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A Pontryagin-based Game-theoretic Approach for Robust Nonlinear Model Predictive Control

Michele Pagone, Lorenzo Zino, Carlo Novara

***Abstract*— A Pontryagin-based differential game approach to solve a class of robust Nonlinear Model Predictive Control is proposed. The methodology defines an optimal control policy that takes into account non-accurate predictions of the system dynamics due to modeling errors and/or unknown exogenous disturbance, which may seriously compromise the controller performances. To this end, we propose a Pontryagin-based solution to the nonlinear min-max problem, which can be viewed as a zero-sum differential game, where the two players are the controlled input and the system's uncertainty/external disturbance. We show that, under suitable assumptions on system's dynamics, the game admits a Nash equilibrium, whose knowledge drastically decreases the high algorithmic complexity usually required for min-max optimization schemes. Finally, the theoretical results are confirmed by numerical simulations, performed on the Van der Pol nonlinear oscillator.**

I. INTRODUCTION

In recent decades, Nonlinear Model Predictive Control (NMPC) has gained an extraordinary attention within the industrial and academic communities as a reliable and flexible control tool for a wide range of practical applications [1]–[6]. The reason for such a success relies on its capability to deal with nonlinear dynamics and to provide optimal control commands for multivariable systems in the presence of input, output, and state constraints [7].

NMPC is based on three operations: i) prediction of the system's behavior along a finite-time horizon; ii) optimization of a suitable performance index; and iii) receding horizon strategy. Focusing on the first operation, the system's prediction is usually performed by employing a simplified and/or approximated model of the system's dynamics, when the latter is highly complex and/or not completely known. In these situations, non-accurate predictions due to modeling errors may seriously hamper the performance of the NMPC. Moreover, it may happen that uncertainties do not only affect the dynamics, but also the input and state constraints which, in turn, depend on the state variables themselves.

This calls for the development of robust versions of classic NMPC, to which we shall refer to with the acronym RNMPC. In this regard, many recent works have been devoted to defining different methodologies to address this key problem. A common nonlinear approach, proposed in [8], consists in the so-called tube-based RNMPC (see also [9], [10]). Nevertheless, as remarked in [8] and [11], tube-based control is based on the determination of both a nominal or reference trajectory and an ancillary controller that constraints deviations of the state of the uncertain systems from the nominal trajectory. Indeed, the ancillary controller forces the trajectories of the uncertain system to remain in a tube surrounding the reference trajectory. Unfortunately, the identification of such neighborhood is, unlike the linear case, a challenging problem.

Other types of RNMPC are learning-based [12]–[14], stochastic [15], [16], and H_∞-based approaches [17]–[20]. These methods, however, can suffer from a high computational complexity, making them rather unsuitable for on-line applications. Moreover, learning-based RNMPC methods are often highly dependent on the quality of the training dataset which, in turn, can be rarely available for some specific applications (e.g., satellite attitude and orbit control), and they lack of a solid formal theoretical framework in terms of closed-loop stability. On the other hand, stochastic RNMPC methods require an accurate modeling of uncertainties and disturbances. If they are not representative of the actual system behavior, the controller might overestimate or underestimate their effect, ultimately jeopardizing the performance of the controller. Finally, the H_∞-based RNMPC is based on an offline computation of a pre-compensation H_∞ control law —as solution of the differential Riccati equation— and then summed to a second control term as solution of the min-max optimal control problem.

Here, we aim at extending this range of approaches by proposing a novel methodology to address this problem. In our approach, we define a min-max RNMPC scheme, together with the Pontryagin Minimum (maximum, in the original version) Principle (PMP) [21]. Robust versions of PMP have been widely discussed, in a general form, in [22], [23]. However, these works propose a min-max approach that may be hardly implementable in practical applications. Here, we follow a different approach, inspired by the well-established methodology that deals with the min-max problem as a zero-sum differential game, whose players are the controlled input and the uncertainty/disturbance (here, termed adversary input) [24]–[26]. A general PMP-based game-theoretic framework is proposed in [27] where the min-max problem is cast as a zero-sum differential game with a Nash equilibrium (NE). However, in order to find the NE, one has to take into account two different Hamiltonians and to simultaneously, one for the input and one for the

The authors are with the Department of Electronics and Telecommunications, Politecnico di Torino, Corso Duca degli Abruzzi, 24, 10129 Torino, Italy {michele.pagone, lorenzo.zino, carlo.novara}@polito.it. M. Pagone is funded by the initiative "PNRRNextGenerationEU," Mission 4, "Sustainable Mobility Center (Centro Nazionale per la Mobilità Sostenibile — CNMS)," Spoke n.2 — Sustainable road vehicle — with grant agreement no. CN 00000023. This manuscript reflects only the authors' views and opinions, neither the European Union nor the European Commission can be considered responsible for them.

uncertainty, with each solution depending on the other, dramatically increasing its computational burden. To overcome this limitation, [28] and [29] defined a min-max solution employing a single covector set, by observing that, in some special cases, a joint Hamiltonian can be defined, such that it is convex in the input and concave in the uncertainty.

In this paper, we take a step further, joining the PMP-based NMPC approach presented in [30], together with some of the results depicted in [28] and [29]. Specifically, we present a PMP-based RNMPC whose min-max problem is resembled as a differential game which, when employing some special classes of functionals, admits a NE, corresponding to the saddle point condition on the Hamiltonian. By exploiting such property, the covector referred to the uncertainty is equal in modulus and opposite in sign to the one of the control input. This result drastically simplify the min-max problem that can be solved by defining a single Hamiltonian and set of covectors evolving along the prediction horizon. Then, we focus on the game separability. As observed in [31], the separability of the Hamiltonian does not necessarily imply the separability of the cost function itself, which is non-convex. To this end, we will show that, if the system's dynamics is affine in the input and uncertainty, the optimal uncertainty input policy does not depend on the optimal control policy, so that the adversary input plays as "first" and it does not change its behavior during the game.

The novel contribution in this work is two-fold. First, a PMP-based game-theoretic approach is applied to a RNMPC. The RNMPC min-max problem takes advantage of the existence of a NE which allows to drastically simplify the classic solution of the min-max problems. Second, unlike the existing works concerning PMP-based min-max [26]–[29], we do not assume a-priori that the separability of the Hamiltonian imply the separability of the original cost function. A discussion on this topic is proposed. To the best of our knowledge, this is the first work that takes into account these aspects in a receding horizon scheme. The generality of the proposed approach, whose effectiveness is demonstrated in a case study, paves the way for a broad range of real-world applications, including the implementation of RNMPC schemes in automotive and aerospace problems —application fields where NMPC is gaining more and more attention in recent years [3], [32]–[35]— where a fast and reliable robust controller is required for real-time implementation of RNMPC schemes on embedded processors.

The rest of the paper is organized as follows. In Section II, we introduce the problem statement. In Section III, we present the Pontryagin-based differential game solution. The numerical example is discussed in Section IV. Section V concludes the paper and outlines the future research.

A. Notation

We denote the set of real and strictly positive integer numbers as $\mathbb{R}$ and $\mathbb{N}_+$, respectively. Given $n, m \in \mathbb{N}_+$, a (column) real-valued vector is denoted as $x \in \mathbb{R}^n$, $x^\top$ denotes its transpose, and $\mathbf{A} \in \mathbb{R}^{n\times m}$ denotes a real-valued matrix. Given $z \in \mathbb{R}^n$ and $\mathbf{W} = \mathrm{diag}(w_1, ..., w_n),\ w_i \geq 0$, $\|z\|^2_{\mathbf{W}} \doteq z^T\mathbf{W}z$ is the (square) weighted norm of z. Finally, $\nabla_z(\cdot)$ is the gradient operator with respect to the variable z.

II. Problem Statement

Consider the following affine-in-the-input nonlinear system, affected by an affine uncertainty and/or disturbance:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) + \ell(x(t))\mu(t) \tag{1}$$

where $x(t) \in \mathcal{X} \equiv \mathbb{R}^{n_x}$ is the state vector at time $t \in \mathbb{R}$ $u(t) \in \mathcal{U} \subseteq \mathbb{R}^{n_u}$ is the input vector (where $\mathcal{U} \subseteq \mathbb{R}^{n_u}$ is a compact set such that $0 \in \mathcal{U}$). On the other hand, $\mu(t) \in \mathcal{W} \subset \mathbb{R}^{n_w}$ is a parameter of the system. Note that, f, g, and ℓ are generic nonlinear functions. Below, we will specify some mild regularity assumptions for these functions.

We assume that the measurements of the state vector are sampled with period $T_S > 0$. At each sampling time $t = t_k$, a prediction of the system state $\hat{x}(t)$ over the time interval $[t_k, t_k + T_p]$ is performed —where $T_p \geq T_S$ is the prediction horizon. Such a prediction is obtained by integrating an approximated model of the plant

$$\dot{\hat{x}}(t) = f(\hat{x}(t)) + g(\hat{x}(t))u(t) + \ell(\hat{x}(t))\hat{\mu}(t) \tag{2}$$

coming from an incomplete knowledge of the system.

In (2), $\hat{x}$ is the (predicted) model state vector, which evolves according to the same nonlinear functions f, g, and ℓ. In the following, we will assume that $\hat{x}(t) \in \mathcal{X}$. Briefly, this means that the state space $\mathcal{X}$ (or, at least, a super-set of it) is known a-priori, which is a reasonable assumption. Note that, at the initial time $t = t_k$, it holds $x(t_k) = \hat{x}(t_k)$.

Assumption 1: Assume that $f \in \mathcal{C}^1(\mathcal{X} \to \mathcal{X})$, $g \in \mathcal{C}^1(\mathcal{X} \to \mathcal{X} \times \mathcal{U})$, and $\ell \in \mathcal{C}^1(\mathcal{X} \to \mathcal{X} \times \mathcal{W})$ with Γ being the Lipschitz constant of the system.

Remark 1: Let us define $\hat{\mu}(t) = \mu(t) - w(t)$. In detail, the quantity $w(t)$ can represent, alternatively, a system uncertainty, a parametric uncertainty, or an additive disturbance. Henceforth, we deal with $w(t)$ as a system uncertainty coming from an incomplete/imperfect knowledge of the dynamics and we shall refer to it as *adversary input*.

The min-max problem is formulated by defining a cost function to be minimized with respect the control signal u over the worst-case adversary input w. The optimal pair (u^*, w^*) is the solution of the following problem:

$$(u^*, w^*) = \arg\min_u \max_w J(x, u, w) \tag{3}$$

subject to the following constraints:

$$\begin{aligned} &\dot{\hat{x}}(\tau) = f(\hat{x}(\tau)) + g(\hat{x}(\tau))u(\tau) + \ell(x(\tau))\mu(t),\\ &\hat{x}(t_k) = x(t_k)\\ &u(\tau) \in \mathcal{U},\ w(\tau) \in \mathcal{W}, \forall \tau \in [t_k, t_k + T_p]\\ &u(\cdot) \in \mathcal{KC}\big([t_k, t_k + T_p]\big). \end{aligned} \tag{4}$$

Here, $\mathcal{U}$, and $\mathcal{W}$ are compact sets describing constraints on the the input and the adversary input, respectively, and $\mathcal{KC}$ is the space of piece-wise continuous functions. Note that, both $\mathcal{U}$ and $\mathcal{W}$ are required to be convex.

Remark 2: Here, for the sake of simplicity, we neglect constraints on the state variable, that is to assume $x(t) \in \mathcal{X} \equiv \mathbb{R}^{n_x}$. A possible fashion of implementing state constraints within the optimization problem consist of augmenting the cost function J with a suitable penalty function, as described in [30]. In this latter case, one has to consider that the NE of the game changes with respect to the unconstrained case. Indeed, some of the results that will be presented in Theorem 1 may not be valid anymore. The implementation of state constraints within the proposed Pontryagin-based differential game framework is a topic of ongoing research.

We thus define the finite-horizon NMPC quadratic cost function as

$$J(x,u,w) = \int_{t_k}^{t_k+T_p} \|u(\tau)\|^2_{\mathbf{R}} \mathrm{d}\tau + \|\tilde{x}(t_k+T_p)\|^2_{\mathbf{P}} \quad (5)$$

where $\tilde{x}(t) = x_r - \hat{x}(t)$ is the predicted tracking error, x_r is a (constant) reference signal of interest[1], and $\mathbf{R}$ and $\mathbf{P}$ are positive diagonal matrices weighting the contributions of the entries of the two vectors. Briefly, the first term of (5) accounts for the control signal over the prediction horizon, the second for the error at the end of the prediction horizon.

A. Robust NMPC Cost Function

In the previous section we highlighted that $\mathcal{W} \subset \mathbb{R}^{n_w}$. Here, we take a step further by encapsulating the bounds on w within the cost function J. In the following, we assume that the set $\mathcal{W}$ is defined as the ball $\mathcal{W} = \{w : \|w\|_2 \leq \bar{w}\}$, where $\bar{w}$ is defined by the knowledge of $\mathcal{W}$.

Hence, the bounds on $\|w\|_2$ can be included within the cost function J. Hence, there exists an arbitrary and unique $\gamma > 0$, $\forall t \geq 0$, such that the following min-max problem:

$$(u^*, w^*) = \arg\min_u \max_w J_w(x,u,w) \quad (6)$$

$$J_w(x,u,w) = \int_{t_k}^{t_k+T_p} \left(\|u(\tau)\|^2_{\mathbf{R}} - \gamma \|w(\tau)\|^2_2\right) \mathrm{d}\tau + + \|\tilde{x}(t_k+T_p)\|^2_{\mathbf{P}}. \quad (7)$$

and subject to the same constraints in (4), is equivalent to the the min-max problem (3)–(5).

To sum up, the optimal control problem in (6), with cost function (7), can be seen as a zero-sum differential game, whose players are the control input u and the uncertainty w, respectively, with payoff functions equal to J_w and $-J_w$, respectively. In other words, the goal of the control input u is to minimize the payoff function, while the goal of the uncertainty w is to maximize it (i.e., minimize its opposite). In this particular setting, the pair (u^*, w^*) is the corresponding saddle point of the game, i.e., the NE, and the optimal value of J_w corresponds to the value of the payoff for the control input at such NE. Furthermore, we shall remark that, in this symmetric scenario, each one of the two players has to make their own choice without any a-priori information on the strategy taken by the opponent.

[1]For simplicity, we assume that the reference x_r is constant. In practice, the controller can be also applied for slowly changing reference signals.

III. PMP-based Differential Game Solution

According to the NE definition [27], the pair (u^*, w^*) is the solution of the min-max problem if and only if the following conditions are simultaneously satisfied:

$$u^* = \arg\min_u J_w(u,w) \quad (8a)$$

$$w^* = \arg\min_w -J_w(u,w) = \arg\max_w J_w(u,w) \quad (8b)$$

over the choice of all inputs $u(t) \in \mathcal{U}$ and adversary inputs $w(t) \in \mathcal{W}$, for all $t \in [t_k, t_k+T_p]$.

By taking into account (1), (7), (8a), and (8b), the pair of Hamiltonians $H^{(u)}, H^{(w)} \in \mathcal{C}^1(\mathcal{X} \times \mathcal{X} \times \mathcal{U} \times \mathcal{W} \to \mathbb{R})$ are defined as

$$H^{(u)} = \|u\|^2_R - \gamma \|w\|^2_2 + + \lambda^{(u)\top}\left[f(x(t)) + g(x(t))u(t) + \ell(x(t))(\hat{\mu} + w(t))\right] \quad (9a)$$

$$H^{(w)} = -\|u\|^2_R + \gamma \|w\|^2_2 + + \lambda^{(w)\top}\left[f(x(t)) + g(x(t))u(t) + \ell(x(t))(\hat{\mu} + w(t))\right] \quad (9b)$$

where $\lambda^{(u)}, \lambda^{(w)} \in \mathbb{R}^{n_x}$ are the covectors of the minimization and maximization problem, respectively. Note that, in (9a)-(9a) the parameter μ in (1) has been replaced with $\hat{\mu} + w(t)$, according with the notion presented in Remark 1. Hence, following [21], the necessary conditions for optimality are given by

$$H^{(u)}(x^*, u^*, w^*, \lambda^{*(u)}, \lambda^{*(w)}) = \min_u H^{(u)}(x^*, u, w^*, \lambda^{*(u)}, \lambda^{*(w)}), \quad (10a)$$

$$H^{(w)}(x^*, u^*, w^*, \lambda^{*(u)}, \lambda^{*(w)}) = \min_w H^{(w)}(x^*, u^*, w, \lambda^{*(u)}, \lambda^{*(w)}), \quad (10b)$$

$$\dot{\lambda}^{(u)} = -\nabla_x H^{(u)}, \quad (10c)$$

$$\dot{\lambda}^{(w)} = -\nabla_x H^{(w)}, \quad (10d)$$

$$\lambda^{(u)}(t_k+T_p) = 2\tilde{x}^\top(t_k+T_p)\mathbf{P}, \quad (10e)$$

$$\lambda^{(w)}(t_k+T_p) = -2\tilde{x}^\top(t_k+T_p)\mathbf{P} \quad (10f)$$

where the minimization operators with respect to u and w in (10a) and (10b) should be intended over all inputs u such that $u(t) \in \mathcal{U}$, for all $t \in [t_k, t_k+T_p]$, and adversary inputs w such that $w(t) \in \mathcal{W}$, for all $t \in [t_k, t_k+T_p]$.

Note that, since the Hamiltonians are convex in u, and concave in w, the min-max Pontryagin problem is separable and the joint Hamiltonian admits a saddle point which coincides with the min-max solution. In this case, there exists a tight correlation between $\lambda^{(u)}$ and $\lambda^{(w)}$, such that the min-max problem resembles in defining a common Hamiltonian and finding the solution (i.e., the saddle point) by minimization of the Hamiltonian with respect to u.

Theorem 1: Consider the min-max necessary conditions in (10a)–(10f) and let Assumption 1 hold. Then, it holds:

$$\lambda^{(w)}(t) = -\lambda^{(u)}(t), \quad \forall t \in [t_k, t_k+T_p]. \quad (11)$$

Proof: For the sake of simplicity and readability, we limit the discussion to the scalar case $x, \lambda^{(u)}, \lambda^{(w)} \in \mathbb{R}$. The extension to the multidimensional case is straightforward, as discussed at the end of the proof.

From (10e) and (10f), one has that $\lambda^{(w)}(t_k + T_p) = -\lambda^{(u)}(t_k + T_p)$. By taking into account (1), (10c) and (10d), the variation of the covectors along the prediction horizon is equal to

$$\dot{\lambda}^{(u)}(t) = \tag{12a}$$
$$-\left(\frac{\partial f}{\partial x}(x(t)) + \frac{\partial g}{\partial x}(x(t))u(t) + \frac{\partial \ell}{\partial x}(x(t))\mu(t)\right)\lambda^{(u)}(t),$$
$$\dot{\lambda}^{(w)}(t) = \tag{12b}$$
$$-\left(\frac{\partial f}{\partial x}(x(t)) + \frac{\partial g}{\partial x}(x(t))u(t) + \frac{\partial \ell}{\partial x}(x(t))\mu(t)\right)\lambda^{(w)}(t),$$

that consist in a set of backward first-order differential equation integration within the TPBVP, and recalling that $\mu(t) = \hat{\mu} - w(t)$. Let us introduce the auxiliary variable

$$\varrho(t) \doteq \frac{\lambda^{(u)}(t)}{\lambda^{(w)}(t)} \tag{13}$$

whose final condition, from (10e) and (10f), is $\varrho(t_k + T_p) = -1$. The time derivative of $\varrho(t)$ is then computed, obtaining

$$\dot{\varrho}(t) = \frac{\dot{\lambda}^{(u)}(t)}{\lambda^{(w)}(t)} - \frac{\lambda^{(u)}(t)\dot{\lambda}^{(w)}(t)}{{\lambda^{(w)}}^2(t)}. \tag{14}$$

Upon substituting (12a)-(12b) into (14), and defining

$$\xi(t) = -\left(\frac{\partial f}{\partial x}(x(t)) + \frac{\partial g}{\partial x}(x(t))u(t) + \frac{\partial \ell}{\partial x}(x(t))\mu(t)\right) \tag{15}$$

one has that

$$\dot{\varrho}(t) = \frac{\xi(t)\lambda^{(u)}(t) - \xi(t)\lambda^{(u)}(t)}{\lambda^{(w)}(t)} = 0. \tag{16}$$

Clearly, holding Assumption 1, from the solution properties of the Lipschitz-continuous differential equations, we have

$$\varrho(t) = \varrho(t_k + T_p) - \int_{t_k+T_p}^{\tau} \dot{\varrho}(\tau)\mathrm{d}\tau \tag{17}$$

which, taking into account (16) and recalling that $\varrho(t_k + T_p) = -1$, leads to the following expression:

$$\varrho(t) = \frac{\lambda^{(u)}(t)}{\lambda^{(w)}(t)} = -1, \quad \forall t \in [t_k, t_k + T_p] \tag{18}$$

yielding the statement. In the multidimensional case, these arguments are still valid, with the understanding that element-wise operations should be used in (13)–(18). ■

We conclude this section by discussing some important consequences of the theoretical findings in Theorem 1.

Remark 3: From Theorem 1, we observe that:

- A joint Hamiltonian H and a common covector λ of the min-max problem can be defined. H can be picked as $H^{(u)}$ and $\lambda(t) = \lambda^{(u)}(t) = -\lambda^{(w)}(t)$, or vice-versa[2]. Hence,

$$u^*(t) = \arg\min_u H(x(t), \lambda(t), u(t), w(t)), \tag{19a}$$
$$w^*(t) = \arg\min_w H(x(t), -\lambda(t), u(t), w(t)) \tag{19b}$$

  for u and w such that $u(t) \in \mathcal{U}$ and $w(t) \in \mathcal{W}$, for all $t \in [t_k, t_k + T_p]$.
- There exists a NE of the min-max problem which coincides with the joint Hamiltonian saddle point. In this situation, a pair (u^*, w^*) exits such that $H(u^*, w) \leq H(u^*, w^*) \leq H(u, w^*)$.
- Being the joint Hamiltonian convex in u and concave in w (i.e., separable), it is sufficient to find only the solution of one TPBVP. Covectors of both problems are opposite and they can be obtained from the first-order necessary conditions from the joint Hamiltonian.

[2]One can pick $H^{(w)}$ as joint Hamiltonian. In this case, the signs of the covectors are inverted.

A. Game Separability

The separability of H generally does not imply the separability of the cost function J, then, the implication $\min_u \max_w J = \max_w \min_u J$ does not always hold [31]. In the following, we further discuss this aspect.

Indeed, from (7), we observe that the cost function J_w includes a term that accounts for the terminal cost ($\|\tilde{x}(t_k + T_p)\|^2_{\mathbf{P}}$), which depends on the state variable and, ultimately, on both inputs u and w. Therefore, the condition for separability —i.e., that the mixed second-order derivative of the cost function with respect to u and w is identically equal to zero [31]— is not met in general.

Nonetheless, in our formulation in (1), we have assumed that the dynamics is affine in u and w. Hence, the policy of the adversary input $w(t)$ does not depend on the input $u(t)$, being the two contributions disjoint, but just on the state $x(t)$. Moreover, it still makes sense to apply our methodology even in more general scenarios, when the dynamics of u and w are related in a more complex and non-affine fashion. In fact, similar to what is assumed in [31], while it makes sense to develop approaches that are robust with respect to the worst-case scenario of the adversary input, it is unrealistic to assume that such an input can change in an adaptive fashion with the input. In other words, even in more complex scenarios, it is always reasonable to assume that the worst-case adversary input is set "at first," and it is not changed as the game evolves as a feedback of the control input.

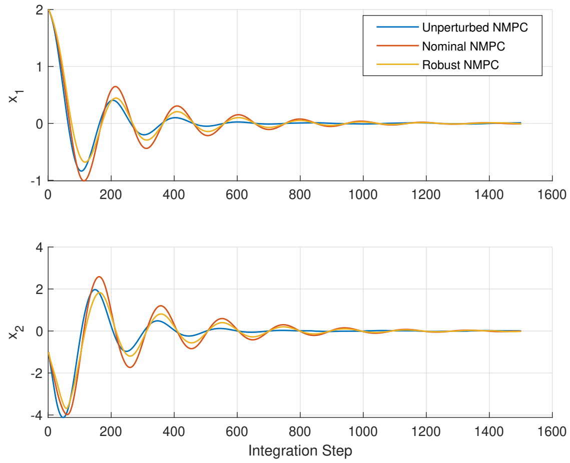
Hence, in view of these considerations, and keeping in mind the inherent complexity of the nonlinear problem under investigation, which forces us to deal with sub-optimal solutions, we believe that the solutions of our PMP-based differential game are still useful. The example reported in the following section help illustrate our method and demonstrate the validity of the proposed approach.

IV. Numerical Example

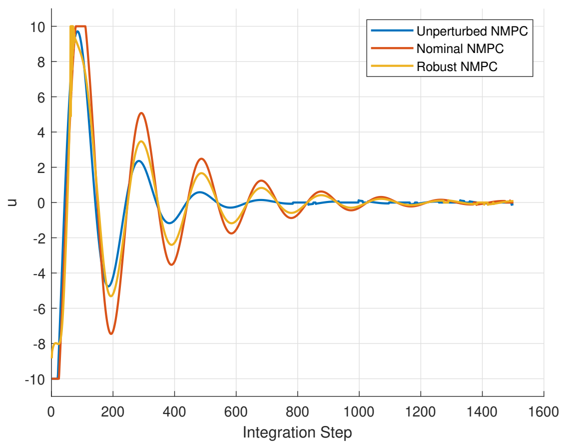
We a Van der Pol oscillator with exogenous input u and uncertainty w affecting the parameter φ, which yield the following planar system of (nonautonomous) equations:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= (\varphi + w)(1 - x_1^2)x_2 - x_1 + u \end{aligned} \tag{20}$$

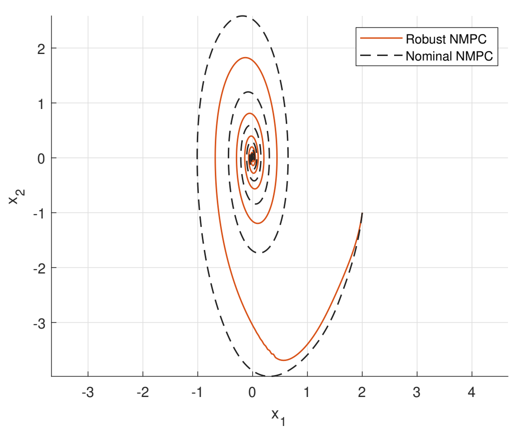
where we set $\varphi = 0.5$ and $w \leq 0.7$. Here, the prediction model assumes $w = 0$. By employing the robust cost



(a)



(b)



(c)

Fig. 1: Evolution of (a) the state variables $x(t) = (x_1(t), x_2(t))$, (b) the input signals $u(t)$, and (c) state-space trajectories for the Van der Pol oscillator discussed in Section IV for the three different NMPC approaches: unperturbed (blue), nominal (orange), and our PMP-based RNMPC (yellow).

function (7), the corresponding Hamiltonians are

$$H^{(u)} = Ru^2 - \gamma w^2 + \lambda_1 \dot{x}_1 + \lambda_2 \dot{x}_2 \tag{21a}$$

$$H^{(w)} = -Ru^2 + \gamma w^2 + \lambda_1 \dot{x}_1 + \lambda_2 \dot{x}_2 \tag{21b}$$

whereas it has been accounted that $u^* = \arg\min_u J$ and $w^* = \arg\max_w J$. The Euler-Lagrange equations are jointly defined as

$$\begin{aligned} \dot{\lambda}_1 &= \lambda_2(2x_1x_2(w+\varphi)+1) \\ \dot{\lambda}_2 &= \lambda_2(w+\varphi)(x_1^2-1) - \lambda_1. \end{aligned} \tag{22}$$

Then, the optimal control and adversary input are, over the corresponding sets, equal to

$$u^* = \mathrm{sat}_{\mathcal{U}}\left(-\frac{\lambda_2}{2R}\right) \tag{23a}$$

$$w^* = \mathrm{sat}_{\mathcal{W}}\left(\lambda_2 x_2 \frac{x_1^2-1}{2\gamma}\right) \tag{23b}$$

respectively, where $\mathrm{sat}(\cdot)$ is the element-wise saturation operator that copes with the constraints [30]. By the saddle condition of the min-max game, $\lambda^{(w)} = -\lambda^{(u)}$.

In our numerical simulations, we set the initial condition as $x_0 = [2, -1]^\top$, while the reference is set to $x_r = [0, 0]^\top$. Concerning the NMPC parameters, we set $T_S = 0.01$, $T_p = 0.2$, $\gamma = 3$, $R = 1$, and $\mathbf{P} = \mathrm{diag}(110, 1)$. Finally, concerning the input and state constraints, we set $\mathcal{X} \equiv \mathbb{R}^{n_x}$ and $|u| \leq 10$. Note that, the extension to the state-constrained problem can be carried out by augmenting the cost function with a suitable penalty, by following the approach in [30].

In Fig. 1, we report the comparison among three different scenarios: i) a nominal NMPC approach where the plant and the prediction model are described by the same equations (unperturbed NMPC); ii) a nominal NMPC approach that does not take into account the parameter uncertainty (nominal NMPC); and iii) our PMP-based RNMPC. All the three approaches are able to track the reference, still, with a different behavior along the trajectory. Trivially, the unperturbed scenario has the best tracking performance, offering lower oscillations during the time evolution. This scenario is shown as a best-case scenario, but it is not achievable if the exact dynamics of the system is not know, as in most real-world scenarios. On the other hand, it is worth noticing that our RNMPC presents less evident overshoots in the oscillations peaks than the nominal approach.

These findings confirm the effectiveness of the proposed PMP-based RNMPC. Indeed, the latter is also able to reach better tracking performance providing a lower control authority with respect the nominal scenario: by considering the overall impulse as $I = \int_{t_0}^{t_f} |u(t)| \mathrm{d}t$, one has that $I_{uNMPC} = 1514.2$, $I_{nNMPC} = 2558.7$, and $I_{rNMPC} = 1978.1$ (for the unperturbed, nominal, and robust scenarios, respectively) highlighting how our RNMPC has better tracking performances with a reduced control impulse.

V. Conclusion

We proposed a novel robust NMPC scheme. By taking advantage of the classic min-max formulation for robust optimization, we obtained a control law by developing an algorithm based on the Pontryagin solution of a zero-sum differential game, turning the min-max problem into a two-points boundary value problem. Hence, the optimal control law was obtained by searching the Hamiltonian saddle point, which is the analogue of the NE of the game. We showed that, although the separability of the Hamiltonian does not strictly imply the separability of the game, under suitable conditions on system dynamics, the adversary input "plays as first" and it does not change its strategy along the game. Thanks to this results, we were able to define a joint Hamiltonian (and, consequently, a single set of covectors) whose minimization leads to the NE of the game, i.e., the solution of the min-max problem. The proposed methodology was, then, applied to the Van der Pol oscillator nonlinear dynamics. The results highlighted the effectiveness of the control algorithm, showing how the robust controller was more effective with respect to the nominal controller, in terms of control effort and reference tracking. Throughout the text, we assumed that all optimization problems are feasible and that the nonlinear system, when the optimal control policy is in closed-loop, is stable. Obtaining conditions for recursive feasibility and robust stability is a topic of on-going research.

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