\Xi_n = (\theta_n, x_n, p_n^1, p_n^2)$$

where θ_n is the vehicle's heading, $x_n \in \mathbb{R}^2$ is the position of the vehicle, and $p_n^i \in \mathbb{R}^2$, $i \in \{1,2\}$ denotes the position of the unknown landmarks. The system's equations are [12]:

$$\begin{cases} \theta_n = \theta_{n-1} + \omega_n \\ x_n = x_{n-1} + u_n R(\theta_{n-1}) e_1 \\ p_n^i = p_{n-1}^i \\ y_n^i = R(\theta_n)^T (p_n^i - x_n) \end{cases}, \tag{28}$$

with $u_n, \omega_n \in \mathbb{R}$ the control inputs, $R(\theta)$ is the rotation matrix encoding the planar rotation of angle θ, and y_n^i is the observation of the i-th landmark's position in the vehicle frame. Letting the error variables ξ_n be classically defined as:

$$\xi_n^\theta = \bar{\theta} - \theta, \quad \xi_n^x = \bar{x}_n - x_n, \quad \xi_n^i = \bar{p}_n^i - p_n^i, \tag{29}$$

the system can be linearized as follows:

$$\xi_n = F_n \xi_{n-1} \text{ and } z_n = \bar{y}_n - y_n = H_n \xi_n \tag{30}$$

with

$$\begin{aligned} F_n &= \begin{pmatrix} 1 & 0_{2,1} & 0_{2,1} & 0_{2,1} \\ u_n J R_n e_1 & I_{2,2} & 0_{2,2} & 0_{2,2} \\ 0_{2,1} & 0_{2,2} & I_{2,2} & 0_{2,2} \\ 0_{2,1} & 0_{2,2} & 0_{2,2} & I_{2,2} \end{pmatrix}, \\ H_n &= \begin{pmatrix} J R_n^T (p_n^1 - x_n) & -R_n^T & R_n^T & 0_{2,2} \\ J R_n^T (p_n^2 - x_n) & -R_n^T & 0_{2,2} & R_n^T \end{pmatrix}, \end{aligned} \tag{31}$$

Using these error variables, it is not obvious at the first sight that the Gramian does not depend on the controls and that all trajectories are equally informative as ω_n implicitly acts on R_n and x_n. However, the system (28) obviously fits into the class considered here, and applying Theorem 2, we find that all trajectories lead to identical objective. This is similar to the linear case, where the Gramian depends neither on the controls nor on the state. Figure 4 confirms the result.

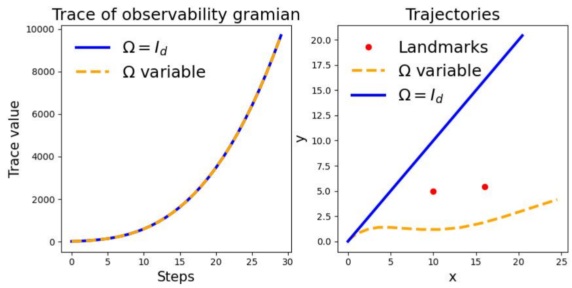


Fig. 4. On the left, the traces of the OG for the two trajectories are identical over time. On the right, the two trajectories are plotted with the two unknown landmarks.

VII. Conclusion

In this paper we have considered the problem of optimal active sensing, and we have derived the optimal trajectories in closed form for a large subclass of two-frame systems, that model vehicles that navigate in 2D and in 3D, leveraging the properties of invariant filtering and the isometry properties of rotation matrices. Although the results are not surprising (straight lines with full acceleration allow for distinguishing a heading error best), they have the merit of being quite general. They are nontrivial to derive, in that it is rare to solve explicitly optimal control problems outside of the LQR framework, and they have indeed necessitated the combination of various ingredients: a) the discrete time, which is not widespread in Gramian-based active sensing (in continuous time we generally have no closed form for the Gramian itself), b) a recent general theory, which highlights systems whose properties are akin to those of linear systems, c) a proof which is not straightforward and heavily relies on the properties of scalar and cross products, d) the detour via (5) while directly maximizing the trace–as is done by numerical methods–would have been more difficult analytically.

In the future, we would like to derive the counterpart for the max-min criterion (27) for the same class of systems. Besides, we would like to consider more general problems that do not perfectly fit into the invariant filtering theory, such as navigation with online calibration of gyrometers' and accelerometers' biases. In this case, the trajectory remains partially state-trajectory independent, which may lead to speedups of numerical methods, in the spirit of [19].

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