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Exploiting Manifold Turnpikes in Model Predictive Path Following without Terminal Constraints

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***Abstract*— Model predictive path-following control (MPFC) considers geometric reference paths in output spaces without pre-assigned timing information. It combines trajectory generation and tracking into one receding-horizon optimal control problem. In this paper, we discuss MPFC without terminal constraints from a geometric point of view. Specifically, we consider implicitly parameterized paths and the recently introduced notion of manifold turnpikes to propose sufficient conditions for practical convergence of the system output towards a neighborhood of the reference path. We draw upon a simulation example to demonstrate the efficacy of the proposed scheme.**

I. INTRODUCTION

Most control tasks may be categorized into set-point stabilization, trajectory tracking, or path following. While set-point stabilization considers a single point reference, trajectory tracking and path following consider a curve in the output space. Unlike path following, trajectory tracking imposes a strict timing requirement on when to be where on the reference curve. Moreover, path following is shown to remove fundamental performance limitations for non-minimum-phase systems introduced by trajectory tracking [1]. Several schemes have been proposed to address path following problems. Transverse feedback linearization, first introduced in [2], is a geometric control scheme that transforms the original system dynamics into transversal and tangential subsystems which simplify tackling the problem [22], [23]. Despite their appeal, these schemes fall short when it comes to handling system constraints. To this end, the use of nonlinear model predictive control (NMPC) has been proposed in the continuous-time setting [6] and labeled model predictive path-following (MPFC), see also [3], [7], [11], [20]. The discrete-time counterpart of MPFC is called model predictive contouring control [16].

Many results on the successful implementation of MPFC have been reported. In [25], MPFC is successfully applied to a highly automated vehicle, while [28] combines continuous re-planning to control a robot in dynamically changing environments. In [17] an n-trailer vehicle is considered, while [27] focuses on the geometric reformulation of robot dynamics and experimental validation. In [24] optimization-based path following is implemented for acoustic levitation, while [3] applies MPFC to a laboratory tower crane, and [21] uses it for automated aerial videography. Moreover, [19] combines MPFC with admittance control.

Many proposed MPFC variants admit closed-loop path convergence guarantees via stabilizing terminal constraints, see [6], [7]. Nevertheless, the design of such terminal regions is often challenging. Moreover, explicit inclusion of such constraints in the optimization increases the computational burden. Existing results on MPFC without explicit terminal region constraints, e.g., [11], [20], rely on cost-controllability assumptions [14]. Moreover, [20] focuses on nonholonomic mobile robots, and [11] is limited to differentially flat systems.

The present paper reports results from the thesis [15]. We propose a new route towards closed-loop guarantees for MPFC schemes without any terminal constraints or terminal penalties. Building upon ideas from [23], we reformulate the output path-following problem as a constrained manifold stabilization problem in the state space. Earlier works on constrained manifold stabilization using NMPC are, e.g., [3], [4]. While [3] provides no formal stability guarantees, [4] utilizes terminal constraints. In contrast, we draw upon the concept of manifold turnpikes [8] to show that the reformulated MPFC optimal control problem (OCP) exhibits the turnpike phenomenon with respect to the path manifold.

The remainder of the paper is structured as follows: Section II recalls constrained output path-following, manifold stabilization, and transversal coordinates. Section III presents the proposed MPFC scheme and the main result. In Section IV, we draw upon a nonholonomic wheeled mobile robot example to illustrate the efficacy of the scheme. Finally, the paper ends with conclusions in Section V.

Notation: The set of k-times continuously differentiable functions is denoted $\mathscr{C}^k$. The set of measurable locally Lebesgue integrable functions from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}^{n_u}$ is denoted $\mathscr{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^{n_u})$. The distance between a point $y \in \mathbb{R}^{n_y}$ and a closed set $\mathscr{A} \subset \mathbb{R}^{n_y}$ is denoted $||y||_{\mathscr{A}} := \inf_{w \in \mathscr{A}} ||y - w||$, where $||\cdot||$ denotes the 2-norm. The Jacobian of a real-valued function $\lambda'(x)$ is denoted $\frac{\partial \lambda'}{\partial x} := (\frac{\partial \lambda'}{\partial x_1}, \ldots, \frac{\partial \lambda'}{\partial x_{n_x}})$.

II. PROBLEM STATEMENT AND PRELIMINARIES

A. Constrained output path-following

We consider nonlinear control-affine systems of the form

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{n_u} g_i(x(t)) u_i(t), \quad x(t_0) = x_0 \tag{1a}$$

$$y(t) = [h_1(x), \ldots, h_{n_y}(x)]^\top, \tag{1b}$$

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where the state x is restricted to the closed connected set $\mathbb{X} \subseteq \mathbb{R}^{n_x}$, the input u is constrained to the compact set $\mathbb{U} \subset \mathbb{R}^{n_u}$, and the output is $y \in \mathbb{R}^{n_y}$. The initial condition x_0 belongs to a compact set $\mathbb{X}_0 \subseteq \mathbb{X}$. The vector fields f, $g_i \ \forall\, i \in \{1,\ldots,n_u\}$, and the functions $h_i \ \forall\, i \in \{1,\ldots,n_y\}$ are assumed to be $\mathscr{C}^\infty$ functions. We use the shorthands $g := [g_1,\ldots,g_{n_u}]^\top$ and $h := [h_1,\ldots,h_{n_y}]^\top$. Additionally, for any admissible input $u(\cdot)$ such that $u(t) \in \mathbb{U}$, $\forall t \geq t_0$ and any initial condition $x_0 \in \mathbb{X}_0$, (1a) is assumed to admit a unique absolutely continuous solution denoted $x(\cdot; x_0, u(\cdot))$.

The reference path $\mathscr{P}$ is defined implicitly via $\mathscr{P} = \{y \in \mathbb{R}^{n_y} : \sigma(y) = 0\}$, where $\sigma : \mathbb{R}^{n_y} \to \mathbb{R}$ is a $\mathscr{C}^\infty$ function.

Problem 1 (Constrained output path-following)**.** *Given the system (1) and a path $\mathscr{P}$, design a controller such that:*

i) The output of the closed-loop system converges asymptotically to the path $\mathscr{P}$, i.e., $\lim_{t\to\infty} ||y(t)||_{\mathscr{P}} = 0$.
ii) The input and state constraints are satisfied for all times, i.e., $x(t) \in \mathbb{X}$, $u(t) \in \mathbb{U}$, $\forall t \geq t_0$.
iii) Additionally, it might be required to follow the path with a desired speed and direction. □

B. Manifold stabilization

Manifold stabilization can be considered a generalization of the ubiquitous set-point stabilization problem. While the latter concerns stabilizing a single point in the state space, the former actually considers a manifold in the state space as reference. Similar to [22], [23], we are interested in the case where the path manifold is a closed (embedded) manifold of the state space $\mathbb{R}^{n_x}$. To this end, consider the set $\Gamma := \{x \in \mathbb{R}^{n_x} : y = h(x) \in \mathscr{P}\}$ which consists of all points in the state space whose image under the output map lie exactly on the path $\mathscr{P}$.

Definition 1 (Path manifold [22], [23])**.** *The path manifold $\Gamma^\star$ is the largest closed controlled-invariant (i.e., for all $x \in \Gamma^\star$ there exists a feasible smooth input $\tilde{u} : \mathbb{R}_{\geq 0} \to \mathbb{U}$ rendering it an invariant set) $n^\star$-dimensional submanifold of Γ.* □

Assumption 1 (Basic feasibility)**.** *The path manifold $\Gamma^\star$ is a non-empty set and $\Gamma^\star \subset \mathbb{X}$.* □

The above assumption allows viewing Problem 1 as a constrained manifold stabilization problem.

Problem 2 (Constrained manifold stabilization)**.** *Given the system (1) and a path manifold $\Gamma^\star$, design a controller such that:*

i) The state of the closed-loop system asymptotically converges to $\Gamma^\star$, i.e., $\lim_{t\to\infty} ||x(t)||_{\Gamma^\star} = 0$.
ii) The input and state constraints are satisfied for all times, i.e., $x(t) \in \mathbb{X}$, $u(t) \in \mathbb{U}$, $\forall t \geq t_0$.
iii) A desired tangential motion on $\Gamma^\star$ is achieved. □

Notice that the additional speed and direction requirements of Problem 1 translate to the desired tangential motion in Problem 2.

C. Suitable Local Coordinates

It is well-known that suitable coordinate representations simplify the analysis of path-following problems [7], [22], [23]. Specifically, one usually transforms the original system dynamics (1) with respect to the path manifold $\Gamma^\star$ into two interconnected subsystems composed of transversal and tangential coordinates. The transversal directions denoted by ξ describe the dynamics moving the system transversely towards (or away from) the path manifold $\Gamma^\star$. On the other hand, the tangential directions η describe the system dynamics on the path manifold $\Gamma^\star$ (i.e., when $\xi = 0$). The existence of suitable coordinates can be guaranteed via the notion of a well-defined vector relative degree of the considered output.

Definition 2 (Vector relative degree)**.** *System* (1) *has a well-defined vector relative degree $\{r_1,\ldots,r_{n_y}\}$ locally at $x^0 \in \mathbb{X}$ if the following two conditions hold:*

i) For all $j \in \{1,\ldots,n_u\}$, $i \in \{1,\ldots,n_y\}$, $k \in \{0,\ldots,r_i - 2\}$, and for all x in an open neighborhood of x^0: $L_{g_j} L_f^k h_i(x) = 0$.
ii) The $n_y \times n_u$ matrix

$$A(x) = \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \ldots & L_{g_{n_u}} L_f^{r_1-1} h_1(x) \\ L_{g_1} L_f^{r_2-1} h_2(x) & \ldots & L_{g_{n_u}} L_f^{r_2-1} h_2(x) \\ \ldots & \ldots & \ldots \\ L_{g_1} L_f^{r_{n_y}-1} h_{n_y}(x) & \ldots & L_{g_{n_u}} L_f^{r_{n_y}-1} h_{n_y}(x) \end{bmatrix}$$

is full-rank at x^0. □

The following assumption guarantees the existence of a local coordinate transformation such that on some neighborhood $\mathscr{N} \supseteq \Gamma^\star$ Problem 2 can be described in transversal and tangential coordinates [22], [23].

Assumption 2 (Existence of transversal coordinates)**.** *Given the nonlinear control system (1) and an $n^\star$-dimensional path manifold $\Gamma^\star \subseteq \mathbb{X}$, there exists an output function $\lambda : \mathscr{N} \to \mathbb{R}^\rho$, defined on some neighborhood $\mathscr{N} \supseteq \Gamma^\star$ where ρ is a positive integer such that the following hold:*

i) $\Gamma^\star \subseteq \{x \in \mathbb{R}^{n_x} : \lambda_i(x) = 0, i = 1,\ldots,\rho\}$,
ii) The system (1a) *with output $y' = (\lambda_1(x),\ldots,\lambda_\rho(x))^\top$ has a well-defined vector relative degree $\{r_1,\ldots,r_\rho\}$ on $\Gamma^\star$ such that $r_1 + \ldots + r_\rho = n_x - n^\star$.* □

Assumption 2 allows transforming the original system state into transverse and tangential coordinates. Thus, the path manifold $\Gamma^\star$ can be represented as follows

$$\Delta(\Gamma^\star) = \{(\xi,\eta) \in \mathbb{R}^{n^\star} \times \mathbb{R}^{n_x - n^\star} : \xi = 0\}, \tag{2}$$

where $\Delta : \mathscr{N} \to \mathbb{R}^{n^\star} \times \mathbb{R}^{n_x - n^\star}, \Delta(x) = (\xi^\top, \eta^\top)^\top$ is a suitable diffeomorphic coordinate transformation defined on some neighborhood of $\Gamma^\star$. The transverse coordinates ξ are

$$\xi = \left[\lambda(x),\, L_f\lambda(x),\, \ldots,\, L_f^{n_x - n^\star - 1}\lambda(x)\right]^\top, \tag{3}$$

where $L_f^i \lambda(x) := L_f(L_f^{i-1}\lambda(x))$ denotes the i-th iterated Lie derivative of λ in the direction of f.

III. MAIN RESULTS

A. NMPC setup

We adopt a sampled-data NMPC formulation, see, e.g., [12], [13]. In other words, we assume a constant inter-sampling time $\delta := t_{k+1} - t_k > 0, \forall k \in \mathbb{N} \cup \{0\}$. The constrained OCP that is solved at each sampling instant reads

$$\min_{u(\cdot|t_k)\in\mathscr{L}^\infty([t_k,t_k+T],\mathbb{R}^{n_u})} J(x(t_k), u(\cdot|t_k))$$

$$\text{s.t. } \dot{x}(\tau|t_k) = f(x(\tau|t_k)) + g(x(\tau|t_k), u(\tau|t_k)) \quad (4a)$$

$$x(t_k|t_k) = x(t_k) \quad (4b)$$

$$u(\tau|t_k) \in \mathbb{U},\ x(\tau|t_k) \in \mathbb{X} \quad (4c)$$

where the system dynamics (4a) and the input and state constraints (4c) must hold for all $\tau \in [t_k, t_k+T]$ for some finite time horizon $T \in \mathbb{R}_{>0}$. The measured state comes in at every sampling instant through (4b). Note that, the predicted state and input trajectories starting at time instant t_k are denoted $x(\cdot|t_k)$ and $u(\cdot|t_k)$ respectively. The cost functional J to be minimized reads

$$J(x(t_k), u(\cdot|t_k)) := \int_{t_k}^{t_k+T} \ell(x(\tau|t_k), u(\tau|t_k))\, d\tau, \quad (5)$$

where $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ is the stage cost. We detail the choice of the stage cost ℓ after introducing the notion of manifold turnpikes below.

B. The manifold turnpike property

In optimal control, the (state) turnpike property with respect to an equilibrium point x_e describes the phenomenon when optimal (state) trajectories stay close to that equilibrium point "most of the time" and for increasing horizons and for different initial conditions, see, e.g., [9]. In the present paper, we are not interested in stabilizing an equilibrium point, but rather the path manifold $\Gamma^\star$.

Definition 3 (Measure manifold turnpike [8])**.** *The optimal state trajectories $x^\star(\cdot;x_0,u^\star(\cdot))$ of OCP (4) are said to have a measure manifold turnpike property with respect to $\Gamma^\star \subseteq \mathbb{X}$ if there exists a function $\hat{\nu} : (0,\infty) \to [0,\infty)$ such that for all initial conditions $x_0 \in \mathbb{X}_0$, and for all time horizons $T > 0$ we have*

$$\mu[\hat{\Theta}_{\varepsilon,T}] < \hat{\nu}(\varepsilon)\ \forall\, \varepsilon > 0, \quad (6)$$

where μ is the Lebesgue measure on the real line and

$$\hat{\Theta}_{\varepsilon,T} = \{t \in [0,T] : ||x^\star(t)||_{\Gamma^\star} > \varepsilon\}. \qquad \square$$

It is well-known that the turnpike property is closely related to strict dissipativity, see, e.g., [9]. With respect to an equilibrium point, an OCP is said to be strictly dissipative if there exists a storage function $S : \mathbb{X} \to \mathbb{R}_{\geq 0}$ that is non-negative and bounded, such that for all $x_0 \in \mathbb{X}_0$ a strict dissipation inequality holds. A natural generalization of this property to the case of manifolds is as follows (cf. [26]):

Definition 4 (Manifold strict dissipativity [8])**.**
The OCP (4) *is said to be strictly dissipative with respect to the manifold $\Gamma^\star \subseteq \mathbb{X}$ if there exists a non-negative bounded storage function $S : \mathbb{X} \to \mathbb{R}_{\geq 0}$, such that along all optimal pairs $z^\star(\cdot;x_0) := (x^\star(\cdot;x_0,u^\star(\cdot)), u^\star(\cdot))$ and for all $x_0 \in \mathbb{X}_0$ the inequality*

$$S(x^\star(T)) - S(x(0)) \leq \int_0^T \ell(x^\star(t),u^\star(t)) - \alpha(||x^\star(t)||_{\Gamma^\star})\, dt, \quad (7)$$

holds with $x^\star(T) = x(T;x_0,u^\star(\cdot))$, $x(0) = x_0$, $\alpha \in \mathscr{K}$. $\square$

Note that the previous definition implicitly assumes that the stage cost ℓ on the manifold is zero, i.e., $\forall x \in \Gamma^\star\ \exists u \in \mathbb{U}$ such that $\ell(x,u) = 0$.

It is well-known that strict dissipativity combined with finite-time reachability of the optimal steady state implies the turnpike property (see, e.g., [9]). The extension of this result to the case of manifolds is straightforward.

Lemma 1. *If the OCP* (4) *is strictly dissipative with respect to the manifold $\Gamma^\star \subseteq \mathbb{X}$, and if for all $x_0 \in \mathbb{X}_0$ there exists an admissible control $u(\cdot)$ such that the manifold $\Gamma^\star$ is reachable in finite time, then the OCP admits a measure state turnpike with respect to $\Gamma^\star$.* $\square$

Proof: The proof follows similarly to [10, Theorem 1] or [8, Theorem 1]. The finite-time-reachability assumption implies that the integral $\int_0^T \ell(x^\star(t),u^\star(t))dt$ is bounded from above, and by definition, the storage function is also bounded. Thus, there exists a finite upper bound $\hat{K}$ such that

$$S(x(0)) - S(x(T)) + \int_0^T \ell(x^\star(t),u^\star(t))dt \leq \hat{K}.$$

Moreover, it follows directly from strict dissipativity that $A := \int_0^T \alpha(||x^\star(t)||_{\Gamma^\star})dt \leq \hat{K}$. Now, since for all $t \in \hat{\Theta}_{\varepsilon,T}$, $\alpha(||x^\star(t)||_{\Gamma^\star}) > \alpha(\varepsilon)$, taking the integral over $\hat{\Theta}_{\varepsilon,T}$ yields $B := \int_{\hat{\Theta}_{\varepsilon,T}} \alpha(||x^\star(t)||_{\Gamma^\star})dt > \mu(\hat{\Theta}_{\varepsilon,T})\alpha(\varepsilon)$. Since $\alpha \in \mathscr{K}$, we have $A \geq B$ and hence we arrive at $\mu(\hat{\Theta}_{\varepsilon,T}) < \hat{K}/\alpha(\varepsilon) := \hat{\nu}(\varepsilon)$. ■

C. Proposed control scheme

Next, we propose an MPFC scheme without terminal constraints or a terminal penalty. The key idea is to exploit the manifold turnpike property to prove the convergence of the scheme. Unlike geometric control schemes, our approach accommodates system constraints explicitly by using NMPC. To this end, and in contrast to [4], we use the original system dynamics (1) and not the transformed coordinates as our NMPC model. What we take from Section II-C is the representation of the path manifold $\Gamma^\star$ in $\mathscr{N}$ as given in (2). We exploit the transverse coordinates defined in (3) to design the stage cost ℓ for the MPFC scheme.

Assumption 3. *For all $(x,u) \in \mathscr{N} \times \mathbb{U}$, the stage cost satisfies*

$$\ell(x,u) = \ell(\Delta^{-1}(\xi,\eta),u) \geq \beta(||\xi||),\ \beta \in \mathscr{K},$$

where $\Delta^{-1}(\xi,\eta) = x$. $\square$

This stage cost choice renders the OCP (4) strictly dissipative with respect to the path manifold $\Gamma^\star$ in a neighborhood of $\Gamma^\star$. In fact, observe that, in $\mathscr{N}$, for any constant storage function the strict dissipativity inequality with respect to $\Gamma^\star$ (7) reduces to the inequality given in Assumption 3. In addition, we know from Lemma 1 that strict dissipativity, combined with a finite-time-reachability of the path manifold $\Gamma^\star$, implies the turnpike property with respect to the path manifold. This fact will be combined with a local controllability assumption with respect to the path manifold to prove the convergence of the NMPC to a closed neighborhood of the path manifold $\Gamma^\star$. The open ε-neighborhood of a closed set $\mathscr{A} \subseteq \mathbb{X}$ denoted $\mathscr{N}_\varepsilon(\mathscr{A})$ is defined as follows

$$\mathscr{N}_\varepsilon(\mathscr{A}) := \{x \in \mathbb{X} : ||x||_{\mathscr{A}} < \varepsilon\},$$

see [12]. The closed ε-neighborhood of $\mathscr{A}$ is denoted by $\bar{\mathscr{N}_\varepsilon}(\mathscr{A})$.

The notion of small-time local controllability with respect to some point $x^\diamond \in \operatorname{int}\mathbb{X}$ implies that there exists $\varepsilon < \gamma$ such that for all $\gamma, \hat{T} > 0$ [18]

$$\mathscr{R}^{\mathscr{N}_\gamma(x^\diamond)}(x^\diamond, \leq \hat{T}) = \mathscr{N}_\varepsilon(x^\diamond) \subset \mathscr{N}_\gamma(x^\diamond),$$

where $\mathscr{R}^{\mathscr{N}_\gamma(x^\diamond)}(x^\diamond, \leq \hat{T})$ denotes the set of states reachable from $x^\diamond$ in at most $\hat{T}$ units of time, whose trajectories remains inside the open neighborhood $\mathscr{N}_\gamma(x^\diamond) := \{x \in \mathbb{X} : ||x - x^\diamond|| < \gamma\}$. This notion was utilized together with the turnpike property in [5] to prove practical convergence to a closed neighborhood of an equilibrium point. In order to generalize this result with respect to the path manifold, we require the following technical assumption.

Assumption 4 (Manifold local controllability)**.** *For some $\delta > 0, \gamma > \varepsilon > 0$, and any $\tilde{x} \in \mathscr{N}_\varepsilon(\Gamma^\star)$, there exist admissible controls $u_{\varepsilon 1}(\cdot), u_{\varepsilon 2}(\cdot)$ such that*

$$\begin{cases} x(\delta/2; \tilde{x}, u_{\varepsilon 1}(\cdot)) = \hat{x} \in \Gamma^\star \\ x(t; \tilde{x}, u_{\varepsilon 1}(\cdot)) \in \mathscr{N}_\gamma(\Gamma^\star) \subseteq \mathbb{X};\ 0 \leq t \leq \delta/2 \\ x(\delta; \hat{x}, u_{\varepsilon 2}(\cdot)) = \tilde{x} \\ x(t; \hat{x}, u_{\varepsilon 2}(\cdot)) \in \mathscr{N}_\gamma(\Gamma^\star) \subseteq \mathbb{X};\ \delta/2 \leq t \leq \delta \end{cases}.$$

□

This assumption requires the control $u_{\varepsilon 1}(\cdot), u_{\varepsilon 2}(\cdot)$ to satisfy all input constraints and leads to constraint-consistent state trajectories such that a feasible periodic orbit is constructed when the state x is ε-close to the path manifold $\Gamma^\star$.

Theorem 1 (Closed-loop path convergence)**.** *Given the nonlinear control system (1), a path $\mathscr{P}$, and let Assumptions 1-4 hold. Suppose that, for all $x_0 \in \mathscr{N}$, the path-following manifold $\Gamma^\star$ is reachable in finite time. Then, there exists a finite time horizon T such that, for all $x_0 \in \mathscr{N}$, the NMPC scheme has the following properties:*

i) If the initial OCP is feasible, then the NMPC scheme is recursively feasible.

ii) The closed-loop system state under the NMPC feedback asymptotically converges to a closed $\hat{\rho}$-neighborhood of the path manifold $\Gamma^\star$, i.e.,

$$\lim_{t \to \infty} ||x(t)||_{\bar{\mathscr{N}_{\hat{\rho}}}(\Gamma^\star)} = 0.$$

Moreover, the region of attraction for the scheme contains all $x_0 \in \mathscr{N}$ for which the initial OCP is feasible. □

Proof: The proof is similar to the main result in [5]. The main difference is that we consider a closed manifold in the state space and not an equilibrium point. We comment on the differences. The choice of the stage cost according to Assumption 3 ensures that for all $x \in \mathscr{N}$, manifold strict dissipativity of the OCPs holds. This fact, combined with the finite-time-reachability of $\Gamma^\star$ for all $x_0 \in \mathscr{N}$, implies the existence of an initial manifold turnpike with respect to $\Gamma^\star$. Then, one can prove the existence of manifold turnpikes in the sequence of OCPs of the NMPC scheme (see [5, Lemma 2]). Recursive feasibility at t_{k+1} can be proved by constructing sub-optimal inputs using the optimal control obtained at the previous time instant $u^\star(\cdot, x(t_k))$ and $u_{\varepsilon 1}$ and $u_{\varepsilon 2}$ from Assumption 4. More precisely, assuming the initial OCP is feasible, and since it exhibits the manifold turnpike property, we split $u^\star(\cdot, x(t_0))$ at some ε-close point to the manifold and insert $u_{\varepsilon 1}$ and $u_{\varepsilon 2}$. The final part is proving practical convergence. This can be proven similar to Proposition 2 of [5]. The main difference is that the closed-ball is taken with respect to $\xi = 0$ and not an equilibrium point, which corresponds to convergence to a closed neighborhood of the path manifold, $\bar{\mathscr{N}_{\hat{\rho}}}(\Gamma^\star)$. ■

Though the notion of practical convergence might seem somehow weak, we illustrate in the next example a core advantage of the manifold turnpike approach to the path-following problem. Namely, and in contrast to [7], [11], it is not necessary to design a control law to stay in the path manifold, which simplifies the control design at the price of losing a bit of path-following accuracy.

IV. MOBILE ROBOT EXAMPLE

We consider the kinematic model for a nonholonomic wheeled mobile robot

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad x(t_0) = x_0$$

with output $y = [x_1, x_2]^\top$, where the state $x = [x_1, x_2, x_3]^\top \in \mathbb{R}^3$ represents respectively the horizontal and vertical positions (in meters) in the plane, and the orientation of the robot with respect to the positive horizontal axis. The input u refers to the angular velocity of the robot in rad/s and v is the speed of the robot in m/s.

The considered path following problem entails forcing the system output y to approach and follow the unit circle centered at the origin in the output space with a desired speed in a desired direction. The path $\mathscr{P}$ is implicitly given as $\mathscr{P} = \{y \in \mathbb{R}^2 : y_1^2 + y_2^2 - 1 = 0\}$. In contrast to [22], [23], where transverse feedback linearization is utilized to solve a similar problem, in our example, we use NMPC which allows us to explicitly consider the following input constraints: $u \in \mathbb{U} := [-1, 1]$, $v \in \mathbb{V} := [-10, 10]$. Additionally, we do not assume a constant speed v of the robot, however, the desired path-following speed v_{ref} will be ensured by the addition of an appropriate term to the cost functional. This allows the controller to freely manipulate the robot's speed before following the path with the desired speed.

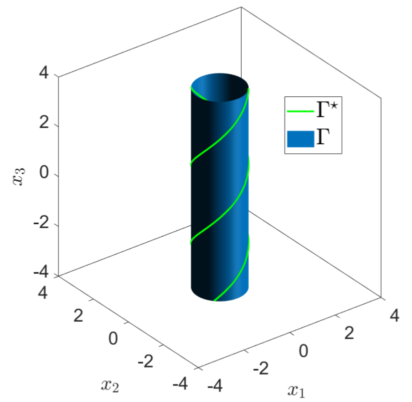
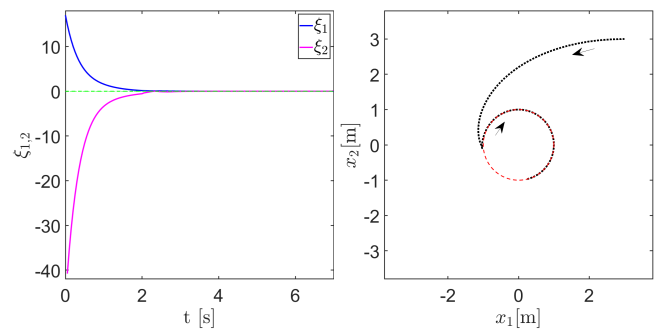


Fig. 1: Visualization of Γ and the path manifold $\Gamma^\star$.



(a) Manifold turnpike (b) Optimal trajectory

Fig. 2: The turnpike phenomenon.

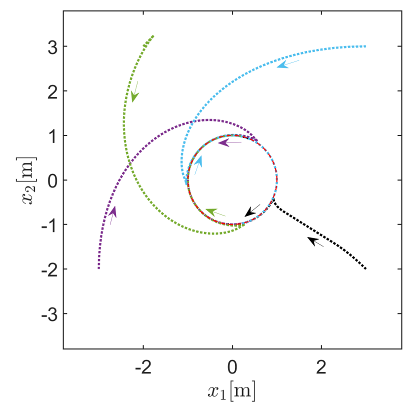


Fig. 3: MPFC with no enforced direction.

The set of all states corresponding to the path $\mathscr{P}$, Γ, forms a cylinder in the state space (Fig. 1) $\Gamma = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 - 1 = 0\}$. To find the largest controlled-invariant submanifold of Γ, we constrain the velocity vector $[\dot{x}_1, \dot{x}_2]^\top$ to be always tangential to the path $\mathscr{P}$, i.e., $[x_1, x_2] \cdot [\dot{x}_1, \dot{x}_2]^\top = 0$. Thus, the path manifold $\Gamma^\star \subset \Gamma$ is defined for all $v \neq 0$ as follows

$$\Gamma^\star = \left\{ x \in \mathbb{R}^3 : \begin{bmatrix} x_1^2 + x_2^2 - 1 = 0 \\ x_1 \cos x_3 + x_2 \sin x_3 = 0 \end{bmatrix} \right\}.$$

The path manifold $\Gamma^\star$ defines a 1-dimensional closed manifold in $\mathbb{R}^3$ consisting of two disconnected helices as shown in Fig. 1. One is generally interested in stabilizing one of the helices depending on the desired path-following direction (i.e., clockwise or counter-clockwise). In the geometric analysis of the problem, we only consider the control u. The reasons for this are twofold: first, this simplifies the analysis; second, it is clear that regardless of the velocity of the robot, one mainly need to control the steering through the angular velocity u to approach and follow the path. The desired behavior of the speed v is straightforward (starting high and eventually converging to v_{ref}). Thus, here, we assume $f(x,t) = [\cos x_3 \ \sin x_3 \ 0]^\top v(t)$, i.e., we consider the speed contribution to the dynamics as a time-varying drift. Then, we can easily satisfy the conditions of Assumption 2. More precisely, for $\sigma(x) := x_1^2 + x_2^2 - 1$, since $L_g \sigma(x) = 0$ and

$$L_g L_f \sigma(x) = 2v(t)(x_2 \cos x_3 - x_1 \sin x_3) \neq 0$$

for all $x \in \mathscr{N} := \{x \in \mathbb{R}^3 : x_2 \cos x_3 \neq x_1 \sin x_3\}$ and $v \neq 0$. Thus, σ yields a well-defined relative degree $r = 2 = n_x - n^\star$ in a neighborhood $\mathscr{N} \supset \Gamma^\star$. Thus, on $\mathscr{N}$, the path manifold $\Gamma^\star$ can be represented via (2). According to (3), the transverse coordinates are $\xi = [\sigma(x), L_f \sigma(x)]^\top = [x_1^2 + x_2^2 - 1, 2v(t)(x_1 \cos x_3 + x_2 \sin x_3)]^\top$. Then, we design the stage cost ℓ as follows

$$\ell(x,u) = \xi^\top Q_\xi \xi + R_v (v - v_{ref})^2 + R_u u^2,$$

where $Q_\xi = Q_\xi^\top \succ 0$, $R_u, R_v > 0$. The interpretation of this choice is as follows: the first term ensures the satisfaction of the manifold strict dissipativity (7), the second term ensures the desired speed is eventually reached, and the third term is a regularization term that sets a compromise between the control effort of u and the "exactness" of path following. For this example, we choose $Q_\xi = \mathrm{diag}(100, 10)$, $R_v = 1$, and $R_u = 10^{-5}$. Notice that, when the robot follows the path very closely with the desired speed, the cost $\ell(x,u)$ is almost negligible. The manifold turnpike property (in ξ-coordinates) is depicted in Fig. 2a for $x_0 = [3,3,0]^\top$, $T = 7$ s, and $v_{ref} = 1$. The corresponding optimal trajectory is shown in Fig. 2b. As expected and verified by our simulations, the larger T is, the longer the optimal trajectory spends near the path manifold. Note that the aforementioned choice of the prediction horizon T is arbitrary, nevertheless, it is of interest, albeit outside the scope of this paper, to find the minimum stabilizing T. The corresponding MPFC simulation is shown for different initial conditions in Fig. 3. Although stabilizing, it is clear that the path-following direction depends on the initial condition. Enforcing a desired path-following direction may be achieved by introducing the following virtual constraint

$$\mathrm{sgn}([x_1(k), x_2(k)]^\top \cdot [x_1(k+1), x_2(k+1)]^\top) \geq 0,$$

where sgn is the sign function. The simulation results of the MPFC scheme with enforced counter-clockwise path-following direction are shown in Fig. 4. The corresponding state and input time-evolutions for the initial condition $x_0 = [3,3,0]^\top$ are shown in Fig. 5.

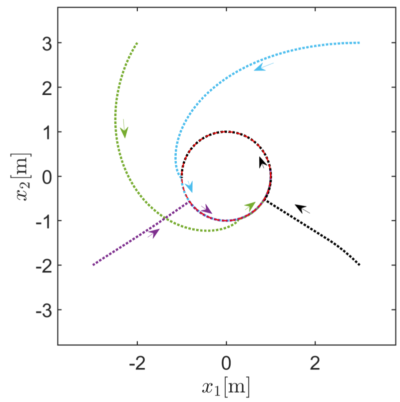


Fig. 4: MPFC with counter-clockwise enforced direction.

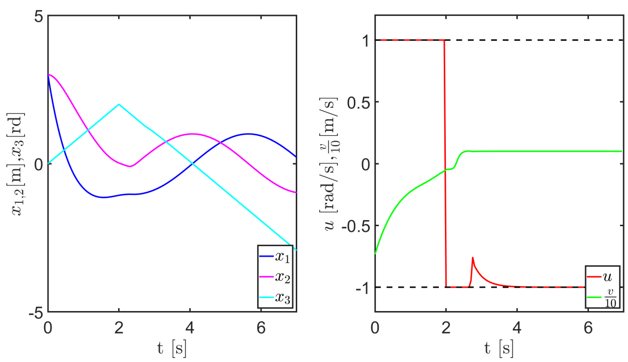


Fig. 5: State and input trajectories of the NMPC scheme

V. CONCLUSIONS

This paper proposed and analyzed a turnpike approach towards the analysis of model predictive path-following control without terminal constraints. Leveraging a turnpike property with respect to the path manifold, which is implied by a suitable stage cost design, we have shown practical closed-loop convergence to the path manifold. It is of practical interest to extend this result by considering smooth regular parametrized paths, e.g., generated by spline-based path or motion planning.

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